# 7. COMMISSION DE LA MECANIQUE CELESTE

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## INTRODUCTION

Since the Moscow meeting the Commission has lost by death its members Professor M. Milankovitch, who died on 12 December 1958, and Dr R. Vernic, who died on 20 October 1958.

During the past three years the activity in celestial mechanics has been at an exceptionally high level. A considerable part of the current literature in celestial mechanics has been concerned with the problem of the motion of artificial satellites. A second contributing circumstance has been the greater use of high-speed calculators for the solution of problems in celestial mechanics.

## ARTIFICIAL SATELLITE MOTION

The literature on this subject is so vast that a comprehensive bibliography of the subject would be beyond the scope of this report. Even a critical analysis of what has been accomplished in this field would be a major undertaking which, at the present time, might be premature. For these reasons only a few comments will be recorded.

Some investigations on the subject of artificial satellite motion were primarily concerned with the analytical solution of the drag-free motion of a small particle under the gravitational attraction of a non-spherical primary, others with the evaluation of the coefficients of the higher terms in the Earth's potential, and still others with the evaluation of the density of the air at great heights. A great deal of new information has been gained from these various aspects of the problem.

The equations of motion of the drag-free problem show a considerable resemblance with the equations of motion of the main problem of the lunar theory, although the character of the disturbing function is, of course, quite different. It is not surprising, therefore, that among the methods that have been used for the solution of the artificial satellite problem many correspond to methods that were used for the solution of the problem of the Moon's motion.

Numerous contributions to the subject have been produced by scientists trained in applied mathematics and general dynamics, but with little previous acquaintance with the literature of celestial mechanics. This has resulted in a freshness of approach, but also in some confusion of terminology and notation. The problem has also attracted the attention of mathematicians who have dealt with questions concerning the convergence of series and whether or not definite bounds can be established for variables that occur in the solution. This revival of interest holds promise for the future development of the field, if it should be extended to more difficult problems.

As a result of all this activity the character of the analytical solution of the problem of the drag-free motion of a particle in the gravitational field of a non-spherical planet with rotational symmetry has been thoroughly explored. The solution is generally obtained in a form similar to that of the solution of the main problem of the lunar theory, except for its greater simplicity. The arguments of sine and cosine terms that appear in the solution are linear combinations of



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two fundamental arguments, as against four in the main problem of the lunar theory. The solution is developed in powers of the coefficients that appear in the development of the Earth's potential in spherical harmonics, but developments in powers of the eccentricity or inclination are not needed to perform the integration. If $J$ designates the coefficient of the principal oblateness term in the potential, and if all further terms are at least of $O(J^2)$, then the solution is usually obtained to $O(J)$ for the periodic terms and to $O(J^2)$ in the motions of perigee and node. This limitation is probably adequate for current applications.

In general, it has been a simple matter to establish agreement among the various published solutions to the first power of $J$. Apparent disagreements of the coefficients of $J^2$ in the secular motions have in many cases been explained as the consequence of differences in the definition of the constants of integration, especially the inclination constant.

The solution described holds for any inclination except in a region near the critical inclination ($\tan I_c = 2$), because of the occurrence of divisors $1-5\cos^2 I$ in the solution. For orbits in the vicinity of the critical inclination a solution in powers of $J^{\frac{1}{2}}$ can be obtained, and has been the subject of several contributions (**1, 2**).

A different form of solution was obtained by Vinti (**3**), who showed that if the force function, written in the form

$$U = \frac{\mu}{r}\left[1 - \sum_{n=1}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\sin\beta)\right] \quad ,$$

satisfies the conditions

$$J_{2n} = (-1)^{n+1} J_2^{\,n} \quad , \quad J_{2n+1} = (-1)^n J_1 J_2^{\,n} \quad ,$$

the solution can be achieved by separation of variables. It is expressed with the aid of elliptic integrals.

Vinti's solution is evidently the solution of a special potential, for which the critical inclination does not exist. It is found that if $J_4 = -J_2^2$, as required by Vinti, all the divisors of the form $1 - 5\cos^2 I$ vanish from the general solution in which terms with $J_2$ and $J_4$ as factors are included.

The odd harmonics with coefficients $J_3$ and $J_5$ were first introduced by O'Keefe, Eckels and Squires (**4**). Subsequently King-Hele (**5**) and Kozai (**6**) included them in their solutions. The three solutions with their mean errors are:

**All quantities are in units of $10^{-6}$**

| | O'Keefe *et al.* | King-Hele | Kozai |
|---|---|---|---|
| $J_2$ | $+1082{\cdot}5 \pm 0{\cdot}1$ | $+1082{\cdot}79 \pm 0{\cdot}15$ | $+1082{\cdot}19 \pm 0{\cdot}04$ |
| $J_3$ | $-\ 2{\cdot}4 \pm 0{\cdot}4$ | $-\ 2{\cdot}4 \pm 0.3$ | $-\ 2{\cdot}28 \pm 0{\cdot}03$ |
| $J_4$ | $-\ 1{\cdot}7 \pm 0{\cdot}1$ | $-\ 1{\cdot}4 \pm 0{\cdot}2$ | $-\ 2{\cdot}12 \pm 0{\cdot}06$ |
| $J_5$ | $-\ 0{\cdot}05 \pm 0{\cdot}1$ | $-\ 0{\cdot}1 \pm 0{\cdot}1$ | $-\ 0{\cdot}23 \pm 0{\cdot}03$ |
| $J_6$ | — | $+\ 0{\cdot}9 \pm 0{\cdot}8$ | — |

A weighted mean value for the ratio $-J_4/J_2^2$ is $+1{\cdot}70 \pm 0{\cdot}05$, significantly exceeding unity, which would apply to Vinti's special potential. Nevertheless, Vinti's solution as an intermediate orbit may have advantages, as suggested by Izsak (**7**).

The many publications dealing with artificial satellite motion have exhibited a great variety of notations for the coefficients of the terms in the Earth's potential. A recommendation by the

Commission may lead to greater uniformity in future work. The notation adopted in this report has the objection that $J_k$ is the standard designation for Bessel Functions.

It may suffice to add a few general references to the literature by remarking that many papers on artificial satellite motion have appeared in the *Astronomical Journal*, the *Astronomical Journal, Moscow*, the *Bulletin of the Institute of Theoretical Astronomy* (Leningrad), as well as in various series of special reports. Among these are the series 'Artificial Earth Satellites' (Moscow), the 'Bulletin of the Stations for the Observation of Artificial Satellites' (Moscow), the series 'Research in Space Science, Special Reports' issued by the Smithsonian Astrophysical Observatory, and the reports issued by the Royal Aircraft Establishment, Farnborough, England. An early paper by King-Hele (**8**) appeared in the *Proceedings of the Royal Society*.

## USE OF HIGH-SPEED COMPUTERS

At the Institute of Theoretical Astronomy in Leningrad Gontkovskaya (**9**) and Polozova (**10**) made a study of the use of high-speed computers in the application to various forms of general planetary theories, while Shor (**11**) considered the solution of the restricted problem of three bodies by the Hill-Brown method. In this connection it may be remarked that the Theory of Mars by Clemence (**12**) is an early example of a development of a general planetary theory by Hansen's method, carried out with the use of high-speed computers.

Many applications of high-speed computers pertain to solutions by numerical integration. The motions of Trojan-type orbits received particular attention. Examples are the papers by Chebotarev and Bozhkova (**13**) who studied the motions of two Trojan asteroids, Patroclus and Anchises over a period of 700 years. Chebotarev and Volkov (**14**) also integrated the motion of Trojan asteroids by Proto-Jupiter with Jupiter's mass put at 1/20 the Sun's mass, to explore Kuiper's (**15**) theory of the origin of the Trojans.

Rabe reports on an as yet unpublished investigation of the well-known sequence of long-period libration orbits associated with the triangular solutions in the restricted problem of three bodies. By making use of recurrence formulas proposed by Steffenson (**16**) he succeeded in obtaining nearly periodic variation orbits. An iteration process was then used to obtain a periodic orbit to the desired degree of accuracy. Rabe remarks that his results, to a high degree of computational accuracy, refute the claims of Thüring (**17**), that these periodic orbits do not exist.

Myachin (**18**) has investigated the problem of the accumulation of errors of rounding in numerical integration; his theory confirms qualitatively the result obtained by Brouwer (**19**) that after $k$ steps, $k$ being sufficiently large, the mean error is proportional to $k^{3/2}$. A further paper on this subject, by Sochilina (**20**), concerns applications of the theory.

It is impossible here to note all significant research undertakings that involved extensive use of high-speed computers. An exception may be made of the work by Kovalevsky (**36**) to which reference is made in a later section of this report.

## THE PROBLEM OF THREE BODIES

Merman (**21**) has examined the problem concerned with the "final motions" in the problem of three bodies when the energy constant is negative. Sufficient conditions are given in order that the motion decomposes into two almost independent motions which are both close to Keplerian motion. In a further paper (**22**) he deals with the representation of the solution of the problem of three bodies by convergent series. The method is distinctly different from Sundman's method, although related to it. On the basis of results obtained by Sundman he

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concludes that, provided that all constants of angular momentum are not zero together, the Mittag-Leffler star region of the general solution contains the whole real $w$-axis. As a consequence of this fact the general solution can be expanded in series in terms of polynomials of $w$, convergent for all values of $w$.

### PLANETARY THEORY

Clemence reports that his new general theory of the motion of Mars, constructed according to the adaptation of Hansen's method that G. W. Hill used for Jupiter and Saturn, is now regarded as complete. It has been compared with a numerical integration of the orbit by Herget with the Naval Ordnance Research Calculator for the years 1919–54. After adjustment of the elements of the orbit the root mean square value for 162 differences in orbital longitude is $0''\cdot013$, and for the latitude $0''\cdot002$. It clearly represents a most significant step forward in the construction of an accurate planetary theory.

Rectangular co-ordinates of Mars referred to mean equinox and equator of 1950·0 with an interval of 4 days for the period 1950 to 2000, based on the new theory, have been published by Duncombe and Clemence (**23**). These are based on provisional elements derived from 87 observations made in the years 1802–39 and 1931–50. Further improvement in accuracy must await the derivation of definitive elements, now being done by Duncombe. In the meantime, the published rectangular co-ordinates are a decided improvement over the values published in the current ephemerides.

Lyttleton (**24**), following up ideas of Littlewood, has shown how the position of Neptune in 1846 could have been predicted with comparatively little trouble and with greater accuracy than was achieved by Adams and Leverrier. The most important improvement results from combining the data in such a way that the unknown eccentricity has very little effect on the equations actually used, and consequently on the predicted longitude.

A paper by Lyakh (**25**) deals with a form of development of the disturbing function suitable for any value of the mutual inclination and more convenient than Tisserand's method.

Bazhenov (**26**) has published two methods, based on Chebyshev's approximation, for computing general perturbations of minor planets. Jarow-Jarowoi (**27**) has constructed an approximate theory of Ceres that proved to be in essential agreement with observations in the years 1801–1937.

### SATELLITE THEORY

Petrowskaja (**28**), by a generalization of Wintner's method, has shown that Hill's series representing the variation orbit are actually convergent for

$$|\mathrm{m}| \leqslant 0\cdot21, \qquad \mathrm{m} = n/(n-n')$$

compared with

$$|\mathrm{m}| \leqslant \tfrac{1}{7}\ \text{(Lyapunov, 1896)} \quad \leqslant \tfrac{1}{12}\ \text{(Wintner, 1929)} \quad \leqslant 0\cdot18\ \text{(Merman, 1952)}$$

A letter from Joachim Schubart reports that, at the suggestion of Professor C. L. Siegel at Göttingen, he has examined periodic solutions of Hill's lunar problem which correspond to Poincaré's "Solutions de la deuxième et troisième sorte". The method is similar to that used by Siegel (**29**) for the treatment of variational orbits.

Hori (**30**) has developed a theory of the orbit of Jupiter's ninth satellite. He used the method developed by Brown and Brouwer for the theory of Jupiter's eighth satellite (**31**). A comparison with observations yields satisfactory agreement.

Kozai has studied the effects of the attraction of Saturn's ring on the motion of the inner satellites, and finds the mass of the ring to be of the order $10^{-4}$ times the mass of Saturn.

Duboshin (32) has worked out the principles of a new theory of motions of Saturn's satellites; some numerical applications of the theory have been published by Rybakov (33), while Kosachevsky (34) has used the theory in an investigation of the motions of the satellites of Mars.

Grebenikov (35) has published a theory of the motion of Iapetus which yields satisfactory agreement with observations 1898–1958.

Kovalevsky (36) has explored the possibility of using the data obtained in a numerical integration for the construction of a general theory of the motion of a satellite perturbed by the Sun. He applied the method to the motion of Jupiter's eighth satellite. The problem consists of obtaining the periods of the four fundamental arguments present in the theory, and the coefficients of the principal terms by harmonic analysis. In the motion of Jupiter's eighth satellite a difficulty presents itself on account of the slow motion of the perijove with period of the order of 25,000 years, while the numerical integration data produced and used by Kovalevsky cover only 100 years. The escape from the difficulty is to ignore the motion of the perijove, which is permissible if a representation for a few centuries is required. By a judiciously chosen procedure the author succeeds in reducing the harmonic analysis part of the work to a manageable scope. The comparison of the approximate theory so obtained with the numerical integration data yields 190″ for the r.m.s. deviation in geocentric position. This is about one-fifth of the r.m.s. deviation of the best analytical theory previously constructed. It is further shown that these deviations must be ascribed mainly to omitted terms, not evaluated by the process of harmonic analysis. The concluding section deals with a method of iteration in which the approximate theory is used to compute the right-hand members of the equations of the variation of the elements.

## MISCELLANEOUS CONTRIBUTIONS

A determination of the mass of Mercury by Makover and Bokhan (37) from the motion of Comet Encke-Backlund in the years 1898–1911 gives for the reciprocal of the mass

$$m^{-1}(\text{Mercury}) = 5\,980\,000 \pm 170\,000.$$

Short notes by Rabe (38) deal with the suitability of the orbit of (1362) Griqua for obtaining accurately the mass of Jupiter and the orbit of (1011) Laodamia for the mass of Mars.

Duncombe (39) published a discussion of the observations of Venus (1750–1949). An important result of this discussion is that the discordance that Newcomb found between the observed and computed motion of the node is not confirmed. Duncombe writes: 'The reason for Newcomb's discrepancy has eluded explanation, but is probably attributable to large systematic errors in the older observations.'

Jeffreys (40) has studied the motion of a pendulum under small harmonic disturbing force. When the forcing period is longer than the period of small free oscillations there are three periodic solutions, two of which are large while the other approximates to the elementary solution. When the periods are sufficiently close the latter coalesces with one of the others and the two disappear, while the other large solution passes continuously into the elementary solution when the forcing period is the shorter. A conclusion will be that if a system passes through resonance there is a radical difference of behaviour according as the forcing period increases or decreases through the elementary free period. If the force is $O(a)$ and the period increases the elementary solution attains amplitude $O(a^{1/3})$ near coincidence and ceases to be small when the difference of the two periods becomes large. If the period diminishes, the stable solution attains $O(a^{1/3})$ and is followed by a stage of libration, the amplitude again becoming $O(a)$ when the difference of periods becomes large.

Littlewood (41) has discussed the convergence and asymptotic properties of solutions of the problem of three bodies, with special reference to Lagrange's particles. Bounds are given for the errors. It remains doubtful whether the series ever converge except in special cases such as periodic orbits. In particular a conjecture of E. T. Whittaker appears to be false.

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