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THE HIGHER ORDER COMMUTATORS OF THE FRACTIONAL INTEGRALS ON HARDY SPACES

SHUNCHAO LONG✉ and JIAN WANG

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Abstract

In this paper we investigate the boundedness on Hardy spaces for the higher order commutator $T^{\tau}_{b,m}$ generated by the *BMO* function b and fractional integral type operator T^{τ}, and establish the boundedness theorems for $T^{\tau}_{b,m}$ from $H^{p_1,q_1,s}_{b,m}$ to L^{p_2} and to H^{p_2} $(0 < p_1 \le 1)$, and from $H\dot{K}^{\alpha,p_1,s}_{q_1,b,m}$ to $\dot{K}^{\alpha,p_2}_{q_2}$ and to $H\dot{K}^{\alpha,p_2}_{q_2}$, respectively, for certain ranges of α, p_1, q_1, p_2, q_2 and s.

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1. Introduction

Let T be a linear operator and b a *BMO* function. The higher order commutator operators are defined as

$$T_{b,m}f(x) = T((b(\cdot) - b(x))^m f(\cdot))(x), \quad m = 0, 1, 2, \ldots.$$

Obviously, $T_{b,0} = T$, $T_{b,1} = [b, T]$ which is the commutator in [6], and

$$T_{b,m} = [b, T_{b,m-1}], \quad m = 1, 2, \ldots.$$

Coifman, Rochberg and Weiss [6] stated that if T is a Calderón-Zygmund singular integral operator, then $T_{b,1} = [b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo [4] extended this result to the fractional integral. Subsequently, many authors have

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studied the L^p–boundedness of the commutator $T_{b,1} = [b, T]$ (see, for example, [2, 3, 10, 22]) and the higher order commutators $T_{b,m}$, $m = 0, 1, \ldots$, (see [8, 12, 15, 23]). The case $0 < p \le 1$ was also considered by many authors. When T is a Calderón-Zygmund singular integral operator, Perez [21], Pluszynski [20] and Alvarez [1] showed that $[b, T]$ does not map $H^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. However, Perez [21] proved that $T_{b,m}$ maps the modified spaces $H^{1,\infty,0}_{b,m}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, Alvarez [1] obtained that $[b, T]$ maps $H^{p,\infty,0}_{b,1}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, and Long and Wang [15] proved that $T_{b,m}$ maps $H^{p,q,s}_{b,m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ and to $H^p(\mathbb{R}^n)$. In this paper, we will extend these results to the higher order commutators $T^\tau_{b,m}$ of fractional integral type operators T^τ and consider their boundedness from $H^{p_1,q_1,s}_{b,m}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$ and to $H^{p_2}(\mathbb{R}^n)$ for $0 < p_1 \le 1$.

On the other hand, Herz type Hardy spaces were recently studied by many authors (see [5, 9, 11, 12, 16–19]). The boundedness of some operators on Herz spaces and Herz type Hardy spaces can be found in [11, 12, 14–17, 19]. If T is a standard Calderón-Zygmund operator, $[b, T]$ is bounded on Herz space $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ (or $K^{\alpha,p}_q(\mathbb{R}^n)$) for $-n/q \le \alpha < n(1 - 1/q)$, but not for $\alpha \ge n(1 - 1/q)$ (see [11, 19]). It is not bounded even from $H\dot{K}^{\alpha,p,0}_q(\mathbb{R}^n)$ (or $HK^{\alpha,p,0}_q(\mathbb{R}^n)$) into $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ (or $K^{\alpha,p}_q(\mathbb{R}^n)$) for $\alpha \ge n(1 - 1/q)$. However, $[b, T]$ is bounded from $H\dot{K}^{\alpha,p,0}_{q,b}(\mathbb{R}^n)$ (or $HK^{\alpha,p,0}_{q,b}(\mathbb{R}^n)$) into $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ (or $K^{\alpha,p}_q(\mathbb{R}^n)$) for $n(1 - 1/q) \le \alpha < n(1 - 1/q) + \gamma$ (see [19]), just as the cases involving the standard Hardy space $H^1(\mathbb{R}^n)$ and the Lebesgue space $L^1(\mathbb{R}^n)$. Long and Wang [15] obtained the boundedness of the higher order commutators $T_{b,m}$ of the Calderón-Zygmund singular integral operators T from $H\dot{K}^{\alpha,p,s}_{q,b,m}(\mathbb{R}^n)$ (or $HK^{\alpha,p,s}_{q,b,m}(\mathbb{R}^n)$) into $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ (or $K^{\alpha,p}_q(\mathbb{R}^n)$), and from $H\dot{K}^{\alpha,p,s}_{q,b,m}(\mathbb{R}^n)$ (or $HK^{\alpha,p,s}_{q,b,m}(\mathbb{R}^n)$) into $H\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ (or $HK^{\alpha,p}_q(\mathbb{R}^n)$) for some ranges of p, q, s and α. The boundedness of higher order commutators of fractional integrals on Herz spaces was obtained for a range of α in [12]. Here, we will also investigate the boundedness for the higher order commutators $T^\tau_{b,m}$ of the fractional integral type operators T^τ from the Herz type Hardy spaces $H\dot{K}^{\alpha,p_1,s}_{q_1,b,m}(\mathbb{R}^n)$ (or $HK^{\alpha,p_1,s}_{q_1,b,m}(\mathbb{R}^n)$) to Herz spaces $\dot{K}^{\alpha,p_2}_{q_2}(\mathbb{R}^n)$ (or $K^{\alpha,p_2}_{q_2}(\mathbb{R}^n)$) and from $H\dot{K}^{\alpha,p_1,s}_{q_1,b,m}(\mathbb{R}^n)$ (or $HK^{\alpha,p_1,s}_{q_1,b,m}(\mathbb{R}^n)$) to $H\dot{K}^{\alpha,p_2}_{q_2}(\mathbb{R}^n)$ (or $HK^{\alpha,p_2}_{q_2}(\mathbb{R}^n)$) for certain ranges of α, p_1, q_1, p_2, q_2 and s.

Let us introduce some definitions below.

DEFINITION 1. Let $0 \le \tau < n$, $0 < \gamma \le 1$, $s \in \mathbb{N} \cup \{0\}$, $1 < q_1 \le q_2 < \infty$ be such that $1/q_1 - 1/q_2 = \tau/n$. T^τ is said to be a $(q_1, \tau; s, \gamma)$-fractional integral type operator if T^τ is a bounded singular integral operator from $L^{q_1}(\mathbb{R}^n)$ into $L^{q_2}(\mathbb{R}^n)$ with kernel $K(x, y)$, which is C^∞ away from the origin and satisfies the following conditions:

(i) $T^\tau f(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, dy$, if $x \ne y$;

(ii) $\left| \dfrac{\partial^\zeta K(x, y)}{\partial y^\zeta} - \dfrac{\partial^\zeta K(x, y')}{\partial y^\zeta} \right| \le C_\zeta \dfrac{|y - y'|^\gamma}{|x - y|^{n-\tau+s+\gamma}}$, if $|x - y| \ge 2|y - y'|$, where $\zeta = (\zeta_1, \ldots, \zeta_n)$ is any multi-index and $s = |\zeta| = \zeta_1 + \cdots + \zeta_n$.

Denote by $[r]$ the integer part of the real number r. For $\beta = (\beta_1, \dots, \beta_n) \in (\mathbb{N} \cup \{0\})^n$, $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ and $|\beta| = \beta_1 + \cdots + \beta_n$. Let

$$\|f\|_{L^q(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^q \, dx \right)^{1/q}.$$

Denote by T^* the conjugate operator of T.

DEFINITION 2. Let $0 < p \le 1 \le q \le \infty$, $p < q$, $[n(1/p - 1)] \le s < \infty$. A function $a(x)$ is said to be a $(p, q, s; b)$-atom of order m if there exists a ball B for which

(i) $\operatorname{supp} a \subseteq B = B(x_0, r) = \{x : |x - x_0| < r\}$;
(ii) $\|a\|_{L^q(\mathbb{R}^n)} \le |B|^{1/q - 1/p}$;
(iii) $\int_{\mathbb{R}^n} a(x) b^i(x) x^\beta dx = 0$; $|\beta| \le s$, $i = 0, 1, \dots, m$.

DEFINITION 3. Let $0 < p \le 1 \le q \le \infty$, $p < q$, $[n(1/p - 1)] \le s < \infty$. We define $f \in H_{b,m}^{p,q,s}(\mathbb{R}^n)$ if and only if $f(x) = \sum_{k \in \mathbb{N}} \lambda_k a_k(x)$, where each a_k is a $(p, q, s; b)$-atom of order m, $\sum_{k \in \mathbb{N}} |\lambda_k|^p < +\infty$, and $\|f\|_{H_{b,m}^{p,q,s}(\mathbb{R}^n)} \sim \left(\sum_{k \in \mathbb{N}} |\lambda_k|^p \right)^{1/p}$.

Obviously, $H_{b,m}^{1,\infty,0}(\mathbb{R}^n)$ are the spaces $H_{b,m}^1(\mathbb{R}^n)$ which were introduced by Perez in [21]. By the atomic decomposition theory of Coifman and Weiss [7, 13, 25], if $0 < p \le 1 \le q \le \infty$, $p < q$ and $[n(1/p - 1)] \le s < \infty$, it is easy to see that $H_{b,0}^{p,q,s}(\mathbb{R}^n) = H^p(\mathbb{R}^n)$, the classical Hardy spaces.

THEOREM 1.1. *Let* $0 < p_1 \le 1$, $1/p_2 = 1/p_1 - \tau/n$, $0 < \gamma \le 1$, $s > [n(1/p_1 - 1)]$, $1 < q_1 \le \infty$, *and let* T^τ *be a* $(q_1, \tau; s, \gamma)$*-fractional integral type operator (as in Definition* 1) *and* $b \in BMO$. *If* $n/(n - \tau + s + \gamma) < p_2 < +\infty$ *and* $0 \le \tau < n$, *then* $T_{b,m}^\tau$ *maps* $H_{b,m}^{p_1,q_1,s}(\mathbb{R}^n)$ *into* $L^{p_2}(\mathbb{R}^n)$.

REMARK 1. Theorem 1.1 is equivalent to [21, Theorem 1.9] when $\tau = 0$, $p_1 = p_2 = 1$, $s = 0$, $q_1 = \infty$; and to [1, Theorem 1.5] when $\tau = 0$, $m = 1$, $s = 0$.

THEOREM 1.2. *Let* $0 < p_1 \le 1$, $1/p_2 = 1/p_1 - \tau/n$, $0 < \gamma \le 1$, $s > [n(1/p_1 - 1)]$, $1 < q_1 \le \infty$, *and let* T^τ *be a* $(q_1, \tau; s, \gamma)$*- fractional integral type operator and* $b \in BMO$. *Assume that* $(T^\tau)^*(g_{i,\beta}) = C$ *(a constant)*, $g_{i,\beta}(x) = b^i(x) x^\beta$, $|\beta| \le s$, $i = 0, 1, \dots, m$. *If* $n/(n - \tau + s + \gamma) < p_2 \le 1$ *and* $0 \le \tau < \gamma$, *then* $T_{b,m}^\tau$ *maps* $H_{b,m}^{p_1,q_1,s}(\mathbb{R}^n)$ *into* $H^{p_2}(\mathbb{R}^n)$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $C_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, where χ_{C_k} is the characteristic function of set C_k.

DEFINITION 4. Let $0 < \alpha < \infty$, $0 < p < \infty$, $1 \le q < \infty$. The Herz spaces are defined by

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(a) $\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < +\infty\}$ (homogeneous space), where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left(\sum_{k\in\mathbb{Z}} |B_k|^{\alpha p/n} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p\right)^{1/p},$$

(b) $K_q^{\alpha,p}(\mathbb{R}^n) = L^q(\mathbb{R}^n) \cap \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (non-homogeneous space), and $\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \|f\|_{L^q(\mathbb{R}^n)} + \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$.

Obviously, $\dot{K}_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) = K_p^{0,p}(\mathbb{R}^n)$ for all $0 < p \le \infty$.

DEFINITION 5. Let $-\infty < \alpha < \infty$, $0 < p < \infty$, $1 \le q < \infty$ and $s \in \mathbb{N} \cup \{0\}$. A function $a(x)$ is said to be an mth order central $H\dot{K} - (\alpha, q, ; b)_s$-atom if $a(x)$ satisfies

(i) $\operatorname{supp} a \subseteq B(0,r) = \{x \in \mathbb{R}^n : |x| < r,\ r > 0\}$;
(ii) $\|a\|_{L^q(\mathbb{R}^n)} \le |B(0,r)|^{-\alpha/n}$;
(iii) $\int_{\mathbb{R}^n} a(x) b^i(x) x^\beta\, dx = 0$, $|\beta| \le s$, $i = 0, 1, \ldots, m$.

A function $a(x)$ is said to be an mth order central $HK - (\alpha, q, ; b)_s$-atom if $a(x)$ satisfies (ii), (iii) and

(i′) $\operatorname{supp} a \subseteq \overline{B(0,r)} = \{x \in \mathbb{R}^n : |x| < r,\ r > 1\}$.

DEFINITION 6. Let $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \le \alpha < \infty$ and $s \in \mathbb{N}\cup\{0\}$. We define $f \in H\dot{K}_{q,b,m}^{\alpha,p,s}(\mathbb{R}^n)$ (or $HK_{q,b,m}^{\alpha,p,s}(\mathbb{R}^n)$) if and only if $f(x) = \sum_{k\in\mathbb{Z}} \lambda_k a_k(x)$ $\big($or $f(x) = \sum_{k\ge 0} \lambda_k a_k(x)\big)$, where each a_k is an m order central $H\dot{K}-$ (or $HK-$) $(\alpha, q, ; b)_s$-atom with the support B_k, $\sum_{k\in\mathbb{Z}} |\lambda_k|^p < +\infty$ $\big($or $\sum_{k\ge 0} |\lambda_k|^p < +\infty\big)$, and

$$\|f\|_{H\dot{K}_{q,b,m}^{\alpha,p,s}(\mathbb{R}^n)} \sim \left(\sum_{k\in\mathbb{Z}} |\lambda_k|^p\right)^{1/p} \quad \left(\text{or } \|f\|_{HK_{q,b,m}^{\alpha,p,s}(\mathbb{R}^n)} \sim \left(\sum_{k\ge 0} |\lambda_k|^p\right)^{1/p}\right).$$

If $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \le \alpha < \infty$ and $s \ge [\alpha - n(1-1/q)]$, it is easy to see that $H\dot{K}_{q,b,0}^{\alpha,p,s}(\mathbb{R}^n) = H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, $HK_{q,b,0}^{\alpha,p,s}(\mathbb{R}^n) = HK_q^{\alpha,p}(\mathbb{R}^n)$ (see [11, 17] or [18, 19]). For $0 < p < \infty$, $H\dot{K}_p^{0,p}(\mathbb{R}^n) = HK_p^{0,p}(\mathbb{R}^n)$ which are the usual Hardy spaces $H^p(\mathbb{R}^n)$. In particular, $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ when $p > 1$. When $1 < q < \infty$, $-n/q < \alpha < n(1-1/q)$ and $0 < p \le \infty$, $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $HK_q^{\alpha,p}(\mathbb{R}^n) = K_q^{\alpha,p}(\mathbb{R}^n)$ (see [11]).

THEOREM 1.3. *Let* $0 \le \tau < n$, $s \in \mathbb{N} \cup \{0\}$, $0 < \gamma \le 1$, $0 < p_1 \le p_2 < \infty$, $1 < q_1 < \infty$, $1/q_2 = 1/q_1 - \tau/n$, $n(1-1/q_1) \le \alpha < s + \gamma + n(1-1/q_1)$, *and let* $b \in BMO$ *and* T^τ *be a* $(q_1, \tau; s, \gamma)$*-fractional integral type operator. Then* $T_{b,m}^\tau$ *maps* $H\dot{K}_{q_1,b,m}^{\alpha,p_1,s}(\mathbb{R}^n)$ *into* $\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$ *and* $HK_{q_1,b,m}^{\alpha,p_1,s}(\mathbb{R}^n)$ *into* $K_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$, *respectively.*

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REMARK 2. Theorem 3 is [19, Theorem 3.1] when $\tau = 0$, $s = 0$, $m = 1$.

THEOREM 1.4. *Let p_1, p_2, q_1, q_2, τ, α, s, γ, T^τ and b be as in Theorem 1.3. Assume that $(T^\tau)^*(g_{i,\beta}) = C$, $g_{i,\beta}(x) = b^i(x)x^\beta$, $|\beta| \le s$, $i = 0, 1, 2, \ldots, m$. Then $T^\tau_{b,m}(\mathbb{R}^n)$ maps $H\dot{K}^{\alpha,p_1,s}_{q_1,b,m}(\mathbb{R}^n)$ into $H\dot{K}^{\alpha,p_2}_{q_2}(\mathbb{R}^n)$ and $HK^{\alpha,p_1,s}_{q_1,b,m}(\mathbb{R}^n)$ into $HK^{\alpha,p_2}_{q_2}(\mathbb{R}^n)$, respectively.*

Denote by BMO the space of measurable functions b such that

$$\int_B |b(x) - b_B|\,dx < C|B|$$

holds for all balls B, where $b_B = |B|^{-1}\int_B b(x)\,dx$ with a constant C independent of B.

Throughout this paper, C always means a constant independent of the main parameters involved, but which may be different from line to line. For any power exponent p with $1 \le p \le \infty$, we denote the conjugate exponent $p/(p-1)$ by p'.

2. Proofs of the theorems

First we prove two lemmas.

LEMMA 2.1. *Let $1 < q \le \infty$, T^τ be the $(q_1, \tau; s, \gamma)$-fractional integral type operator defined as above and $b(x) \in BMO$. If for $i = 0, 1, \ldots, m$, $a(x)b^i(x)$ satisfy s order vanishing moments with* supp $a \subset B$ *with the center at $x_0 = 0$ and $x \in (2B)^c$, then*

$$|T^\tau_{b,m}a(x)| \le C|B|^{(s+\gamma)/n+1/q'}\|a\|_{L^q(\mathbb{R}^n)}\left(\frac{|b(x)-b_B|^m}{|x|^{n-\tau+s+\gamma}} + \frac{\|b\|^m_{BMO}}{|x|^{n-\tau+s+\gamma}}\right).$$

PROOF. Using the sth order vanishing moments of $a(x)b^i(x)$, $i = 0, 1, \ldots, m$, we have

$$\begin{aligned} T^\tau_{b,m}a(x) &= \int_{\mathbb{R}^n} (b(x) - b(y))^m K(x,y)a(y)\,dy \\ &= \int_{\mathbb{R}^n} (b(x) - b(y))^m (K(x,y) - P(x,y))a(y)\,dy, \end{aligned}$$

where $P(x, y)$ is the $(s-1)$th order Taylor's expansion for $K(x, y)$ as a function of y at $y = 0$. Again, using the sth order vanishing moments of $a(x)b^i(x)$, $i = 0, 1, \ldots, m$,

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we have

$$T^{\tau}_{b,m}a(x) = \int_{\mathbb{R}^n} \sum_{|\zeta|=s} C_{s,\zeta} \frac{\partial^{\zeta} K(x, y_0)}{\partial y^{\zeta}} y^{\zeta}(b(x) - b(y))^m a(y)\, dy$$
$$= \int_{\mathbb{R}^n} \sum_{|\zeta|=s} C_{s,\zeta} \left(\frac{\partial^{\zeta} K(x, y_0)}{\partial y^{\zeta}} - \frac{\partial^{\zeta} K(x, 0)}{\partial y^{\zeta}}\right) y^{\zeta}(b(x) - b(y))^m a(y)\, dy,$$

where y_0 is a point on the line segment connecting y and 0. Thus, by (ii) in Definition 1, and since $y_0 \in B$, $y \in B$, $|\zeta| = s$, $\|a\|_{L^1(\mathbb{R}^n)} \le |B|^{1/q'}\|a\|_{L^q(\mathbb{R}^n)}$ and

$$\int_{\mathbb{R}^n} |b(y) - b_B|^m |a(y)|\, dy \le \|a\|_{L^q(\mathbb{R}^n)} \left(\int_B |b(x) - b_B|^{mq'}\, dy\right)^{1/q'}$$
$$\le C\|b\|^m_{BMO}|B|^{1/q'}\|a\|_{L^q(\mathbb{R}^n)},$$

we have

$$|T^{\tau}_{b,m}a(x)| \le C\frac{|B|^{(s+\gamma)/n}}{|x|^{n-\tau+s+\gamma}} \int_{\mathbb{R}^n} |b(x) - b(y)|^m |a(y)| dy$$
$$\le C\frac{|B|^{(s+\gamma)/n}}{|x|^{n-\tau+s+\gamma}} \left(|b(x) - b_B|^m \int_{\mathbb{R}^n} |a(y)|\, dy + \int_{\mathbb{R}^n} |b(y) - b_B|^m |a(y)|\, dy\right)$$
$$\le C|B|^{(s+\gamma)/n+1/q'}\|a\|_{L^q(\mathbb{R}^n)} \left(\frac{|b(x) - b_B|^m}{|x|^{n-\tau+s+\gamma}} + \frac{\|b\|^m_{BMO}}{|x|^{n-\tau+s+\gamma}}\right). \qquad \square$$

LEMMA 2.2. *Let $q > 0$, $m \in \mathbb{N} \cup \{0\}$, $B = B(0, r)$. Then*

$$\text{II} = \int_{\mathbb{R}^n \setminus 4B} \frac{|b(x) - b_B|^{mq}}{|x|^{(n-\tau+s+\gamma)q}} |x|^{\delta}\, dx$$
$$\le C\|b\|^{mq}_{BMO}|B|^{-((n-\tau+s+\gamma)q-\delta-n)/n} \sum_{j=2}^{\infty} (j + 1)^{mq} 2^{-j((n-\tau+s+\gamma)q-\delta-n)}.$$

PROOF.

$$\text{II} = \sum_{j=2}^{\infty} \int_{2^j r<|x|<2^{j+1} r} \frac{|b(x) - b_B|^{mq}}{|x|^{(n-\tau+s+\gamma)q-\delta}}\, dx$$
$$\le \sum_{j=2}^{\infty} \frac{1}{|2^j r|^{(n-\tau+s+\gamma)q-\delta}} \int_{2^{j+1}B} |b(x) - b_B|^{mq}\, dx$$
$$= C\sum_{j=2}^{\infty} \frac{1}{(2^j r)^{(n-\tau+s+\gamma)q-\delta-n}} \frac{1}{(2^{j+1}r)^n} \int_{2^{j+1}B} |b(x) - b_B|^{mq}\, dx.$$

When $mq \geq 1$, we have

$$\begin{aligned}
&\frac{1}{(2^{j+1}r)^n}\int_{2^{j+1}B}|b(x)-b_B|^{mq}\,dx\\
&\quad\leq C\frac{1}{(2^{j+1}r)^n}\int_{2^{j+1}B}|b(x)-b_{2^{j+1}B}|^{mq}\,dx + C\left(\sum_{i=0}^{j}|b_{2^iB}-b_{2^{i+1}B}|\right)^{mq}\\
&\quad\leq C\|b\|_{BMO}^{mq} + C\left(\sum_{i=0}^{j}\|b\|_{BMO}\right)^{mq}\\
&\quad= C(j+1)^{mq}\|b\|_{BMO}^{mq}.
\end{aligned}$$

When $0 < mq < 1$, by Hölder's inequality, we have

$$\begin{aligned}
&\frac{1}{(2^{j+1}r)^n}\int_{2^{j+1}B}|b(x)-b_B|^{mq}\,dx\\
&\quad\leq \frac{1}{(2^{j+1}r)^n}\left(\int_{2^{j+1}B}|b(x)-b_B|dx\right)^{mq}|2^{i+1}B|^{1-mq}\\
&\quad\leq \left(\frac{1}{(2^{j+1}r)^n}\int_{2^{j+1}B}|b(x)-b_{2^{j+1}B}|\,dx + \sum_{i=0}^{j}|b_{2^iB}-b_{2^{i+1}B}|\right)^{mq}\\
&\quad\leq C(j+1)^{mq}\|b\|_{BMO}^{mq}.
\end{aligned}$$

When $mq = 0$,

$$\frac{1}{(2^{j+1}r)^n}\int_{2^{j+1}B}|b(x)-b_B|^{mq}\,dx = C.$$

Thus, we have finished the proof of Lemma 2.2. □

PROOF OF THEOREM 1.1. We need to prove that there exists a constant C such that for each function f in $H_{b,m}^{p_1,q_1,s}(\mathbb{R}^n)$,

$$\|T_{b,m}^{\tau}f\|_{L^{p_2}(\mathbb{R}^n)} \leq C\|f\|_{H_{b,m}^{p_1,q_1,s}(\mathbb{R}^n)},$$

where q_1, q_2 are as in Definition 1, that is, $1 < q_1 \leq q_2 \leq \infty$ such that $1/q_1 - 1/q_2 = \tau/n (= 1/p_1 - 1/p_2)$. By a standard argument, it is enough to show that there exists a constant C independent of a such that $\|T_{b,m}^{\tau}a\|_{L^{p_2}(\mathbb{R}^n)} \leq C$ for each $(p_1, q_1, s; b)$-atom a of order m.

To prove this, without loss of generality, we may suppose that supp $a \subset B$ with center at the origin. Then

$$\begin{aligned}
\int_{\mathbb{R}^n}|T_{b,m}^{\tau}a(x)|^{p_2}\,dx &= \left(\int_{4B}+\int_{\mathbb{R}^n\setminus 4B}\right)|T_{b,m}^{\tau}a(x)|^{p_2}\,dx\\
&= \mathrm{I}+\mathrm{II}.
\end{aligned}$$

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By the boundedness of $T^{\tau}_{b,m}$ from $L^{q_1}(\mathbb{R}^n)$ into $L^{q_2}(\mathbb{R}^n)$ and Hölder's inequality, we have

$$\begin{aligned}
\mathrm{I} &\le C\left(|B|^{1/p_2-1/q_2}\left(\int_{4B}|T^{\tau}_{b,m}a(x)|^{q_2}\,dx\right)^{1/q_2}\right)^{p_2}\\
&\le C\left(\|b\|^m_{BMO}|B|^{1/p_1-1/q_1}\|a\|_{L^{q_1}(\mathbb{R}^n)}\right)^{p_2}\\
&\le C\left(\|b\|^m_{BMO}|B|^{1/p_1-1/q_1}|B|^{1/q_1-1/p_1}\right)^{p_2}\\
&= C\|b\|^{mp_2}_{BMO}.
\end{aligned}$$

For II, using Lemma 2.1 ($q = q_1$), Lemma 2.2 ($\delta = 0$, $q = p_2$) and $\|a\|_{L^{q_1}(\mathbb{R}^n)} \le |B|^{1/q_1-1/p_1}$, we have

$$\begin{aligned}
\mathrm{II} &\le C|B|^{(s+\gamma)p_2/n+p_2/q_1'}\|a\|^{p_2}_{L^{q_1}(\mathbb{R}^n)}\\
&\quad\times\left(\int_{\mathbb{R}^n\setminus 4B}\frac{|b(x)-b_B|^{mp_2}}{|x|^{(n-\tau+s+\gamma)p_2}}\,dx+\int_{\mathbb{R}^n\setminus 4B}\frac{\|b\|^{mp_2}_{BMO}}{|x|^{(n-\tau+s+\gamma)p_2}}\,dx\right)\\
&\le C|B|^{(s+\gamma)p_2/n+p_2/q_1'}|B|^{(1/q_1-1/p_1)p_2}\\
&\quad\times\|b\|^{mp_2}_{BMO}|B|^{-(n-\tau+s+\gamma)p_2/n+1}\sum_{j=2}^{\infty}(j+1)^{mp_2}2^{-j((n-\tau+s+\gamma)p_2-n)}\\
&= C\|b\|^{mp_2}_{BMO},
\end{aligned}$$

since $n/(n-\tau+s+\gamma) < p_2$. This concludes the proof of Theorem 1.1. □

PROOF OF THEOREM 1.2. By the atom-molecule theory of Coifman and Weiss (see [7, 13, 25]) we need only to prove that there exists a constant C such that for each $(p_1, q_1, s; b)$-atom a of order m, $T^{\tau}_{b,m}a$ is a $(p_2, q_2, s, \varepsilon)$-molecule, that is,

(i) $|x|^{nd}T^{\tau}_{b,m}a(x) \in L^{q_2}(\mathbb{R}^n)$;
(ii) $N_{q_2}(T^{\tau}_{b,m}a) = \|T^{\tau}_{b,m}a\|^{c/d}_{L^{q_2}(\mathbb{R}^n)}\||x|^{nd}T^{\tau}_{b,m}a(x)\|^{1-c/d}_{L^{q_2}(\mathbb{R}^n)} = C < \infty$;
(iii) $\int_{\mathbb{R}^n} T^{\tau}_{b,m}a(x)x^{\beta}\,dx = 0$, $|\beta| \le s$,

where $s \ge s_0 = [n(1/p_2 - 1)]$,

$$\frac{s-\tau+\gamma}{n} > \varepsilon > \max\left\{\frac{s}{n}, \frac{1}{p_2}-1\right\}$$

($(s-\tau+\gamma)/n > 1/p_2 - 1$ since $p_2 > n/(n-\tau+s+\gamma)$), $c = 1 - 1/p_2 + \varepsilon$, $d = 1 - 1/q_2 + \varepsilon$, $0 < p_1 \le 1$, $0 < \gamma \le 1$, $s \ge 0$, $1/q_1 - 1/q_2 = \tau/n$.

Let $\operatorname{supp} a \subset B$ with the center at the origin. By the boundedness of $T^{\tau}_{b,m}$ from $L^{q_1}(\mathbb{R}^n)$ into $L^{q_2}(\mathbb{R}^n)$, we have $\|T^{\tau}_{b,m}a\|_{L^{q_2}(\mathbb{R}^n)} \le C\|b\|^m_{BMO}\|a\|_{L^{q_1}(\mathbb{R}^n)}$. Let

$$\begin{aligned}
\int_{\mathbb{R}^n}|x|^{q_2nd}|T^{\tau}_{b,m}a(x)|^{q_2}\,dx &= \left(\int_{4B}+\int_{\mathbb{R}^n\setminus 4B}\right)|x|^{q_2nd}|T^{\tau}_{b,m}a(x)|^{q_2}\,dx\\
&= \mathrm{I}+\mathrm{II}.
\end{aligned}$$

For I, by the $L^q(\mathbb{R}^n)$-boundedness of $T^\tau_{b,m}$ and $\|a\|_{L^{q_1}(\mathbb{R}^n)} \le |B|^{1/q_1-1/p_1}$, we have

$$\mathrm{I} \le C|B|^{q_2 d}\|T^\tau_{b,m}a\|^{q_2}_{L^{q_2}(\mathbb{R}^n)} \le C|B|^{q_2 d}\|b\|^{q_2 m}_{BMO}\|a\|^{q_2}_{L^{q_1}(\mathbb{R}^n)}.$$

For II, by Lemma 2.1 and Lemma 2.2 ($q = q_2$, $\delta = q_2 nd$), we have

$$\begin{aligned}
\mathrm{II} &\le C|B|^{q_2(s+\gamma)/n+q_2/q_1'}\|a\|^{q_2}_{L^{q_1}(\mathbb{R}^n)} \\
&\quad \times \left(\int_{\mathbb{R}^n\setminus 4B} \frac{|b(x)-b_B|^{q_2 m}}{|x|^{q_2(n-\tau+s+\gamma)-q_2 nd}}\,dx + \int_{\mathbb{R}^n\setminus 4B} \frac{\|b\|^{q_2 m}_{BMO}}{|x|^{q_2(n-\tau+s+\gamma)-q_2 nd}}\,dx\right) \\
&\le C|B|^{q_2(s+\gamma)/n+q_2/q_1'}\|a\|^{q_2}_{L^{q_1}(\mathbb{R}^n)} \\
&\quad \times \|b\|^{q_2 m}_{BMO}|B|^{-q_2(n-\tau+s+\gamma)/n+q_2 d+1}\sum_{j=2}^{\infty}(j+1)^{q_2 m}2^{-j(q_2((n-\tau+s+\gamma)-nd)-n)} \\
&= C\|b\|^{q_2 m}_{BMO}|B|^{q_2 d}\|a\|^{q_2}_{L^{q_1}(\mathbb{R}^n)},
\end{aligned}$$

since $q_2(n-\tau+s+\gamma) - q_2 nd - n = q_2(s+\gamma-\tau-n\varepsilon) > 0$.

Thus,

$$\begin{aligned}
N_{q_2}(T^\tau_{b,m}a) &\le C\|b\|^{mc/d}_{BMO}\|a\|^{c/d}_{L^{q_1}(\mathbb{R}^n)}\|b\|^{m(1-c/d)}_{BMO}|B|^{d(1-c/d)}\|a\|^{1-c/d}_{L^{q_1}(\mathbb{R}^n)} \\
&= C\|b\|^{m}_{BMO}\|a\|_{L^{q_1}(\mathbb{R}^n)}|B|^{1/p_2-1/q_2} \le C.
\end{aligned}$$

We have proved (i) and (ii). By $(T^\tau)^*(g_{i,\beta}) = C$, $|\beta| \le s$, $i = 0, 1, 2, \ldots, m$, and the vanishing moments of a, (iii) is obvious.

This concludes the proof of Theorem 1.2. □

PROOF OF THEOREM 1.3. Let $f \in H\dot{K}^{\alpha,p_1,s}_{q_1,b,m}(\mathbb{R}^n)$, that is, $f(x) = \sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$, where each a_j is an m-order dyadic central $H\dot{K} - (\alpha, q_1; b)_s$-atom with support B_j, and $\|f\|_{H\dot{K}^{\alpha,p_1,s}_{q_1,b,m}(\mathbb{R}^n)} \sim \left(\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\right)^{1/p_1}$. Then,

$$\begin{aligned}
\|T^\tau_{b,m}f\|^{p_1}_{\dot{K}^{\alpha,p_2}_{q_2}(\mathbb{R}^n)} &\le C\left(\sum_{k=-\infty}^{\infty}2^{k\alpha p_2}\|(T^\tau_{b,m}f)\chi_k\|^{p_2}_{L^{q_2}(\mathbb{R}^n)}\right)^{p_1/p_2} \\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\|(T^\tau_{b,m}f)\chi_k\|^{p_1}_{L^{q_2}(\mathbb{R}^n)} \\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\|(T^\tau_{b,m}a_j)\chi_k\|_{L^{q_2}(\mathbb{R}^n)}\right)^{p_1} \\
&\quad + C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\|(T^\tau_{b,m}a_j)\chi_k\|_{L^{q_2}(\mathbb{R}^n)}\right)^{p_1} \\
&= I_1 + I_2.
\end{aligned}$$

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For I_2, by the $L^q(\mathbb{R}^n)$-boundedness of $T^\tau_{b,m}$ and Hölder's inequality, we have

$$
\begin{aligned}
I_2 &\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\left(\sum_{j=k-1}^{\infty}\|b\|^m_{BMO}|\lambda_j|\|a_j\|_{L^{q_1}(\mathbb{R}^n)}\right)^{p_1}\\
&\le C\|b\|^{p_1 m}_{BMO}\sum_{k=-\infty}^{\infty}\left(\sum_{j=k-1}^{\infty}2^{(k-j)\alpha}|\lambda_j|\right)^{p_1}\\
&\le \begin{cases} C\|b\|^{p_1 m}_{BMO}\displaystyle\sum_{k=-\infty}^{\infty}\sum_{j=k-1}^{\infty}2^{(k-j)p_1\alpha}|\lambda_j|^{p_1} & \text{if } 0<p_1\le 1\\ C\|b\|^{p_1 m}_{BMO}\displaystyle\sum_{k=-\infty}^{\infty}\left(\sum_{j=k-1}^{\infty}2^{(k-j)p_1\alpha/2}|\lambda_j|^{p_1}\right) & \\ \quad\times\left(\displaystyle\sum_{j=k-1}^{\infty}2^{(k-j)p_1'\alpha/2}\right)^{p_1/p_1'} & \text{if } 1<p_1<\infty\end{cases}\\
&\le \begin{cases} C\|b\|^{p_1 m}_{BMO}\displaystyle\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\sum_{k=-\infty}^{j+1}2^{(k-j)p_1\alpha} & \text{if } 0<p_1\le 1\\ C\|b\|^{p_1 m}_{BMO}\displaystyle\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\sum_{k=-\infty}^{j+1}2^{(k-j)p_1\alpha/2} & \text{if } 1<p_1<\infty\end{cases}\\
&\le C\|b\|^{p_1 m}_{BMO}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}.
\end{aligned}
$$

That is, $I_2 \le C\|b\|^{p_1 m}_{BMO}\|f\|^{p_1}_{H\dot{K}^{\alpha,p_1,s}_{q_1,b,m}(\mathbb{R}^n)}$. For I_1, since

$$
\begin{aligned}
|b(x)-b_{B_j}|^{q_2 m} &\le C|b(x)-b_{B_k}|^{q_2 m} + C\left(\sum_{i=j}^{k-1}|b_{B_i}-b_{B_{i+1}}|\right)^{q_2 m}\\
&\le C|b(x)-b_{B_k}|^{q_2 m} + C(k-j)^{q_2 m}\|b\|^{q_2 m}_{BMO},
\end{aligned}
$$

and $\int_{C_k}|b(x)-b_{B_k}|^{q_2 m}\,dx \le |B_k|\|b\|^{q_2 m}_{BMO} \le C2^{kn}(k-j)\|b\|^{q_2 m}_{BMO}$, for $k>j$, we have

$$
\begin{aligned}
&\|(T^\tau_{b,m}a_j)\chi_k\|^{q_2}_{L^{q_2}(\mathbb{R}^n)}\\
&\quad\le C|B_j|^{(s+\gamma)q_2/n+q_2/q_1'}\|a_j\|^{q_2}_{L^{q_1}(\mathbb{R}^n)}\\
&\qquad\times\left(\int_{C_k}\frac{|b(x)-b_{B_j}|^{q_2 m}}{|x|^{q_2(n-\tau+s+\gamma)}}\,dx + \int_{C_k}\frac{\|b\|^{q_2 m}_{BMO}}{|x|^{q_2(n-\tau+s+\gamma)}}\,dx\right)\\
&\quad\le C|B_j|^{(s+\gamma)q_2/n+q_2/q_1'}\|a_j\|^{q_2}_{L^{q_1}(\mathbb{R}^n)}\\
&\qquad\times\left(\int_{C_k}\frac{|b(x)-b_{B_k}|^{q_2 m}}{|x|^{q_2(n-\tau+s+\gamma)}}\,dx + \int_{C_k}\frac{(k-j)^{q_2 m}\|b\|^{q_2 m}_{BMO}}{|x|^{q_2(n-\tau+s+\gamma)}}\,dx\right)
\end{aligned}
$$

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$$\le C2^{j((s+\gamma)q_2+nq_2-nq_2/q_1)-j\alpha q_2}2^{-k(n-\tau+s+\gamma)q_2}\times 2^{kn}(k-j)^{q_2m}\|b\|_{BMO}^{q_2m}$$
$$= C2^{(j-k)((s+\gamma+n)q_2-nq_2/q_1)}2^{-j\alpha q_2}(k-j)^{q_2m}\|b\|_{BMO}^{q_2m}.$$

Hence

$$I_1 \le C\|b\|_{BMO}^{p_1m}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}(k-j)^m 2^{(j-k)(n+s+\gamma-n/q_1-\alpha)}|\lambda_j|\right)^{p_1}$$
$$\le \begin{cases} C\|b\|_{BMO}^{p_1m}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}(k-j)^{mp_1}2^{(j-k)(n+s+\gamma-n/q_1-\alpha)p_1}|\lambda_j|^{p_1} & \text{if } 0<p_1\le 1\\ C\|b\|_{BMO}^{p_1m}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{(j-k)(n+s+\gamma-n/q_1-\alpha)p_1/2}|\lambda_j|^{p_1}\right) & \\ \quad\times\left(\sum_{j=-\infty}^{k-2}(k-j)^{mp_1'}2^{(j-k)(n+s+\gamma-n/q_1-\alpha)p_1'/2}\right)^{p_1/p_1'}. & \text{if } 1<p_1<\infty \end{cases}$$
$$\le \begin{cases} C\|b\|_{BMO}^{p_1m}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\sum_{k=j+2}^{\infty}(k-j)^{mp_1}2^{(j-k)(n+s+\gamma-n/q_1-\alpha)p_1} & \text{if } 0<p_1\le 1\\ C\|b\|_{BMO}^{p_1m}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\sum_{k=j+2}^{\infty}2^{(j-k)(n+s+\gamma-n/q_1-\alpha)p_1/2} & \text{if } 1<p_1<\infty \end{cases}$$
$$\le C\|b\|_{BMO}^{p_1m}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}$$

for $\gamma > \alpha + n/q_1 - n - s$. That is, $I_1 \le C\|b\|_{BMO}^{p_1m}\|f\|_{H\dot{K}_{q_1,b,m}^{\alpha,p_1,s}(\mathbb{R}^n)}^{p_1}$.

Thus, we have proved the case of homogeneous spaces of Theorem 1.3 The proof of the non-homogeneous case is similar to that of homogeneous case. This concludes the proof of Theorem 1.3. □

PROOF OF THEOREM 1.4. By the atom-molecule theory of $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (see [18, Theorem 2.5]), we need only to prove that there exists a constant C such that for each dyadic central m-order $H\dot{K}$-$(\alpha, q_1; b)$-atom a_k with support B_k, $T_{b,m}^{\tau}a_k$ is a dyadic central $(\alpha, q_2, s, \varepsilon)_k$-molecule, that is,

(i) $\|T_{b,m}^{\tau}a_k\|_{L^{q_2}(\mathbb{R}^n)} \le 2^{-k\alpha}$;
(ii) $N(T_{b,m}^{\tau}a_k) = \|T_{b,m}^{\tau}a_k\|_{L^{q_2}(\mathbb{R}^n)}^{c/d}\||x|^{nd}T_{b,m}^{\tau}a_k(x)\|_{L^{q_2}(\mathbb{R}^n)}^{1-c/d} = C < \infty$;
(iii) $\int_{\mathbb{R}^n} T_{b,m}^{\tau}a_k(x)x^{\beta}\,dx = 0, |\beta| \le s$,

where $s \ge [\alpha + n(1/q_1 - 1)]$, $(s+\gamma-\tau)/n > \varepsilon > \max\{s/n, \alpha/n + 1/q_1 - 1\}$ (since $(s+\gamma-\tau)/n > \alpha/n + 1/q_1 - 1$), $c = 1 - 1/q_1 - \alpha/n + \varepsilon$, $d = 1 - 1/q_2 + \varepsilon$.

(i) By the $L^q(\mathbb{R}^n)$-boundedness of $T_{b,m}^{\tau}$, we have

$$\|T_{b,m}^{\tau}a_k\|_{L^{q_2}(\mathbb{R}^n)} \le C\|b\|_{BMO}^m\|a_k\|_{L^{q_1}(\mathbb{R}^n)} \le C\|b\|_{BMO}^m 2^{-k\alpha}.$$

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(ii)

$$\int_{\mathbb{R}^n} |x|^{q_2 nd} |T^\tau_{b,m} a_k(x)|^{q_2}\, dx = \left(\int_{B_{k+2}} + \int_{\mathbb{R}^n\setminus B_{k+2}}\right) |x|^{q_2 nd} |T^\tau_{b,m} a_k(x)|^{q_2}\, dx$$
$$= \mathrm{I} + \mathrm{II}.$$

For I, by the $L^q(\mathbb{R}^n)$-boundedness of $T^\tau_{b,m}$, we have

$$\mathrm{I} \le C2^{kndq_2} \| T^\tau_{b,m} a_k \|^{q_2}_{L^{q_2}(\mathbb{R}^n)} \le C2^{kndq_2} \|b\|^{q_2 m}_{BMO} \|a_k\|^{q_2}_{L^{q_1}(\mathbb{R}^n)} \le C\|b\|^{q_2 m}_{BMO} 2^{kq_2(nd-\alpha)}.$$

For II, by Lemma 2.1 and Lemma 2.2 ($\delta = q_2 nd$), we have

$$\begin{aligned}
\mathrm{II} &\le C|B_k|^{(s+\gamma)q_2/n + q_2/q_1'} \|a_k\|^{q_2}_{L^{q_1}(\mathbb{R}^n)} \\
&\quad \times \left(\int_{\mathbb{R}^n\setminus B_{k+2}} \frac{|b(x) - b_{B_k}|^{q_2 m}}{|x|^{q_2(n-\tau+s+\gamma) - q_2 nd}}\, dx + \int_{\mathbb{R}^n\setminus B_{k+2}} \frac{\|b\|^{q_2 m}_{BMO}}{|x|^{q_2(n-\tau+s+\gamma)-q_2 nd}}\, dx \right) \\
&\le C2^{nk((s+\gamma)q_2/n + q_2/q_1')} \|a_k\|^{q_2}_{L^{q_1}(\mathbb{R}^n)} \\
&\quad \times \|b\|^{q_2 m}_{BMO} 2^{-nk((n-\tau+s+\gamma)q_2/n - q_2 d - 1)} \sum_{j=2}^{\infty} (j+1)^{q_2 m} 2^{-j(q_2((n-\tau+s+\gamma)-nd)-n)} \\
&= C\|b\|^{q_2 m}_{BMO} 2^{kq_2 nd} \|a_k\|^{q_2}_{L^{q_1}(\mathbb{R}^n)} \sum_{j=2}^{\infty} (j+1)^{q_2 m} 2^{-j(s+\gamma-\tau-\varepsilon n)q_2} \\
&= C\|b\|^{q_2 m}_{BMO} 2^{kq_2(nd-\alpha)},
\end{aligned}$$

since $(s+\gamma-\tau-\varepsilon n)q_2 > 0$. Therefore,

$$N(T^\tau_{b,m} a_k) \le C2^{-k\alpha c/d} 2^{k(nd-\alpha)(1-c/d)} = C.$$

(iii) By $(T^\tau)^*(g_{i,\beta}) = C$, $|\beta| \le s$, $i = 0, 1, \ldots, m$, and the vanishing moments of a_k, the vanishing moments of $T^\tau_{b,m} a_k$ are obvious.

Thus, we have proved the case of homogeneous spaces of Theorem 1.4. The proof of the non-homogeneous case is similar to that of homogeneous case. This conclude the proof of Theorem 1.4. □

3. Fractional integrals

Let $0 \le \tau < n$, we define the fractional integrals of T by

$$T^\tau f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\tau}}\, dx.$$

The proof of the boundedness of T^τ from $L^{q_1}(\mathbb{R}^n)$ into $L^{q_2}(\mathbb{R}^n)$ for $1 < q_1 < q_2 < \infty$ such that $1/q_1 - 1/q_2 = \tau/n$ can be found in [24]. Thus, the fractional integrals T^τ as above satisfy the condition in Definition 1 for $\gamma = 1$ and any $s \ge 0$. Hence, as the special case of Theorems 1.1–1.4, we have the following corollaries.

COROLLARY 3.1. *Let* $0 < p_1 \leq 1$, $1/p_2 = 1/p_1 - \tau/n$, $s > [n(1/p_1 - 1)]$, $1 < q_1 \leq \infty$, *and let* T^τ *be a fractional integral operator (as above) and* $b \in BMO$. *If* $n/(n - \tau + s + 1) < p_2 < +\infty$ *and* $0 \leq \tau < n$, *then* $T^\tau_{b,m}$ *maps* $H^{p_1,q_1,s}_{b,m}(\mathbb{R}^n)$ *into* $L^{p_2}(\mathbb{R}^n)$.

The case $p_1 = 1$ and $m = 0$ of Corollary 3.1 was proved by Stein and Weiss.

COROLLARY 3.2. *Let* $0 < p_1 \leq 1$, $1/p_2 = 1/p_1 - \tau/n$, $s > [n(1/p_1 - 1)]$, $1 < q_1 \leq \infty$, *and let* T^τ *be a fractional integral operator and* $b \in BMO$. *Assume that* $(T^\tau)^*(b^i(x)x^\beta) = C$, $|\beta| \leq s$, $i = 0, 1, \ldots, m$. *If* $n/(n - \tau + s + 1) < p_2 \leq 1$ *and* $0 \leq \tau < 1$, *then* $T^\tau_{b,m}$ *maps* $H^{p_1,q_1,s}_{b,m}(\mathbb{R}^n)$ *into* $H^{p_2}(\mathbb{R}^n)$.

The case $0 < p_1 < p_2 \leq 1$ and $m = 0$ of Corollary 3.2 was proved by Taibleson and Weiss [25].

COROLLARY 3.3. *Let* $0 \leq \tau < n$, $s \geq 0$, $0 < p_1 \leq p_2 < \infty$, $1 < q_1 < \infty$, $1/q_2 = 1/q_1 - \tau/n$, $n(1 - 1/q_1) \leq \alpha < s + 1 + n(1 - 1/q_1)$, *and let* T^τ *be a fractional integral and* $b \in BMO$. *Then* $T^\tau_{b,m}$ *maps* $H\dot{K}^{\alpha,p_1,s}_{q_1,b,m}(\mathbb{R}^n)$ *into* $\dot{K}^{\alpha,p_2}_{q_2}(\mathbb{R}^n)$ *and* $HK^{\alpha,p_1,s}_{q_1,b,m}(\mathbb{R}^n)$ *into* $K^{\alpha,p_2}_{q_2}(\mathbb{R}^n)$, *respectively.*

COROLLARY 3.4. *Let* $p_1, p_2, q_1, q_2, \tau, \alpha, s, T^\tau$ *and* b *be as in Corollary* 3.3. *Assume that* $(T^\tau)^*(b^i(x)x^\beta) = C$, $|\beta| \leq s$, $i = 0, 1, \ldots, m$. *Then* $T^\tau_{b,m}$ *maps* $H\dot{K}^{\alpha,p_1,s}_{q_1,b,m}(\mathbb{R}^n)$ *into* $H\dot{K}^{\alpha,p_2}_{q_2}(\mathbb{R}^n)$ *and* $HK^{\alpha,p_1,s}_{q_1,b,m}(\mathbb{R}^n)$ *into* $HK^{\alpha,p_2}_{q_2}(\mathbb{R}^n)$, *respectively.*

The case $m = 0$ of Corollary 3.3 and Corollary 3.4 can be found in [14, 18].

The case $m = 1$ of Corollary 3.3 can be found in [19].

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Mathematics Department
Xiangtan University
Xiangtan, 411105
P. R. China
e-mail: sclong@xtu.edu.cn, jwang@hnedu.com