COMPOSITIO MATHEMATICA

Ascending chain condition for F-pure thresholds on a fixed strongly F-regular germ

Kenta Sato

Compositio Math. **155** (2019), 1194–1223.

doi:10.1112/S0010437X19007358

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

Compositio Math. 155 (2019) 1194–1223
doi:10.1112/S0010437X19007358

Ascending chain condition for F-pure thresholds on a fixed strongly F-regular germ

Kenta Sato

Abstract

In this paper, we prove that the set of all F-pure thresholds on a fixed germ of a strongly F-regular pair satisfies the ascending chain condition. As a corollary, we verify the ascending chain condition for the set of all F-pure thresholds on smooth varieties or, more generally, on varieties with tame quotient singularities, which is an affirmative answer to a conjecture given by Blickle, Mustaţă and Smith.

1. Introduction

In characteristic zero, Shokurov [Sho92] conjectured that the set of all log canonical thresholds on varieties of any fixed dimension satisfies the ascending chain condition. This conjecture was partially solved by de Fernex *et al.* in [dFEM10] and [dFEM11] using generic limit, and finally settled by Hacon *et al.* in [HMX14] using global geometry.

In this paper, we deal a positive characteristic analogue of this problem. Let $(R, \mathfrak{m})$ be a Noetherian normal local ring of characteristic $p > 0$ and Δ be an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$. We further assume that R is F-finite, that is, the Frobenius morphism $F : R \longrightarrow R$ is a finite ring homomorphism. For a proper ideal $\mathfrak{a} \subsetneq R$ and a real number $t \geqslant 0$, we consider the test ideal $\tau(R, \Delta, \mathfrak{a}^t)$, which is defined in terms of the Frobenius morphism (see Definition 2.3 below). Since we have $\tau(R, \Delta, \mathfrak{a}^t) \subseteq \tau(R, \Delta, \mathfrak{a}^s)$ for every real numbers $0 \leqslant s \leqslant t$, for a given $\mathfrak{m}$-primary ideal $I \subseteq R$, we define the F-jumping number of $(R, \Delta; \mathfrak{a})$ with respect to I as

$$\operatorname{fjn}^I(R, \Delta; \mathfrak{a}) := \inf\{t \geqslant 0 \mid \tau(R, \Delta, \mathfrak{a}^t) \subseteq I\} \in \mathbb{R}.$$

When $I = \mathfrak{m}$ and (R, Δ) is *strongly F-regular*, that is, $\tau(R, \Delta) = R$, we denote it by $\operatorname{fpt}(R, \Delta; \mathfrak{a})$ and call it the *F-pure threshold* of $(R, \Delta; \mathfrak{a})$.

Since test ideals in positive characteristic enjoy several important properties which hold for multiplier ideals in characteristic zero, it is natural to ask whether or not the set of F-pure thresholds satisfies the ascending chain condition. Blickle, Mustaţă, and Smith conjectured the following.

Conjecture 1.1 [BMS09, Conjecture 4.4]. Fix an integer $n \geqslant 1$, a prime number $p > 0$ and a set $\mathcal{D}^{\mathrm{reg}}_{n,p}$ such that every element of $\mathcal{D}^{\mathrm{reg}}_{n,p}$ is an n-dimensional F-finite Noetherian regular local ring of characteristic p. The set

$$\mathcal{T}^{\mathrm{reg}}_{n,p,\mathrm{pr}} := \{\operatorname{fpt}(A; \mathfrak{a}) \mid A \in \mathcal{D}^{\mathrm{reg}}_{n,p}, \mathfrak{a} \subsetneq A \text{ is a principal ideal}\}$$

satisfies the ascending chain condition.

Received 4 June 2018, accepted in final form 14 March 2019, published online 28 May 2019.
2010 Mathematics Subject Classification 14B05 (primary), 13A35, 14L30 (secondary).
Keywords: ascending chain condition, F-jumping number, F-pure threshold, tame quotient singularities, non-standard extension.

This problem has been considered by several authors [BMS09, HNWZ16], and [HNW17]. We give an affirmative answer to this conjecture.

Theorem 1.2 (Corollary 5.10). *With the notation above, the set*

$$\mathcal{T}_{n,p}^{\mathrm{reg}} := \{\mathrm{fpt}(A;\mathfrak{a}) \mid A \in \mathcal{D}_{n,p}^{\mathrm{reg}}, \mathfrak{a} \subsetneq A \text{ is an ideal}\}$$

satisfies the ascending chain condition.

Employing the strategy in [dFEM10], we can also verify the ascending chain condition for F-pure thresholds on tame quotient singularities.

Theorem 1.3 (Proposition 5.12). *Fix an integer $n \geqslant 1$, a prime number $p > 0$ and a set $\mathcal{D}_{n,p}^{\mathrm{quot}}$ such that every element of $\mathcal{D}_{n,p}^{\mathrm{quot}}$ is an n-dimensional F-finite Noetherian normal local ring of characteristic p with tame quotient singularities. The set*

$$\mathcal{T}_{n,p}^{\mathrm{quot}} := \{\mathrm{fpt}(R;\mathfrak{a}) \mid R \in \mathcal{D}_{n,p}^{\mathrm{quot}}, \mathfrak{a} \subsetneq R \text{ is an ideal}\}$$

satisfies the ascending chain condition.

Since the F-pure threshold on the ring of the formal power series does not change by any field extension (Lemma 2.11), in order to prove Theorem 1.2, it is enough to show that the set of all F-pure thresholds on a fixed F-finite Noetherian regular local ring satisfies the ascending chain condition. We consider this problem in a more general setting. Let (R,Δ) be a pair, that is, $(R,\mathfrak{m})$ is an F-finite Noetherian normal local ring of characteristic $p > 0$ and Δ is an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$. For a given $\mathfrak{m}$-primary ideal $I \subseteq R$, we define

$$\mathrm{FJN}^I(R,\Delta) := \{\mathrm{fjn}^I(R,\Delta;\mathfrak{a}) \mid \mathfrak{a} \subsetneq R \text{ is an ideal}\} \subseteq \mathbb{R}_{\geqslant 0}.$$

We note that if (R,Δ) is strongly F-regular and $I = \mathfrak{m}$, then the set $\mathrm{FJN}^I(R,\Delta)$ coincides with the set of all F-pure thresholds

$$\mathrm{FPT}(R,\Delta) := \{\mathrm{fpt}(R,\Delta;\mathfrak{a}) \mid \mathfrak{a} \subsetneq R \text{ is an ideal}\}.$$

Main Theorem (Theorem 5.9). *Let (R,Δ) be a pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index not divisible by p, where K_X is a canonical divisor of $X = \operatorname{Spec} R$, and $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Assume that $\tau(R,\Delta)$ is $\mathfrak{m}$-primary or trivial. Then the set $\mathrm{FJN}^I(R,\Delta)$ satisfies the ascending chain condition. In particular, if (R,Δ) is strongly F-regular, then the set $\mathrm{FPT}(R,\Delta)$ satisfies the ascending chain condition.*

For a real number $t > 0$ and a power q of p, we consider the ascending sequence $\{\langle t\rangle_{n,q}\}_{n\in\mathbb{N}}$, where $\langle t\rangle_{n,q} := \lceil tq^n - 1\rceil/q^n$ is the *nth truncation* of t in base q. It is not so hard to prove that the set $\mathrm{FJN}^I(R,\Delta)$ satisfies the ascending chain condition if and only if for every real number $t > 0$, there exists an integer $n_1 > 0$ with the following property: for every ideal $\mathfrak{a} \subseteq R$ and every integer $n \geqslant n_1$, $\tau(R,\Delta,\mathfrak{a}^{\langle t\rangle_{n,q}}) \subseteq I$ if and only if $\tau(R,\Delta,\mathfrak{a}^{\langle t\rangle_{n_1,q}}) \subseteq I$.

In this paper, we define a new ideal $\tau_e^{n,u}(R,\Delta,\mathfrak{a}^t) \subseteq R$ for every integers $u, n \geqslant 0$ in terms of the trace map for the Frobenius morphism so that for every n, the sequence $\{\tau_e^{n,u}(R,\Delta,\mathfrak{a}^t)\}_{u\in\mathbb{N}}$ is an ascending chain which converges to $\tau(R,\Delta,\mathfrak{a}^{\langle t\rangle_{n,q}})$. We investigate the behavior of the ideals $\{\tau_e^{n,u}(R,\Delta,\mathfrak{a}^t)\}_{n\in\mathbb{N}}$ for some fixed $u \geqslant 0$ instead of the ideals $\{\tau(R,\Delta,\mathfrak{a}^{\langle t\rangle_{n,q}})\}_{n\in\mathbb{N}}$. In particular, we prove the following theorem, which plays a crucial role in the proof of the main theorem.

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

Theorem 1.4 (Corollary 5.7). *Let* $(X = \operatorname{Spec} R, \Delta)$ *be a pair such that* $(p^e - 1)(K_X + \Delta)$ *is Cartier for some integer* $e > 0$, *let* $I \subseteq R$ *be an* $\mathfrak{m}$*-primary ideal, let* $l, n_0 \geqslant 0$ *and* $u \geqslant 2$ *be integers, and let* $t > 0$ *be a rational number such that* $t = (s/p^e) + (l/p^e(p^e - 1))$ *for some integers* $s \geqslant 0$ *and* $0 < l < p^e$. *We set* $t_0 := p^{2e}/(p^e - 1)$ *and* $M_0 = (p^{e(n_0+6)} - 1) \cdot \operatorname{emb}(R)/(p^e - 1)$, *where* $\operatorname{emb}(R)$ *is the embedding dimension of* R. *Then there exists an integer* $n_1 > 0$ *with the following property. For any ideal* $\mathfrak{a} \subseteq R$ *such that:*

(i) $p^e > \mu_R(\mathfrak{a}) + \ell\ell_R(R/I) + \operatorname{emb}(R)$, *where* $\mu_R(\mathfrak{a})$ *is the number of a minimal generator of* $\mathfrak{a}$ *and* $\ell\ell_R(R/I) := \max\{m \geqslant 0 \mid \mathfrak{m}^m \subseteq I\}$; *and*
(ii) $\tau_e^{n_0+1,u}(R, \Delta, \mathfrak{a}^{lt_0}) + \mathfrak{m}^{M_0} \cdot \tau(R, \Delta) \supseteq \tau_e^{n_0,u}(R, \Delta, \mathfrak{a}^{lt_0})$,

we have

$$\tau_e^{n,u}(R, \Delta, \mathfrak{a}^t) \subseteq I \quad \text{if and only if } \tau_e^{n_1,u}(R, \Delta, \mathfrak{a}^t) \subseteq I$$

for every integer $n \geqslant n_1$.

Another key ingredient of the proof of the main theorem is the rationality of accumulation points of $\mathrm{FJN}^I(R, \Delta)$. Blickle, Mustaţă, and Smith proved in [BMS09] that the set $\mathcal{T}^{\mathrm{reg}}_{n,p,\mathrm{pr}}$ is a closed set of rational numbers using ultraproduct. Their proof relies on the fact that for any local ring $A \in \mathcal{D}^{\mathrm{reg}}_{n,p}$, any principal ideal $\mathfrak{a} \subsetneq A$, and any integer $e \geqslant 0$, the test ideal $\tau(A, \mathfrak{a}^{1/p^e})$ can be computed by the trace map $\mathrm{Tr}^e : F^e_* A \longrightarrow A$ for the eth Frobenius morphism F^e, that is, we have $\tau(A, \mathfrak{a}^{1/p^e}) = \mathrm{Tr}^e(F^e_* \mathfrak{a})$, which fails if $\mathfrak{a}$ is not principal. In order to extend the result to the non-principal case, we introduce the notion of stabilization exponent for a triple $(R, \Delta, \mathfrak{a}^t)$, which indicates how many times we should compose the trace map for the Frobenius morphism to compute the test ideal $\tau(R, \Delta, \mathfrak{a}^t)$ (see Definition 3.6).

By combining the method used in [BMS09] and some argument about the stabilization exponents, we prove the following theorem.

Theorem 1.5 (Theorem 4.7). *Let* $(X = \operatorname{Spec} R, \Delta)$ *be a pair such that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier with index not divisible by* p, *and let* $I \subseteq R$ *be an* $\mathfrak{m}$*-primary ideal. Then the limit of any sequence in* $\mathrm{FJN}^I(R, \Delta)$ *is a rational number.*

As the consequence of Theorems 1.4 and 1.5, we obtain the main theorem.

2. Preliminaries

2.1 Test ideals

In this subsection, we recall the definition and some basic properties of test ideals.

A ring R of characteristic $p > 0$ is said to be *F-finite* if the Frobenius morphism $F : R \longrightarrow R$ is a finite ring homomorphism. Throughout this paper, all rings will be assumed to be F-finite and of characteristic $p > 0$. If R is an F-finite Noetherian normal ring, then R is excellent [Kun76] and $X = \operatorname{Spec}(R)$ has a *canonical divisor* K_X (see, for example, [ST18, p. 4]).

Definition 2.1. A *pair* (R, Δ) consists of an F-finite Noetherian normal local ring $(R, \mathfrak{m})$ and an effective $\mathbb{Q}$-Weil divisor Δ on $\operatorname{Spec} R$. A *triple* $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet} = \prod_{i=1}^m \mathfrak{a}_i^{t_i})$, consists of a pair (R, Δ) and a symbol $\mathfrak{a}_\bullet^{t_\bullet} = \prod_{i=1}^m \mathfrak{a}_i^{t_i}$, where $m > 0$ is an integer, $\mathfrak{a}_1, \ldots, \mathfrak{a}_m \subseteq R$ are ideals, and $t_1, \ldots, t_m \geqslant 0$ are real numbers.

Definition 2.2. Let $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet} = \prod_{i=1}^m \mathfrak{a}_i^{t_i})$ be a triple. An ideal $J \subseteq R$ is said to be *uniformly* $(\Delta, \mathfrak{a}_\bullet^{t_\bullet}, F)$*-compatible* if $\varphi(F^e_*(\mathfrak{a}_1^{\lceil t_1(p^e-1)\rceil} \cdots \mathfrak{a}_m^{\lceil t_m(p^e-1)\rceil} J)) \subseteq J$ for every $e \geqslant 0$ and every $\varphi \in \operatorname{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R)$.

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

DEFINITION 2.3. Let $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet} = \prod_{i=1}^m \mathfrak{a}_i^{t_i})$ be a triple. Assume that $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ are non-zero ideals. Then we define the *test ideal*

$$\tau(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau\left(R, \Delta, \prod_{i=1}^m \mathfrak{a}_i^{t_i}\right) = \tau(R, \Delta, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_m^{t_m})$$

to be an unique minimal non-zero uniformly $(\Delta, \mathfrak{a}_\bullet^{t_\bullet}, F)$-compatible ideal. The test ideal always exists (see [Sch10, Theorem 6.3]).

When $\mathfrak{a}_i = R$ and $t_i = 0$ for every i, then we denote the ideal $\tau(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ by $\tau(R, \Delta)$. If $\mathfrak{a}_i = 0$ for some i, then we define $\tau(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = (0)$.

LEMMA 2.4. *Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple. Then the following hold.*

(i) *If $t \leqslant t'$ and $\mathfrak{a}' \subseteq \mathfrak{a}$, then $\tau(R, \Delta, (\mathfrak{a}')^{t'}) \subseteq \tau(R, \Delta, \mathfrak{a}^t)$.*

(ii) *[ST14, Lemma 6.1] Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then there exists a real number $\varepsilon > 0$ such that if $t \leqslant t' \leqslant t + \varepsilon$, then $\tau(R, \Delta, \mathfrak{a}^{t'}) = \tau(R, \Delta, \mathfrak{a}^t)$.*

DEFINITION 2.5. Let (R, Δ) be a pair and $\mathfrak{a} \subseteq R$ be an ideal. A real number $t > 0$ is called a *F-jumping number* of $(R, \Delta; \mathfrak{a})$ if

$$\tau(R, \Delta, \mathfrak{a}^{t-\varepsilon}) \neq \tau(R, \Delta, \mathfrak{a}^t),$$

for all $\varepsilon > 0$.

PROPOSITION 2.6 [ST14, Theorem B]. *Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a})$ be a triple such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then the set of all F-jumping numbers of $(R, \Delta; \mathfrak{a})$ is a discrete set of rational numbers.*

DEFINITION 2.7. Let $(R, \Delta, \mathfrak{a})$ be a triple such that $\mathfrak{a} \neq R$, and let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. We define the *F-jumping number* of $(R, \Delta; \mathfrak{a})$ with respect to I as

$$\operatorname{fjn}^I(R, \Delta; \mathfrak{a}) := \inf\{t \in \mathbb{R}_{\geqslant 0} \mid \tau(R, \Delta, \mathfrak{a}^t) \subseteq I\} \in \mathbb{R}_{\geqslant 0}.$$

When $\tau(R, \Delta) = R$ and $I = \mathfrak{m}$, we denote it by $\operatorname{fpt}(R, \Delta; \mathfrak{a})$ and call it the *F-pure threshold* of $(R, \Delta; \mathfrak{a})$. If $\Delta = 0$, then we denote it by $\operatorname{fpt}(R; \mathfrak{a})$.

DEFINITION 2.8. Let $(X = \operatorname{Spec} R, \Delta)$ be a pair and $e \geqslant 0$ be an integer. Assume that $(p^e - 1)(K_X + \Delta)$ is Cartier. Then there exists an isomorphism

$$\operatorname{Hom}_R(F_*^e(R((p^e - 1)\Delta)), R) \cong F_*^e R$$

as $F_*^e R$-modules [Sch09, Lemma 3.1]. We denote by φ_Δ^e a generator of $\operatorname{Hom}_R(F_*^e(R((p^e-1)\Delta)), R)$ as an $F_*^e R$-module.

Remark 2.9. Although a map $\varphi_\Delta^e : F_*^e R \longrightarrow R$ is not uniquely determined, it is unique up to multiplication by $F_*^e R^\times$. When we consider this map, we only need the information about the image of this map. Hence we ignore the multiplication by $F_*^e R^\times$.

Let R be a Noetherian ring of characteristic $p > 0$, let e be a positive integer, and let $\mathfrak{a} \subseteq R$ be an ideal. Then we denote by $\mathfrak{a}^{[p^e]}$ the ideal generated by $\{f^{p^e} \in R \mid f \in \mathfrak{a}\}$. The following proposition seems to be well known to experts, but difficult to find a proof in the literature.

Proposition 2.10. *Let $(R, \mathfrak{m}, k)$ and $(S, \mathfrak{n}, l)$ be F-finite Noetherian normal local rings. Let $R \longrightarrow S$ be a flat local homomorphism, Δ_X be an effective $\mathbb{Q}$-Weil divisor on $X = \operatorname{Spec} R$ and Δ_Y be the flat pullback of Δ_X to $Y = \operatorname{Spec} S$. Assume that $\mathfrak{m}S = \mathfrak{n}$ and that the relative Frobenius morphism $F^e_{l/k} : F^e_* k \otimes_k l \longrightarrow F^e_* l$ is an isomorphism for every $e \geqslant 0$. Then the following hold.*

(i) *The morphism $R \longrightarrow S$ is a regular morphism, that is, every fiber is geometrically regular.*

(ii) *The relative Frobenius morphism $F^e_{S/R} : F^e_* R \otimes_R S \longrightarrow F^e_* S$ is an isomorphism for every $e \geqslant 0$.*

(iii) *For every $e \geqslant 0$, we have*

$$\operatorname{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta_X \rceil), R) \otimes_R S \cong \operatorname{Hom}_S(F^e_* S(\lceil (p^e - 1)\Delta_Y \rceil), S).$$

(iv) *Let $(R, \Delta_X, \mathfrak{a}^{t_\bullet}_\bullet = \prod_{i=1}^m \mathfrak{a}_i^{t_i})$ be a triple. We write $(\mathfrak{a}_\bullet \cdot S)^{t_\bullet} := \prod_i (\mathfrak{a}_i S)^{t_i}$. Then we have*

$$\tau(R, \Delta_X, \mathfrak{a}^{t_\bullet}_\bullet) \cdot S = \tau(S, \Delta_Y, (\mathfrak{a}_\bullet \cdot S)^{t_\bullet}).$$

(v) *If $(p^e - 1)(K_X + \Delta_X)$ is Cartier for some $e > 0$, then $(p^e - 1)(K_Y + \Delta_Y)$ is also Cartier and $\varphi^e_{\Delta_Y} : F^e_* S \longrightarrow S$ coincides with the morphism $\varphi^e_{\Delta_X} \otimes_R S : F^e_* R \otimes_R S \longrightarrow S$ via the isomorphism $F^e_{S/R} : F^e_* R \otimes_R S \longrightarrow F^e_* S$.*

Proof. Since the relative Frobenius morphism $F_{l/k} : F_* k \otimes_k l \longrightarrow F_* l$ is injective, the field extension $k \subseteq l$ is separable by [Mat89, Theorem 26.4]. Then (i) follows from [Mat89, Theorem 28.10] and [And74].

We will prove the assertion in (ii). Fix an integer $e \geqslant 0$. By (i), the morphism $R \longrightarrow S$ is generically separable. It follows from [Mat89, Theorem 26.4] that the relative Frobenius morphism $F^e_{S/R} : F^e_* R \otimes_R S \longrightarrow F^e_* S$ is injective.

We next consider the surjectivity of the map $F^e_{S/R}$. We denote the ring $F^e_* R \otimes_R S$ by R'. We consider the following commutative diagram.

$$\begin{tikzcd}
& F^e_* S \\
S \arrow[ur, "F^e_S"] \arrow[r] & R' \arrow[u, "F^e_{S/R}"'] \\
R \arrow[u] \arrow[r, "F^e_R"'] & F^e_* R \arrow[u]
\end{tikzcd}$$

Since the morphisms $F^e_R : R \longrightarrow F^e_* R$ and $S \longrightarrow R'$ are both finite and $\mathfrak{n} \cap R = \mathfrak{m}$, every maximal ideal of R' contains the maximal ideal $F^e_* \mathfrak{m}$ of $F^e_* R$. Therefore, $I := (F^e_* \mathfrak{m}) \cdot R' \subseteq R'$ is contained in the Jacobson radical of R'. On the other hand, since the finite morphism $F^e_S : F^e_* S \longrightarrow S$ factors through $F^e_{S/R}$, the morphism $F^e_{S/R}$ is also finite. Then the morphism

$$F^e_{S/R} \otimes_{R'} (R'/I) : R'/I \longrightarrow (F^e_* S) \otimes_{R'} (R'/I)$$

coincides with the relative Frobenius morphism $F^e_{l/k} : F^e_* k \otimes_k l \longrightarrow F^e_* l$, and hence it is surjective. Therefore, the map $F^e_{S/R}$ is surjective by Nakayama.

We next prove the assertion in (iii). Since S is flat over R and $F^e_* R(\lceil (p^e - 1)\Delta_X \rceil)$ is a finite R-module, we have

$$\operatorname{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta_X \rceil), R) \otimes_R S \cong \operatorname{Hom}_S(F^e_* R(\lceil (p^e - 1)\Delta_X \rceil) \otimes_R S, S).$$

By (i), the flat pullback of a prime divisor on X to Y is a reduced divisor. Therefore, the Weil divisor $\lceil (p^e-1)\Delta_Y \rceil$ coincides with the flat pullback of $\lceil (p^e-1)\Delta_X \rceil$. It follows from (ii) that $F^e_* R(\lceil (p^e-1)\Delta_X \rceil) \otimes_R S \cong F^e_* S(\lceil p^e-1 \rceil \Delta_Y)$, which completes the proof of (iii).

For (iv), it follows from (iii) that the test ideal $\tau(R, \Delta_X, \mathfrak{a}_\bullet^{t_\bullet}) \cdot S$ is uniformly $(\Delta_Y, (\mathfrak{a}_\bullet \cdot S)^{t_\bullet}, F)$-compatible and $\tau(S, \Delta_Y, (\mathfrak{a}_\bullet \cdot S)^{t_\bullet}) \cap R$ is uniformly $(\Delta_X, \mathfrak{a}_\bullet^{t_\bullet}, F)$-compatible. Therefore, we have

$$\tau(S, \Delta_Y, (\mathfrak{a}_\bullet \cdot S)^{t_\bullet}) \subseteq \tau(R, \Delta_X, \mathfrak{a}_\bullet^{t_\bullet}) \cdot S$$

and

$$\tau(S, \Delta_Y, (\mathfrak{a}_\bullet \cdot S)^{t_\bullet}) \cap R \supseteq \tau(R, \Delta_X, \mathfrak{a}_\bullet^{t_\bullet}),$$

which complete the proof of (iv).

For (v), we assume that $(p^e-1)(K_X+\Delta_X)$ is Cartier. Since the canonical divisor K_Y coincides with the flat pullback of K_X ([Aoy83, Proposition 4.1], see also [Sta18, Lemma 45.22.1]), the Weil divisor $(p^e-1)(K_Y+\Delta_Y)$ is also Cartier. The second assertion in (v) follows from (iii). □

LEMMA 2.11. *Let $k \subseteq l$ be an extension of F-finite fields and $d \geqslant 1$ be an integer. Assume that $(A_k, \mathfrak{m}_k)$ and $(A_l, \mathfrak{m}_l)$ are the rings of the formal power series of dimension d with coefficients k and l, respectively. Then, for any ideal $\mathfrak{a} \subseteq A_k$, we have*

$$\mathrm{fpt}(A_k; \mathfrak{a}) = \mathrm{fpt}(A_l; (\mathfrak{a} A_l)).$$

Proof. The proof is similar to the case where $\mathfrak{a}$ is principal [BMS09, Theorem 3.5(i)]. For every integer $e > 0$, set $\nu_{\mathfrak{a}}(p^e) := \max\{r \in \mathbb{N}_{\geqslant 0} \mid \mathfrak{a}^r \not\subseteq \mathfrak{m}_k^{[p^e]}\}$ and $\nu_{(\mathfrak{a}A_l)}(p^e) := \max\{r \in \mathbb{N}_{\geqslant 0} \mid (\mathfrak{a}A_l)^r \not\subseteq \mathfrak{m}_l^{[p^e]}\}$. Then, it follows from the proof of [BMS08, Corollary 2.30] that $\mathrm{fpt}(A_k; \mathfrak{a}) = \lim_e \nu_{\mathfrak{a}}(p^e)/p^e$ and $\mathrm{fpt}(A_l; (\mathfrak{a}A_l)) = \lim_e (\nu_{(\mathfrak{a}A_l)}(p^e))/p^e$.

On the other hand, since $\mathfrak{m}_k^{[p^e]} A_l = \mathfrak{m}_l^{[p^e]}$ for every e and the extension $A_k \subseteq A_l$ is faithfully flat, we have $\nu_{\mathfrak{a}}(p^e) = \nu_{(\mathfrak{a}A_l)}(p^e)$ for every e, which completes the proof. □

Example 2.12. With the notation above, it follows from the previous lemma that we have

$$\tau(A_k, \mathfrak{a}^t) = A_k \quad \text{if and only if } \tau(A_l, (\mathfrak{a}A_l)^t) = A_l$$

for every $t \geqslant 0$. However, in general, we have

$$\tau(A_k, \mathfrak{a}^t) \cdot A_l \neq \tau(A_l, (\mathfrak{a}A_l)^t).$$

For example, suppose that $k = \mathbb{F}_p(t)$, $l = \mathbb{F}_p(t^{1/p})$ and $d = 2$. Set $f := x^p + ty^p \in A_k$ and $\mathfrak{a} := (f)$. It follows from [BMS08, Proposition 2.5], [BMS09, Lemma 2.1] and the decomposition $A_k = \bigoplus_{0 \leqslant a,b,c < p} A_k^p \cdot t^a x^b y^c$ that we have $\tau(A_k, \mathfrak{a}^{1/p}) = (x, y)$.

On the other hand, if we set $g := x + t^{1/p} y \in A_l$, then we have

$$\tau(A_l, (\mathfrak{a}A_l)^{1/p}) = \tau(A_l, (g)^1) = (g).$$

Therefore, we have $\tau(A_k, \mathfrak{a}^t) \cdot A_l \neq \tau(A_l, (\mathfrak{a}A_l)^t)$.

Let $(R, \mathfrak{m})$ be a Noetherian local ring. For a finitely generated R-module M, we denote by $\mu_R(M)$ the minimal number of generators of M as an R-module. We denote by $\mathrm{emb}(R)$ the embedding dimension $\mu_R(\mathfrak{m})$. If M has finite length, then we denote by $\ell_R(M)$ the length of M as an R-module and define

$$\ell\ell_R(M) := \min\{n \geqslant 0 \mid \mathfrak{m}^n M = 0\}.$$

The following lemma is well known to experts, but we prove it for convenience.

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

LEMMA 2.13. *Suppose that R is a Noetherian ring of characteristic $p>0$, $\mathfrak{c}_1,\ldots,\mathfrak{c}_m \subseteq R$ are ideals, and $M_1,\ldots,M_m$ are positive integers. Set $l := \sum_i \mu_R(\mathfrak{c}_i)$ and $\mathfrak{c} := \mathfrak{c}_1^{M_1}+\cdots+\mathfrak{c}_m^{M_m}$. Let a,b,e be non-negative integers and set $q := p^e$. If $b > q(l-1)$, then we have*

$$\mathfrak{c}^{aq+b} = (\mathfrak{c}^a)^{[q]} \cdot \mathfrak{c}^b.$$

In particular, for any ideal $\mathfrak{q} \subseteq R$ and R-homomorphism $\varphi : F^e_*R \longrightarrow R$, we have

$$\varphi(F^e_*(\mathfrak{c}^{aq+b}\mathfrak{q})) = \mathfrak{c}^a \cdot \varphi(F^e_*(\mathfrak{c}^b\mathfrak{q})).$$

Proof. The second assertion follows from the first one because if $\mathfrak{c}^{aq+b} = (\mathfrak{c}^a)^{[q]} \cdot \mathfrak{c}^b$, then one has

$$\mathfrak{c}^a \cdot \varphi(F^e_*(\mathfrak{c}^b\mathfrak{q})) = \varphi(F^e_*((\mathfrak{c}^a)^{[q]}\mathfrak{c}^b\mathfrak{q})) = \varphi(F^e_*(\mathfrak{c}^{aq+b}\mathfrak{q})).$$

Therefore, it is enough to prove the first assertion.

We first consider the case $m=1$. If $m=M_1=1$, then the assertion follows straightforwardly by taking a minimal generator of $\mathfrak{c}=\mathfrak{c}_1$. If $m=1$ and $M_1 \geqslant 1$ is arbitrary, then it follows from the case $m=M_1=1$ that

$$\begin{aligned}\mathfrak{c}^{aq+b} = \mathfrak{c}_1^{M_1(aq+b)} &= \mathfrak{c}_1^{(aM_1)q+(bM_1)}\\ &= (\mathfrak{c}_1^{aM_1})^{[q]}\mathfrak{c}_1^{bM_1}\\ &= (\mathfrak{c}^a)^{[q]}\mathfrak{c}^b.\end{aligned}$$

We next consider the case $m \geqslant 2$. Set $\mathfrak{b}_i := \mathfrak{c}_i^{M_i}$ and $l_i := \mu_R(\mathfrak{c}_i)$. Then we have

$$\mathfrak{c}^{aq+b} = \sum_{n_1,\ldots,n_m} \prod_{i=1}^m \mathfrak{b}_i^{n_i},$$

where n_i runs through all non-negative integers such that $\sum_i n_i = aq+b$. Fix such integers $(n_i)_{i=1}^m$ and set $a_i := \max\{0, \lceil n_i/q \rceil - l_i\}$ and $b_i := n_i - qa_i$. Then a_i and b_i are non-negative integers such that $n_i = qa_i + b_i$. If $a_i \geqslant 1$, then we have $a_i = \lceil n_i/q \rceil - l_i$, which implies $b_i > q(l_i-1)$. In this case, it follows from the case $m=1$ that $\mathfrak{b}_i^{n_i} = (\mathfrak{b}_i^{a_i})^{[q]}\mathfrak{b}_i^{b_i}$. On the other hand, if $a_i=0$, then $b_i = n_i$ and the equation $\mathfrak{b}_i^{n_i} = (\mathfrak{b}_i^{a_i})^{[q]}\mathfrak{b}_i^{b_i}$ holds too. Therefore, we have

$$\begin{aligned}\prod_i \mathfrak{b}_i^{n_i} &= \left(\prod_i \mathfrak{b}_i^{a_i}\right)^{[q]} \cdot \prod_i \mathfrak{b}_i^{b_i}\\ &\subseteq \left(\mathfrak{c}^{\sum_i a_i}\right)^{[q]} \cdot \mathfrak{c}^{\sum_i b_i}.\end{aligned}$$

On the other hand, since $a_i \geqslant (n_i/q) - l_i$ for every i, we have

$$\sum_i a_i \geqslant \left(\sum_i n_i\right)\Big/ q - l = (aq+b)/q - l > a-1.$$

Combining with $(\sum_i a_i)q + (\sum_i b_i) = aq+b$, we have

$$\left(\mathfrak{c}^{\sum_i a_i}\right)^{[q]} \cdot \mathfrak{c}^{\sum_i b_i} \subseteq (\mathfrak{c}^a)^{[q]} \cdot \mathfrak{c}^b,$$

which implies the assertion. $\square$

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

2.2 Ultraproduct

In this subsection, we define the ultraproduct of a family of sets and recall some properties. We also define the catapower of a Noetherian local ring and prove some properties. The reader is referred to [Sch10] for details.

DEFINITION 2.14. Let $\mathfrak{U}$ be a collection of subsets of $\mathbb{N}$. Then $\mathfrak{U}$ is called an *ultrafilter* if the following properties hold.

(i) We have $\emptyset \notin \mathfrak{U}$.
(ii) For every pair of subsets $A, B \subseteq \mathbb{N}$, if $A \in \mathfrak{U}$ and $A \subseteq B$, then $B \in \mathfrak{U}$.
(iii) For every pair of subsets $A, B \subseteq \mathbb{N}$, if $A, B \in \mathfrak{U}$, then $A \cap B \in \mathfrak{U}$.
(iv) For every subset $A \subseteq \mathbb{N}$, if $A \notin \mathfrak{U}$, then $\mathbb{N} \setminus A \in \mathfrak{U}$.

An ultrafilter $\mathfrak{U}$ is called *non-principal* if the following holds.

(v) If A is a finite subset of $\mathbb{N}$, then $A \notin \mathfrak{U}$.

By Zorn's lemma, there exists a non-principal ultrafilter. From now on, we fix a non-principal ultrafilter $\mathfrak{U}$.

DEFINITION 2.15. Let $\{T_m\}_{m\in\mathbb{N}}$ be a family of sets. We define the equivalence relation $\sim$ on the set $\prod_{m\in\mathbb{N}} T_m$ by

$$(a_m)_m \sim (b_m)_m \quad \text{if and only if } \{m \in \mathbb{N} \mid a_m = b_m\} \in \mathfrak{U}.$$

We define the *ultraproduct* of $\{T_m\}_{m\in\mathbb{N}}$ as

$$\operatorname{ulim}_{m\in\mathbb{N}} T_m := \left(\prod_{m\in\mathbb{N}} T_m\right) \Big/ \sim .$$

If T is a set and $T_m = T$ for all m, then we denote $\operatorname{ulim}_m T_m$ by ${}^*T$ and call it the *ultrapower* of T.

Let $\{T_m\}_{m\in\mathbb{N}}$ be a family of sets and $a_m \in T_m$ for every m. We denote by $\operatorname{ulim}_m a_m$ the class of $(a_m)_m$ in $\operatorname{ulim}_m T_m$. Let $\{S_m\}_m$ be another family of sets and $f_m : T_m \longrightarrow S_m$ be a map for every m. We can define the map

$$\operatorname{ulim}_m f_m : \operatorname{ulim}_m T_m \longrightarrow \operatorname{ulim}_m S_m$$

by sending $\operatorname{ulim}_m a_m \in \operatorname{ulim}_m T_m$ to $\operatorname{ulim}_m f_m(a_m) \in \operatorname{ulim}_m S_m$. If $T_m = T$, $S_m = S$, and $f_m = f$ for every $m \in \mathbb{N}$, then we denote the map $\operatorname{ulim}_m f_m$ by ${}^*f : {}^*T \longrightarrow {}^*S$.

Let $\{R_m\}_{m\in\mathbb{N}}$ be a family of rings and M_m be an R_m-module for every m. Then $\operatorname{ulim}_m R_m$ has the ring structure induced by that of $\prod_m R_m$ and $\operatorname{ulim}_m M_m$ has the structure of a $\operatorname{ulim} R_m$-module induced by its structure as a $\prod_m R_m$-module on $\prod_m M_m$. Moreover, if k_m is a field for every m, then $\operatorname{ulim}_m k_m$ is a field.

PROPOSITION 2.16. *We have the following properties.*

(i) *Let R be a Noetherian ring and M be a finitely generated R-module. Then we have ${}^*M \cong M \otimes_R {}^*R$.*

(ii) *Let k be an F-finite field of positive characteristic. Then the relative Frobenius morphism $F^e_*(k) \otimes_k {}^*k \longrightarrow F^e_*({}^*k)$ is an isomorphism. In particular, ${}^*k$ is an F-finite field.*

Proof. For (i), we consider the natural homomorphism $M \otimes_R {}^*R \longrightarrow {}^*M$. Since the functors ${}^*(-)$ and $(-) \otimes_R {}^*R$ are both right exact, we may assume that M is a free R-module of finite rank. In this case, the assertion is obvious.

For (ii), we consider the natural bijection ${}^*(F^e_*k) \cong F^e_*({}^*k)$. Combining with (i), the relative Frobenius morphism $F^e_*(k) \otimes_k {}^*k \longrightarrow F^e_*({}^*k)$ is an isomorphism. □

Let $\mathfrak{a}_m \subseteq R_m$ be an ideal for every m. Then the natural map $\operatorname{ulim}_m \mathfrak{a}_m \longrightarrow \operatorname{ulim}_m R_m$ is injective, and hence we can consider $\operatorname{ulim}_m \mathfrak{a}_m$ as an ideal of the ring $\operatorname{ulim}_m R_m$. Let $\mathfrak{b}_m \subseteq R_m$ be other ideals. Then $\operatorname{ulim}_m \mathfrak{b}_m \subseteq \operatorname{ulim}_m \mathfrak{a}_m$ if and only if

$$\{m \in \mathbb{N} \mid \mathfrak{b}_m \subseteq \mathfrak{a}_m\} \in \mathfrak{U}.$$

Moreover, we have the equation

$$(\operatorname{ulim}_m \mathfrak{a}_m) + (\operatorname{ulim}_m \mathfrak{b}_m) = \operatorname{ulim}_m(\mathfrak{a}_m + \mathfrak{b}_m).$$

Lemma 2.17. *Let $\{R_m\}_{m\in\mathbb{N}}$ be a family of rings and $\mathfrak{a}_m, \mathfrak{b}_m \subseteq R_m$ be ideals for every m. Assume that there exists an integer $l > 0$ such that $\mu(\mathfrak{a}_m) \leqslant l$ for every m. Then we have*

$$(\operatorname{ulim}_m \mathfrak{a}_m) \cdot (\operatorname{ulim}_m \mathfrak{b}_m) = \operatorname{ulim}_m(\mathfrak{a}_m \cdot \mathfrak{b}_m).$$

Proof. Let $\alpha = \operatorname{ulim}_m a_m \in \operatorname{ulim}_m \mathfrak{a}_m$ and $\beta = \operatorname{ulim}_m b_m \in \operatorname{ulim}_m \mathfrak{b}$. Then we have $\alpha \cdot \beta = \operatorname{ulim}_m(a_m b_m) \in \operatorname{ulim}_m(\mathfrak{a}_m \cdot \mathfrak{b}_m)$. This shows the inclusion $(\operatorname{ulim}_m \mathfrak{a}_m) \cdot (\operatorname{ulim}_m \mathfrak{b}_m) \subseteq \operatorname{ulim}_m(\mathfrak{a}_m \cdot \mathfrak{b}_m)$.

We consider the converse inclusion. By the assumption, there exist $f_{m,1}, \ldots, f_{m,l} \in \mathfrak{a}_m$ such that $\mathfrak{a}_m = (f_{m,1}, \ldots, f_{m,l})$. Then we have $\mathfrak{a}_m \cdot \mathfrak{b}_m = \sum_i f_{m,i} \cdot \mathfrak{b}_m$, and hence we have

$$\operatorname{ulim}_m(\mathfrak{a}_m \cdot \mathfrak{b}_m) = \sum_i f_{\infty,i} \cdot (\operatorname{ulim}_m \mathfrak{b}_m),$$

where $f_{\infty,i} := \operatorname{ulim}_m f_{m,i} \in \operatorname{ulim}_m \mathfrak{a}_m$ for every i, which complete the proof of the lemma. □

Proposition-Definition 2.18 [Gol98, Theorem 5.6.1]. *Let $\{a_m\}_{m\in\mathbb{N}}$ be a sequence of real numbers such that there exist real numbers M_1, M_2 which satisfies $M_1 < a_m < M_2$ for every $m \in \mathbb{N}$. Then there exists an unique real number $w \in \mathbb{R}$ such that for every real number $\varepsilon > 0$, we have*

$$\{m \in \mathbb{N} \mid |w - a_m| < \varepsilon\} \in \mathfrak{U}.$$

We denote this number w by $\operatorname{sh}(\operatorname{ulim}_m a_m)$ and call it the shadow of $\operatorname{ulim}_m a_m \in {}^*\mathbb{R}$.

Let $(R, \mathfrak{m}, k)$ be a local ring. Then, one can show that $({}^*R, {}^*\mathfrak{m}, {}^*k)$ is a local ring. However, even if R is Noetherian, the ultrapower ${}^*R$ may not be Noetherian because we do not have the equation $\bigcap_{n\in\mathbb{N}}({}^*\mathfrak{m})^n = 0$ in general.

Definition 2.19 [Sch10]. Let $(R, \mathfrak{m})$ be a Noetherian local ring and $({}^*R, {}^*\mathfrak{m})$ be the ultrapower. We define the *catapower* $R_\#$ as the quotient ring

$$R_\# := {}^*R \Big/ \Big(\bigcap_n ({}^*\mathfrak{m})^n\Big).$$

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

Proposition 2.20 [Sch10, Theorem 8.1.19]. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of equicharacteristic and $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. We fix a coefficient field $k \subseteq \widehat{R}$. Then we have*

$$R_\# \cong \widehat{R} \mathbin{\widehat{\otimes}}_k ({}^*k).$$

In particular, if $(R, \mathfrak{m})$ is an F-finite Noetherian normal local ring, then so is $R_\#$.

Let $(R, \mathfrak{m})$ be a Noetherian local ring, $R_\#$ be the catapower and $a_m \in R$ for every m. We denote by $[a_m]_m \in R_\#$ the image of $\mathrm{ulim}_m\, a_m \in {}^*R$ by the natural projection ${}^*R \longrightarrow R_\#$. Let $\mathfrak{a}_m \subseteq R$ be an ideal for every $m \in \mathbb{N}$. We denote by $[\mathfrak{a}_m]_m \subseteq R_\#$ the image of the ideal $\mathrm{ulim}_m\, \mathfrak{a}_m \subseteq {}^*R$ by the projection ${}^*R \longrightarrow R_\#$.

Lemma 2.21. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and $\mathfrak{a}_m, \mathfrak{b}_m \subseteq R$ be ideals for every $m \in \mathbb{N}$. If $[\mathfrak{a}_m]_m \subseteq [\mathfrak{b}_m]_m$, then for every $\mathfrak{m}$-primary ideal $\mathfrak{q} \subseteq R$, we have*

$$\{m \in \mathbb{N} \mid \mathfrak{a}_m \subseteq \mathfrak{b}_m + \mathfrak{q}\} \in \mathfrak{U}.$$

Proof. By the definition of the catapower, if $[\mathfrak{a}_m]_m \subseteq [\mathfrak{b}_m]_m$, then we have

$$\mathrm{ulim}_m\, \mathfrak{a}_m \subseteq \mathrm{ulim}_m\, \mathfrak{b}_m + ({}^*\mathfrak{m})^n$$

for every n.

On the other hand, it follows from Lemma 2.17 that $({}^*\mathfrak{m})^n = {}^*(\mathfrak{m}^n)$. Therefore we have

$$\begin{aligned} \mathrm{ulim}\, \mathfrak{a}_m &\subseteq (\mathrm{ulim}\, \mathfrak{b}_m) + {}^*(\mathfrak{m}^n) \\ &= \mathrm{ulim}(\mathfrak{b}_m + \mathfrak{m}^n), \end{aligned}$$

which is equivalent to

$$\{m \in \mathbb{N} \mid \mathfrak{a}_m \subseteq \mathfrak{b} + \mathfrak{m}^n\} \in \mathfrak{U}.$$

This implies the assertion in the lemma. □

3. Variants of test ideals

In this section, we introduce some variants of test ideals by using the trace maps for the Frobenius morphisms and the q-adic expansion of a real number (Definitions 3.3 and 3.11). We also introduce the stabilization exponent (Definition 3.6).

Definition 3.1 (Cf. [HNWZ16, Definitions 2.1, 2.2]). Let $q \geqslant 2$ be an integer, $t > 0$ be a real number and $n \in \mathbb{Z}$ be an integer. We define the *nth digit* of t in base q by

$$t^{(n)} := \lceil tq^n - 1 \rceil - q\lceil tq^{n-1} - 1 \rceil \in \mathbb{Z}.$$

We define the *nth round up* and the *nth truncation* of t in base q by

$$\langle t \rangle^{n,q} := \lceil tq^n \rceil / q^n \in \mathbb{Q},$$

and

$$\langle t \rangle_{n,q} := \lceil tq^n - 1 \rceil / q^n \in \mathbb{Q},$$

respectively.

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

Lemma 3.2. *Let $q \geqslant 2$ be an integer, $t > 0$ be a real number and $n \in \mathbb{Z}$ be an integer. Then the following hold:*

(i) $0 \leqslant t^{(n)} < q$;

(ii) $t^{(n)}$ *is eventually zero for* $n \ll 0$ *and is not eventually zero for* $n \gg 0$;

(iii) $t = \sum_{m \in \mathbb{Z}} t^{(m)} \cdot q^{-m}$;

(iv) $\langle t \rangle_{n,q} = \sum_{m \leqslant n} t^{(m)} \cdot q^{-m}$;

(v) *the sequence* $\{\langle t \rangle^{n,q}\}_{n \in \mathbb{Z}}$ *is a descending chain which converges to* t;

(vi) *the sequence* $\{\langle t \rangle_{n,q}\}_{n \in \mathbb{Z}}$ *is an ascending chain which converges to* t.

Proof. These all follow easily from the definitions. For the assertion in (ii), we note that if $t = s/q^m$ for some integers s and m, then we have $t^{(n)} = q - 1$ for all $n > m$. $\square$

Definition 3.3. Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}_{\bullet}^{t_{\bullet}} = \prod_i \mathfrak{a}_i^{t_i})$ be a triple such that $t_i > 0$ for all i, and let $e > 0$ be an integer such that $(p^e - 1)(K_X + \Delta)$ is Cartier. For every integer $n \geqslant 0$, we define

$$\tau_+^{en}(R, \Delta, \mathfrak{a}_{\bullet}^{t_{\bullet}}) := \varphi_{\Delta}^{en}(F_*^{en}(\mathfrak{a}_1^{\lceil t_1 p^{en} \rceil} \cdots \mathfrak{a}_m^{\lceil t_m p^{en} \rceil} \cdot \tau(R, \Delta))) \subseteq R$$

and

$$\tau_-^{en}(R, \Delta, \mathfrak{a}_{\bullet}^{t_{\bullet}}) := \varphi_{\Delta}^{en}(F_*^{en}(\mathfrak{a}_1^{\lceil t_1 p^{en} - 1 \rceil} \cdots \mathfrak{a}_m^{\lceil t_m p^{en} - 1 \rceil} \cdot \tau(R, \Delta))) \subseteq R.$$

Example 3.4. Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple such that $t > 0$ and that $\mathfrak{a}$ is a principal ideal, and let e be a positive integer such that $(p^e - 1)(K_X + \Delta)$ is Cartier. Then it follows from [BSTZ10, Lemma 5.4] that

$$\tau_+^{en}(R, \Delta, \mathfrak{a}^t) = \tau(R, \Delta, \mathfrak{a}^{\langle t \rangle^{n,q}}),$$

and

$$\tau_-^{en}(R, \Delta, \mathfrak{a}^t) = \tau(R, \Delta, \mathfrak{a}^{\langle t \rangle_{n,q}}).$$

Therefore, in this case, it follows from Proposition 2.6 that the sequence $\{\tau_+^{en}(R, \Delta, \mathfrak{a}^t)\}_n$ is an ascending chain of ideals which converges to $\tau(R, \Delta, \mathfrak{a}^t)$ and the sequence $\{\tau_-^{en}(R, \Delta, \mathfrak{a}^t)\}_n$ is a descending chain of ideals which eventually stabilizes.

Proposition 3.5 (Basic properties). *Let $(R, \Delta, \mathfrak{a}_{\bullet}^{t_{\bullet}})$ and e be as in Definition 3.3. Then the following hold.*

(i) *[BSTZ10, Lemma 3.21] The sequence $\{\tau_+^{en}(R, \Delta, \mathfrak{a}_{\bullet}^{t_{\bullet}})\}_{n \geqslant 0}$ is an ascending chain which converges to the test ideal $\tau(R, \Delta, \mathfrak{a}_{\bullet}^{t_{\bullet}})$.*

(ii) *If $t_1 > 1$, then we have*

$$\tau_+^{en}(R, \Delta, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_m^{t_m}) \supseteq \mathfrak{a}_1 \cdot \tau_+^{en}(R, \Delta, \mathfrak{a}_1^{t_1 - 1} \cdots \mathfrak{a}_m^{t_m}).$$

Moreover, if $t_1 > \mu_R(\mathfrak{a}_1)$, then we have

$$\tau_+^{en}(R, \Delta, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_m^{t_m}) = \mathfrak{a}_1 \cdot \tau_+^{en}(R, \Delta, \mathfrak{a}_1^{t_1 - 1} \cdots \mathfrak{a}_m^{t_m}).$$

(iii) *We have $\varphi_{\Delta}^e(F_*^e(\tau_+^{en}(R, \Delta, \mathfrak{a}_{\bullet}^{p^e \cdot t_{\bullet}}))) = \tau_+^{e(n+1)}(R, \Delta, \mathfrak{a}_{\bullet}^{t_{\bullet}})$, where we set $\mathfrak{a}_{\bullet}^{p^e \cdot t_{\bullet}} := \prod_i \mathfrak{a}_i^{p^e t_i}$.*

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

Proof. The proof of (i) follows as in the case when $m = 1$, see [BSTZ10, Lemma 3.21]. The assertion in (ii) follows from Lemma 2.13 by setting $\mathfrak{c} = \mathfrak{c}_1 := \mathfrak{a}_1$, $a := 1$ and $b := \lceil (t_i - 1)q^n \rceil$. The assertion in (iii) follows from the fact that $\varphi_\Delta^{e(n+1)} = \varphi_\Delta^e \circ F_*^e \varphi_\Delta^{en}$ [Sch09, Theorem 3.11(e)]. □

DEFINITION 3.6. Let $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$ and e be as in Definition 3.3. We define the *stabilization exponent* of $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}; e)$ by

$$\mathrm{stab}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}; e) := \min\{n \geqslant 0 \mid \tau_+^{en}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})\}.$$

PROPOSITION 3.7 (Basic properties). *Let $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet} = \prod_{i=1}^m \mathfrak{a}_i^{t_i})$ and e be as in Definition 3.3. Then the following hold.*

(i) *If $t_1 > \mu_R(\mathfrak{a}_1)$, then we have*

$$\mathrm{stab}(R, \Delta, \mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_m^{t_m}; e) \leqslant \mathrm{stab}(R, \Delta, \mathfrak{a}_1^{t_1 - 1} \cdots \mathfrak{a}_m^{t_m}; e).$$

(ii) *We have*

$$\mathrm{stab}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}; e) \leqslant \mathrm{stab}(R, \Delta, \mathfrak{a}_\bullet^{p^e \cdot t_\bullet}; e) + 1.$$

Proof. The assertions in (i) and (ii) follow from Proposition 3.5(ii) and (iii), respectively. □

The following proposition is also basic, but it may be useful for studies of test ideals.

PROPOSITION 3.8. *Let $(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet} = \prod_{i=1}^m \mathfrak{a}_i^{t_i})$ and e be as in Definition 3.3. Moreover, assume that $t_i > \mu_R(\mathfrak{a}_i)$ and $(p^e - 1)t_i \in \mathbb{N}$ for every i. If*

$$\tau_+^{en}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau_+^{e(n+1)}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$$

for some integer n, then we have

$$\tau_+^{en}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}).$$

In particular, we have $n \geqslant \mathrm{stab}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}; e)$.

Proof. It follows from Proposition 3.5(ii) and (iii) that

$$\begin{aligned}\tau_+^{e(n+1)}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) &= \varphi_\Delta^e(F_*^e(\tau_+^{en}(R, \Delta, \mathfrak{a}_1^{p^e t_1} \cdots \mathfrak{a}_m^{p^e t_m}))) \\ &= \varphi_\Delta^e(F_*^e(\mathfrak{a}_1^{(p^e-1)t_1} \cdots \mathfrak{a}_m^{(p^e-1)t_m} \cdot \tau_+^{en}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}))).\end{aligned}$$

Therefore, if $\tau_+^{en}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau_+^{e(n+1)}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$, then one has $\tau_+^{e(n+1)}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}) = \tau_+^{e(n+2)}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet})$, which completes the proof. □

PROPOSITION 3.9. *Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}_\bullet = \prod_i \mathfrak{a}_i)$ be a triple, and let e be a positive integer such that $(p^e - 1)(K_X + \Delta)$ is Cartier. We define*

$$\widetilde{\mathrm{stab}}(R, \Delta, \mathfrak{a}_\bullet; e) := \sup_{t_1, \ldots, t_m} \{\mathrm{stab}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}; e)\},$$

where every t_i runs through all positive rational numbers such that $(p^e - 1)t_i \in \mathbb{N}$. Then we have $\widetilde{\mathrm{stab}}(R, \Delta, \mathfrak{a}_\bullet; e) < \infty$. Moreover, for every integer $l \geqslant 0$ and rational numbers $t_1, \ldots, t_m > 0$ such that $p^{el}(p^e - 1)t_i \in \mathbb{N}$, we have

$$\mathrm{stab}(R, \Delta, \mathfrak{a}_\bullet^{t_\bullet}; e) \leqslant \widetilde{\mathrm{stab}}(R, \Delta, \mathfrak{a}_\bullet; e) + l.$$

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

Proof. By Proposition 3.7(i), we have

$$\widetilde{\operatorname{stab}}(R,\Delta,\mathfrak{a}_\bullet;e)=\sup_{t_1,\ldots,t_m}\{\operatorname{stab}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet};e)\},$$

where every t_i runs through all positive rational numbers such that $(p^e-1)t_i\in\mathbb{N}$ and $t_i\leqslant\mu_R(\mathfrak{a}_i)$. Hence we have $\widetilde{\operatorname{stab}}(R,\Delta,\mathfrak{a}_\bullet;e)<\infty$.

The second statement follows from Proposition 3.7(ii). □

Example 3.10. Set $R:=\mathbb{F}_3[[x,y]]$, $\Delta:=0$, $\mathfrak{a}:=(x,y^2)\subseteq R$. Then, for every integer $n\geqslant 0$ and every number $t>0$, it follows from [BSTZ10, Proposition 3.10] and [BMS08, Proposition 2.5] that

$$\tau_+^n(R,\mathfrak{a}^t)=(\mathfrak{a}^{\lceil 3^n t\rceil})^{[1/3^n]}=\left(x^{\lfloor a/3^n\rfloor}y^{\lfloor 2b/3^n\rfloor}\mid a,b\in\mathbb{N}_{\geqslant 0},a+b=\lceil 3^n t\rceil\right).$$

Therefore, we have

$$\tau_+^1(R,\mathfrak{a}^{3/2})=\mathfrak{a},\quad \tau_+^2(R,\mathfrak{a}^{3/2})=\mathfrak{m},\quad\text{and}\quad \tau_+^3(R,\mathfrak{a}^{3/2})=\mathfrak{m},$$

which implies $\operatorname{stab}(R,\mathfrak{a}^{3/2})=2$ by Proposition 3.8. Similarly, we have

$$\operatorname{stab}(R,\mathfrak{a}^{1/2})=1,\quad \operatorname{stab}(R,\mathfrak{a}^{1})=1,\quad\text{and}\quad \operatorname{stab}(R,\mathfrak{a}^{2})=1,$$

which shows $\widetilde{\operatorname{stab}}(R,\mathfrak{a};1)=2$.

We next consider the sequence of ideals $\{\tau_-^{en}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet})\}_n$. In general, $\{\tau_-^{en}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet})\}_n$ may not be a descending chain. In order to make a descending chain, we mix the definitions of τ_+ and τ_-, and define the new variants of test ideals as below. In fact, we later see that we can make a descending chain by using these ideals under some mild assumptions (Proposition 3.14).

Definition 3.11. Let $(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}=\prod_i\mathfrak{a}_i^{t_i})$ and e be as in Definition 3.3, $\mathfrak{q}\subseteq R$ be an ideal, and $n,u\geqslant 0$ be integers. We define

$$\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}):=\varphi_\Delta^{e(n+u)}(F_*^{e(n+u)}(\mathfrak{a}_1^{p^{eu}\lceil t_1p^{en}-1\rceil}\cdots\mathfrak{a}_m^{p^{eu}\lceil t_mp^{en}-1\rceil}\cdot\mathfrak{q})).$$

When $\mathfrak{q}=\tau(R,\Delta)$, we denote it by $\tau_e^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet})$.

Proposition 3.12 (Basic properties). *Let $(X=\operatorname{Spec}R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}=\prod_{i=1}^m\mathfrak{a}_i^{t_i})$ be a triple such that $t_i>0$ for every i and $(q-1)(K_X+\Delta)$ is Cartier for some $q=p^e$, let $\mathfrak{q}\subseteq R$ be an ideal, and let $n,u\geqslant 0$ be integers. Then the following hold.*

(i) *For real numbers $0<s_i\leqslant t_i$, we have $\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{s_\bullet})\supseteq\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet})$. Moreover, if $\langle t_i\rangle_{n,q}<s_i\leqslant t_i$ for every i, then we have $\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{s_\bullet})=\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet})$.*

(ii) *For ideals $\mathfrak{b}_i\subseteq\mathfrak{a}_i$ and $\mathfrak{q}'\subseteq\mathfrak{q}$, we have $\tau_{e,\mathfrak{q}'}^{n,u}(R,\Delta,\mathfrak{b}_\bullet^{t_\bullet})\subseteq\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet})$.*

(iii) *If $\mathfrak{a}_1\equiv\mathfrak{b}_1 \bmod J$ for some ideal J and $\mathfrak{a}_i=\mathfrak{b}_i$ for every $i\geqslant 2$, then we have*

$$\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet})\equiv\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{b}_\bullet^{t_\bullet}) \bmod \tau_{e,J\cdot\mathfrak{q}}^{n,u}\left(R,\Delta,\prod_{i=2}^m\mathfrak{a}_i^{t_i}\right).$$

If $\mathfrak{q}\equiv\mathfrak{q}' \bmod J$ for some ideals $\mathfrak{q}'$ and J, then we have

$$\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet})\equiv\tau_{e,\mathfrak{q}'}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}) \bmod \tau_{e,J}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}).$$

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

(iv) *If* $\mathfrak{q} = \mathfrak{a}_{m+1}^{q^u\lceil t_{m+1}q^n-1\rceil}\tau(R,\Delta)$, *then we have* $\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}) = \tau_e^{n,u}(R,\Delta,\prod_{i=1}^{m+1}\mathfrak{a}_i^{t_i})$.

(v) *If* $t_1 > 1$, *then we have* $\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}) \supseteq \mathfrak{a}_1 \cdot \tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_1^{t_1-1}\cdots\mathfrak{a}_m^{t_m})$. *Moreover, if* $t_1 > \mu_R(\mathfrak{a}_1) + (1/q^n)$, *then we have*

$$\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}) = \mathfrak{a}_1 \cdot \tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_1^{t_1-1}\cdots\mathfrak{a}_m^{t_m}).$$

(vi) *We have* $\varphi_\Delta^e(F_*^e(\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{p^e\cdot t_\bullet}))) = \tau_{e,\mathfrak{q}}^{n+1,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet})$.

(vii) *The sequence* $\{\tau_e^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet})\}_{u\in\mathbb{N}}$ *is an ascending chain of ideals which converges to the ideal* $\tau(R,\Delta,\prod_i \mathfrak{a}_i^{\langle t_i\rangle_{n,q}})$.

(viii) *If* $u \geqslant \widetilde{\mathrm{stab}}(R,\Delta,\mathfrak{a}_\bullet;e)$, *then we have*

$$\tau_e^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}) = \tau\left(R,\Delta,\prod_i \mathfrak{a}_i^{\langle t_i\rangle_{n,q}}\right)$$

for every n.

(ix) *Assume that* $q^{u-1} \geqslant \mu_R(\mathfrak{a}_i)$ *and the* n*th digit* $t_i^{(n)}$ *of* t_i *in base* q *is non-zero for every* i. *Then we have* $\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}) = \tau_{e,\mathfrak{q}'}^{n-1,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet})$, *where* $\mathfrak{q}' := \varphi_\Delta^e(F_*^e(\prod_i \mathfrak{a}_i^{q^u\cdot t_i^{(n)}}\mathfrak{q}))$.

Proof. The assertions in (i), (ii), (iii), (iv) and (viii) follow easily from the definitions. The proves of (v), (vi) and (vii) are similar to those of Proposition 3.5. For (ix), set $a_i := q^u\lceil t_iq^{n-1}-1\rceil$ for every i. Then for every i, it follows from Lemma 2.13 by setting $\mathfrak{c} = \mathfrak{c}_1 := \mathfrak{a}_i$ that we have

$$\mathfrak{a}_i^{q^u\lceil t_iq^n-1\rceil} = \mathfrak{a}_i^{q^u(q\lceil t_iq^{n-1}-1\rceil + t_i^{(n)})} = (\mathfrak{a}_i^{a_i})^{[q]}\cdot\mathfrak{a}_i^{q^u\cdot t_i^{(n)}}.$$

Therefore, we have

$$\begin{aligned}
\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}) &= \varphi_\Delta^{e(n+u)}\left(F_*^{e(n+u)}\left(\prod_i \mathfrak{a}_i^{q^u\lceil t_iq^n-1\rceil}\right)\cdot\mathfrak{q}\right)\\
&= \varphi_\Delta^{e(n+u-1)}\left(F_*^{e(n+u-1)}\varphi_\Delta^e\left(F_*^e\left(\prod_i (\mathfrak{a}_i^{a_i})^{[q]}\cdot\mathfrak{a}_i^{q^u\cdot t_i^{(n)}}\right)\cdot\mathfrak{q}\right)\right)\\
&= \varphi_\Delta^{e(n+u-1)}\left(F_*^{e(n+u-1)}\left(\prod_i \mathfrak{a}_i^{a_i}\right)\cdot\left(\varphi_\Delta^e\left(F_*^e\left(\prod_i \mathfrak{a}_i^{q^u\cdot t_i^{(n)}}\right)\cdot\mathfrak{q}\right)\right)\right)\\
&= \varphi_\Delta^{e(n+u-1)}\left(F_*^{e(n+u-1)}\left(\prod_i \mathfrak{a}_i^{a_i}\right)\cdot\mathfrak{q}'\right)\\
&= \tau_{e,\mathfrak{q}'}^{n-1,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}).
\end{aligned}$$

□

We consider variants of Proposition 3.12(v) and (ix).

PROPOSITION 3.13. *Let* $(X = \operatorname{Spec} R,\Delta,\mathfrak{a}_\bullet^{t_\bullet})$ *be a triple such that* $t_i > 0$ *for every* i *and* $(q-1)(K_X+\Delta)$ *is Cartier for some* $q = p^e$, *let* $\mathfrak{q} \subseteq R$ *be an ideal, and let* $n, u \geqslant 0$ *be integers. Assume that for every* j, *there exist ideals* $\mathfrak{b}_{j,1},\ldots,\mathfrak{b}_{j,m_j} \subseteq R$ *and integers* $M_{j,1},\ldots,M_{j,m_j} > 0$ *such that* $\mathfrak{a}_j = \sum_{i=1}^{m_j}\mathfrak{b}_{j,i}^{M_{j,i}}$. *Set* $l_j := \sum_i \mu_R(\mathfrak{b}_{j,i})$. *Then, the following hold.*

(i) *If* $t_1 > l_1 + (1/q^n)$, *then we have*

$$\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_\bullet^{t_\bullet}) = \mathfrak{a}_1\cdot\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}_1^{t_1-1}\cdots\mathfrak{a}_m^{t_m}).$$

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

(ii) *Assume that* $q^{u-1} \geqslant l_j$ *and the* n*th digit* ${t_j}^{(n)}$ *of* t_j *in base* q *is non-zero for every* j. *Then we have* $\tau_{e,\mathfrak{q}}^{n,u}(R, \Delta, \mathfrak{a}_{\bullet}^{t_\bullet}) = \tau_{e,\mathfrak{q}'}^{n-1,u}(R, \Delta, \mathfrak{a}_{\bullet}^{t_\bullet})$, *where* $\mathfrak{q}' := \varphi_\Delta^e(F_*^e(\prod_i \mathfrak{a}_i^{q^u \cdot {t_i}^{(n)}} \mathfrak{q}))$.

Proof. The assertion in (i) follows as in the proof of Proposition 3.12(v) by applying Lemma 2.13 for $\mathfrak{c} := \mathfrak{a}_1$ and $\mathfrak{c}_i := \mathfrak{b}_{1,i}$. For (ii), as in the proof of Proposition 3.12(ix), it follows from Lemma 2.13 by setting $\mathfrak{c} := \mathfrak{a}_j$ and $\mathfrak{c}_i := \mathfrak{b}_{j,i}$ that we have

$$\mathfrak{a}_j^{q^u \lceil t_j q^n - 1 \rceil} = \mathfrak{a}_j^{q^u (q \lceil t_j q^{n-1} - 1 \rceil + t_j^{(n)})} = (\mathfrak{a}_j^{a_j})^{[q]} \cdot \mathfrak{a}_j^{q^u \cdot t_j^{(n)}},$$

for every j, which proves (ii). □

PROPOSITION 3.14. *With the above notation, we further assume that* $u > 0$, $q^{u-1} \geqslant \max_i l_i$ *and* $q(q-1)t_i \in \mathbb{N}$ *for every* i. *Then the sequence* $\{\tau_e^{n,u}(R, \Delta, \mathfrak{a}_{\bullet}^{t_\bullet})\}_{n \geqslant 1}$ *is a descending chain of ideals.*

Proof. Since $q(q-1)t_i \in \mathbb{N}$, the nth digit $t_i^{(n)}$ of t_i in base q is constant for $n \geqslant 2$. By Lemma 3.2(ii), it is non-zero. Therefore, the assertion follows from Propositions 3.12(ii) and 3.13(ii). □

DEFINITION 3.15. Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple with $t > 0$, let I be an $\mathfrak{m}$-primary ideal, let $\mathfrak{b} \subseteq R$ be a proper ideal, and let e be a positive integer such that $(p^e - 1)(K_X + \Delta)$ is Cartier. Then we define

$$\operatorname{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{b}) := \inf\{s > 0 \mid \tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{b}^s) \subseteq I\} \in \mathbb{R}_{\geqslant 0}.$$

PROPOSITION 3.16. *With the above notation, the following hold:*

(i) $0 \leqslant \operatorname{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{b}) \leqslant \ell\ell_R(R/I) + \mu_R(\mathfrak{b})$;

(ii) $p^{en} \cdot \operatorname{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{b}) \in \mathbb{Z}$.

Proof. By Proposition 3.12(v), we have

$$\begin{aligned}\tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{b}^{\ell\ell_R(R/I) + \mu_R(\mathfrak{b})}) &= \mathfrak{b}^{\ell\ell_R(R/I)} \cdot \tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{b}^{\mu_R(\mathfrak{b})}) \\ &\subseteq \mathfrak{b}^{\ell\ell_R(R/I)} \subseteq I,\end{aligned}$$

which proves the assertion in (i).

For (ii), set $\Sigma := \{s \in \mathbb{R}_{\geqslant 0} \mid \tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{b}^s) \subseteq I\} \subseteq \mathbb{R}_{\geqslant 0}$. It follows from Proposition 3.12(i) that if $s \in \Sigma$ and $\langle s \rangle_{n,q} < s'$, then $s' \in \Sigma$. Therefore

$$\operatorname{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{b}) = \inf \Sigma = \min_{s \in \Sigma} \langle s \rangle_{n,q},$$

which proves (ii). □

PROPOSITION 3.17. *Let* $(X = \operatorname{Spec} R, \Delta)$ *be a pair such that* $(p^e - 1)(K_X + \Delta)$ *is Cartier for some positive integer* e, *let* $t > 0$ *be a rational number, and let* $M, \mu > 0$ *and* $u \geqslant 2$ *be positive integers. Assume that:*

(i) $q > \mu + \operatorname{emb}(R)$; *and*

(ii) $q^m(q-1)t \in \mathbb{N}$ *for some integer* m.

Then, there exists a positive integer n_1 *with the following property. For every ideal* $\mathfrak{a}$ *with* $\mu_R(\mathfrak{a}) \leqslant \mu$, *if we set* $\mathfrak{b} := \mathfrak{a} + \mathfrak{m}^M$, *then we have* $\tau_e^{n,u}(R, \Delta, \mathfrak{b}^t) = \tau_e^{n_1,u}(R, \Delta, \mathfrak{b}^t)$ *for every* $n \geqslant n_1$.

Proof. By Proposition 3.12(vi), it is enough to show the assertion in the case when $t > \mu+\mathrm{emb}(R)$ and $(p^e-1)t \in \mathbb{N}$. Set $n_1 := \ell_R(\tau(R,\Delta)/(\mathfrak{m}^{M\lceil t\rceil}\cdot\tau(R,\Delta)))$. We will prove that the assertion holds for this constant n_1.

Let $\mathfrak{a} \subseteq R$ be an ideal such that $\mu_R(\mathfrak{a}) \leqslant \mu$, and set $\mathfrak{b} := \mathfrak{a} + \mathfrak{m}^M$. We consider the sequence of ideals $\{\tau_e^{n,u}(R,\Delta,\mathfrak{b}^t)\}_{n\geqslant 1}$. By Proposition 3.14, the sequence $\{\tau_e^{n,u}(R,\Delta,\mathfrak{b}^t)\}_n$ is a descending chain. Moreover, since $\mathfrak{b} \supseteq \mathfrak{m}^M$, we have

$$\begin{aligned}\tau_e^{n,u}(R,\Delta,\mathfrak{b}^t) &\supseteq \tau_e^{n,u}(R,\Delta,(\mathfrak{m}^M)^t)\\ &\supseteq \tau_e^{n,u}(R,\Delta,(\mathfrak{m}^M)^t)\\ &\supseteq \tau_e^{n,0}(R,\Delta,(\mathfrak{m}^M)^t)\\ &\supseteq \mathfrak{m}^{M\lceil t\rceil}\cdot\tau(R,\Delta).\end{aligned}$$

Since we have

$$\tau(R,\Delta) \supseteq \tau_e^{1,u}(R,\Delta,\mathfrak{b}^t) \supseteq \tau_e^{2,u}(R,\Delta,\mathfrak{b}^t) \supseteq \cdots \supseteq \mathfrak{m}^{M\lceil t\rceil}\cdot\tau(R,\Delta),$$

there exists an integer $1 \leqslant m \leqslant n_1$ such that

$$\tau_e^{m,u}(R,\Delta,\mathfrak{b}^t) = \tau_e^{m+1,u}(R,\Delta,\mathfrak{b}^t).$$

On the other hand, by Proposition 3.13(i), we have

$$\tau_e^{m+1,u}(R,\Delta,\mathfrak{b}^{t'+1}) = \mathfrak{b}\cdot\tau_e^{m,u}(R,\Delta,\mathfrak{b}^{t'})$$

for any real number $t' > \mu + \mathrm{emb}(R)$. Then, as in the proof of Proposition 3.8, we have $\tau_e^{m+1,u}(R,\Delta,\mathfrak{a}^t) = \tau_e^{m+2,u}(R,\Delta,\mathfrak{a}^t)$, which completes the proof. ◻

4. Rationality of the limit of F-pure thresholds

In this section, we give uniform bounds for the denominators of F-jumping numbers (Proposition 4.1) and for the stabilization exponents (Proposition 4.3) of $\mathfrak{m}$-primary ideals with fixed colength. By using these bounds, we will prove Theorem 1.5.

Proposition 4.1. *Let $(X = \operatorname{Spec} R, \Delta)$ be a pair such that $(p^e-1)(K_X+\Delta)$ is Cartier for some integer $e > 0$, and let $M > 0$ be an integer. Then there exists an integer $N > 0$ such that for any ideal $\mathfrak{a} \subseteq R$, if $\mathfrak{a} \supseteq \mathfrak{m}^M$, then any F-jumping number of $(R,\Delta;\mathfrak{a})$ is contained in $(1/N)\cdot\mathbb{Z}$.*

Proof. Set $l := \ell_R(R/\mathfrak{m}^M) + \mu_R(\mathfrak{m}^M)$ and $n := \ell_R(\tau(R,\Delta)/\tau(R,\Delta,\mathfrak{m}^{Ml}))$. We note that the module $\tau(R,\Delta)/\tau(R,\Delta,\mathfrak{m}^{Ml})$ has finite length because the test ideals commute with localization [HT04, Proposition 3.1]. Let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{m}^M \subseteq \mathfrak{a}$ and let $B \subseteq \mathbb{R}_{>0}$ be the set of all F-jumping numbers of $(R,\Delta;\mathfrak{a})$.

Since we have $\mu(\mathfrak{a}) \leqslant l$, it follows from [BSTZ10, Corollary 3.27] that for every element $b \in B \cap \mathbb{R}_{>l}$, we have $b-1 \in B$. It also follows from [BSTZ10, Lemma 3.25] that for every element $b \in B$, we have $p^e b \in B$. Moreover, since $\tau(R,\Delta) \supseteq \tau(R,\Delta,\mathfrak{a}^t) \supseteq \tau(R,\Delta,\mathfrak{m}^{Ml})$ for every $t \leqslant l$, the number of the set $B \cap [0,l]$ is at most n. Then the assertion follows from the lemma below. ◻

Lemma 4.2. *Let $l, n > 0$ and $q \geqslant 2$ be integers. Then there exists an integer $N > 0$ with the following property: if $B \subseteq \mathbb{R}_{\geqslant 0}$ is a subset such that:*

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

(i) *for every element* $b \in B$, *if* $b > l$, *then* $b - 1 \in B$;
(ii) *if* $b \in B$, *then* $q \cdot b \in B$; *and*
(iii) *the number of the set* $B \cap [0, l]$ *is at most* n,

then we have $B \subseteq (1/N) \cdot \mathbb{Z}$.

Proof. The proof is essentially the same as that of [BMS08, Proposition 3.8]. Set $N := q^n(q^{n!}-1)$, where $n!$ is the factorial of n.

For every element $b \in B$ and every integer $m \geqslant 0$, we define $b_m \in B \cap [0, l]$ by

$$b_m := (q^m b - \lfloor q^m b \rfloor) + \min\{l - 1, \lfloor q^m b \rfloor\}.$$

If $b \notin (1/N) \cdot \mathbb{Z}$, then $b_0, b_1, \dots, b_n$ are all distinct and hence contradiction. □

Proposition 4.3. *Let* $(X = \operatorname{Spec} R, \Delta)$ *be a pair such that* $(p^e - 1)(K_X + \Delta)$ *is Cartier for some integer* $e > 0$, *and let* $M > 0$ *be an integer. Then there exists* $u_0 > 0$ *such that for every ideal* $\mathfrak{a} \supseteq \mathfrak{m}^M$, *we have*

$$\widetilde{\operatorname{stab}}(R, \Delta, \mathfrak{a}; e) \leqslant u_0.$$

Proof. Set $l := \ell_R(R/\mathfrak{m}^M) + \mu_R(\mathfrak{m}^M)$ and take an integer $n_0 > 0$ such that $p^{e(n_0-1)} > l$. Let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \supseteq \mathfrak{m}^M$, and let $t > 0$ be a rational number such that $(p^e - 1)t \in \mathbb{N}$.

We first consider the case when $l < t \leqslant lp^{en_0}$. In this case, by Proposition 3.5(i), the sequence $\{\tau_+^{en}(R, \Delta, \mathfrak{a}^t)\}_{n \geqslant 0}$ is an ascending chain such that

$$\tau(R, \Delta) \supseteq \tau_+^{en}(R, \Delta, \mathfrak{a}^t) \supseteq \tau_+^{0}(R, \Delta, \mathfrak{a}^t) = \mathfrak{a}^{\lceil t \rceil} \cdot \tau(R, \Delta) \supseteq \mathfrak{m}^{lMp^{en_0}} \cdot \tau(R, \Delta)$$

for every n. Therefore, there exists an integer $0 \leqslant n < \ell_R(\tau(R, \Delta)/(\mathfrak{m}^{lMp^{en_0}} \cdot \tau(R, \Delta)))$ such that

$$\tau_+^{en}(R, \Delta, \mathfrak{a}^t) = \tau_+^{e(n+1)}(R, \Delta, \mathfrak{a}^t).$$

By Proposition 3.8, we have

$$\operatorname{stab}(R, \Delta, \mathfrak{a}^t; e) \leqslant n \leqslant \ell_R(\tau(R, \Delta)/(\mathfrak{m}^{lMp^{en_0}} \cdot \tau(R, \Delta))).$$

We next consider the case when $t \leqslant l$. Since $l < tp^{en_0} \leqslant lp^{en_0}$, it follows from Proposition 3.7(ii) that

$$\begin{aligned} \operatorname{stab}(R, \Delta, \mathfrak{a}^t; e) &\leqslant \operatorname{stab}(R, \Delta, \mathfrak{a}^{tp^{en_0}}; e) + n_0 \\ &\leqslant \ell_R(\tau(R, \Delta)/(\mathfrak{m}^{lMp^{en_0}} \cdot \tau(R, \Delta))) + n_0. \end{aligned}$$

Therefore, $u_0 := \ell_R(\tau(R, \Delta)/(\mathfrak{m}^{lMp^{en_0}} \cdot \tau(R, \Delta))) + n_0$ satisfies the property. □

Proposition 4.4. *Let* $(X = \operatorname{Spec} R, \Delta)$ *be a pair such that* $(p^e - 1)(K_X + \Delta)$ *is Cartier for some integer* $e > 0$, *let* $\{\mathfrak{a}_m\}_{m \in \mathbb{N}}$ *be a family of ideals of* R, *and let* $t > 0$ *be a real number. Fix a non-principal ultrafilter* $\mathfrak{U}$. *Let* $(R_\#, \mathfrak{m}_\#)$ *be the catapower of the local ring* $(R, \mathfrak{m})$, *let* $\Delta_\#$ *be the flat pullback of* Δ *to* $\operatorname{Spec} R_\#$, *and* $\mathfrak{a}_\infty := [\mathfrak{a}_m]_m \subseteq R_\#$. *If there exists a positive integer* M *such that* $\mathfrak{a}_m \supseteq \mathfrak{m}^M$ *for every* m, *then we have*

$$\tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^t) = [\tau(R, \Delta, \mathfrak{a}_m^t)]_m \subseteq R_\#.$$

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

Proof. We first consider the case when t is a rational number. By enlarging e, we may assume that $p^{en}(p^e-1)t \in \mathbb{Z}$ for some integer $n \geqslant 0$. Take a positive integer u as in Proposition 4.3. Then we have

$$\tau(R,\Delta,\mathfrak{a}_m^t) = \tau_+^{e(n+u)}(R,\Delta,\mathfrak{a}_m^t),$$

for every m. By enlarging u, we may assume that

$$\tau(R_\#,\Delta_\#,\mathfrak{a}_\infty^t) = \tau_+^{e(n+u)}(R_\#,\Delta_\#,\mathfrak{a}_\infty^t).$$

Since $\mu_R(\mathfrak{a}_m) \leqslant \ell_R(R/\mathfrak{m}^M) + \mu_R(\mathfrak{m}^M)$ for every m, it follows from Lemma 2.17 that

$$(\mathfrak{a}_\infty)^s = [(\mathfrak{a}_m)^s]_m$$

for every integer $s > 0$. Combining with Propositions 2.10(v) and 2.16(ii), we have

$$\begin{aligned}
\tau_+^{el}(R_\#,\Delta_\#,\mathfrak{a}_\infty^t) &= \varphi_{\Delta_\#}^{el}(F_*^{el}(\mathfrak{a}_\infty^{\lceil tp^{el}\rceil}\cdot\tau(R_\#,\Delta_\#)))\\
&= \varphi_{\Delta_\#}^{el}(F_*^{el}[\mathfrak{a}_m^{\lceil tp^{el}\rceil}\cdot\tau(R,\Delta)]_m)\\
&= [\varphi_{\Delta}^{el}(F_*^{el}(\mathfrak{a}_m^{\lceil tp^{el}\rceil}\cdot\tau(R,\Delta)))]_m\\
&= [\tau_+^{el}(R,\Delta,\mathfrak{a}_m^t)]_m \subseteq R_\#
\end{aligned}$$

for every integer l. Therefore, we have

$$\tau(R_\#,\Delta_\#,\mathfrak{a}_\infty^t) = [\tau(R,\Delta,\mathfrak{a}_m^t)]_m \subseteq R_\#.$$

We next consider the case when t is not a rational number. For sufficiently large integer n, we have

$$\begin{aligned}
\tau(R_\#,\Delta_\#,\mathfrak{a}_\infty^t) &= \tau_+^{en}(R_\#,\Delta_\#,\mathfrak{a}_\infty^t)\\
&= [\tau_+^{en}(R,\Delta,\mathfrak{a}_m^t)]_m\\
&\subseteq [\tau(R,\Delta,\mathfrak{a}_m^t)]_m \subseteq R_\#.
\end{aligned}$$

For the converse inclusion, by Proposition 2.6, we can take a rational number t' such that $t' < t$ and $\tau(R_\#,\Delta_\#,\mathfrak{a}_\infty^t) = \tau(R_\#,\Delta_\#,\mathfrak{a}_\infty^{t'})$. Then, we have

$$\begin{aligned}
\tau(R_\#,\Delta_\#,\mathfrak{a}_\infty^t) &= \tau(R_\#,\Delta_\#,\mathfrak{a}_\infty^{t'})\\
&= [\tau(R,\Delta,\mathfrak{a}_m^{t'})]_m\\
&\supseteq [\tau(R,\Delta,\mathfrak{a}_m^t)]_m,
\end{aligned}$$

which completes the proof. □

PROPOSITION 4.5. *With the notation above, let $I \subsetneq R$ be an $\mathfrak{m}$-primary ideal. Assume that $\mathfrak{m}^M \subseteq \mathfrak{a}_m \subseteq \mathfrak{m}$ for every m. Then there exists $T \in \mathfrak{U}$ such that for all $m \in T$, we have*

$$\mathrm{fjn}^I(R,\Delta;\mathfrak{a}_m) = \mathrm{fjn}^{I\cdot R_\#}(R_\#,\Delta_\#,\mathfrak{a}_\infty).$$

Proof. Set $t := \mathrm{fjn}^{I\cdot R_\#}(R_\#,\Delta_\#;\mathfrak{a}_\infty) \in \mathbb{R}_{\geqslant 0}$. If $\tau(R,\Delta) \subseteq I$, then we have $\mathrm{fjn}^I(R,\Delta;\mathfrak{a}_m) = 0$ for every $m \in \mathbb{N}$ and $\mathrm{fjn}^{I\cdot R_\#}(R_\#,\Delta_\#,\mathfrak{a}_\infty) = 0$. Therefore, we may assume that $\tau(R,\Delta) \not\subseteq I$. Since $\mathfrak{a}_\infty \neq (0)$, it follows from Lemma 2.4(ii) that $t > 0$.

It follows from Proposition 4.4 that we have

$$[\tau(R,\Delta,\mathfrak{a}_m^t)]_m = \tau(R_\#,\Delta_\#,\mathfrak{a}_\infty^t) \subseteq I\cdot R_\#.$$

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

Since I is $\mathfrak{m}$-primary, it follows from Lemma 2.21 that there exists $S_1 \in \mathfrak{U}$ such that $\tau(R, \Delta, \mathfrak{a}_m^t) \subseteq I$ for every $m \in S_1$. Therefore $\mathrm{fjn}^I(R, \Delta; \mathfrak{a}_m) \leqslant \mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#, \mathfrak{a}_\infty)$ for every $m \in S_1$.

On the other hand, by Proposition 4.1, there exists $0 < t' < t$ such that for every ideal $\mathfrak{b} \supseteq \mathfrak{m}^M$, if $t' < \mathrm{fjn}^I(R, \Delta; \mathfrak{b})$, then $t \leqslant \mathrm{fjn}^I(R, \Delta; \mathfrak{b})$. Since $t' < t$, we have

$$[\tau(R, \Delta, \mathfrak{a}_m^{t'})]_m = \tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^{t'}) \not\subseteq I \cdot R_\#.$$

Hence, we have

$$\mathrm{ulim}_m\, \tau(R, \Delta, \mathfrak{a}_m^{t'}) \not\subseteq {}^*I.$$

Therefore, there exists $S_2 \in \mathfrak{U}$ such that $\tau(R, \Delta, \mathfrak{a}_m^{t'}) \not\subseteq I$ for every $m \in S_2$. Then $T := S_1 \cap S_2$ satisfies the assertion. $\square$

Lemma 4.6 (Cf. [BMS09, Lemma 3.3]). *Let $(X = \operatorname{Spec} R, \Delta)$ be a pair such that $K_X + \Delta$ is $\mathbb{Q}$-Carter, let I be an $\mathfrak{m}$-primary ideal, and let $\mathfrak{a}, \mathfrak{b} \subseteq R$ be proper ideals. Then we have*

$$\mathrm{fjn}^I(R, \Delta; \mathfrak{a} + \mathfrak{b}) \leqslant \mathrm{fjn}^I(R, \Delta; \mathfrak{a}) + \mathrm{fjn}^I(R, \Delta; \mathfrak{b}).$$

Proof. As in the proof of [Tak06, Theorem 3.1], for every real number $c \geqslant 0$, we can show that

$$\tau(R, \Delta, (\mathfrak{a} + \mathfrak{b})^c) = \sum_{u, v \geqslant 0, u+v=c} \tau(R, \Delta, \mathfrak{a}^u \mathfrak{b}^v).$$

Set $t := \mathrm{fjn}^I(R, \Delta; \mathfrak{a})$ and $s := \mathrm{fjn}^I(R, \Delta; \mathfrak{b})$. Then we have

$$\tau(R, \Delta, (\mathfrak{a} + \mathfrak{b})^{t+s}) = \sum_{u, v \geqslant 0, u+v=s+t} \tau(R, \Delta, \mathfrak{a}^u \mathfrak{b}^v) \subseteq \tau(R, \Delta, \mathfrak{a}^t) + \tau(R, \Delta, \mathfrak{b}^s) \subseteq I. \quad \square$$

Theorem 4.7 (Theorem 1.5, cf. [BMS09, Theorem 1.2]). *Let $(X = \operatorname{Spec} R, \Delta)$ be a pair such that $(p^e - 1)(K_X + \Delta)$ is Cartier for some integer $e > 0$, let $(R_\#, \mathfrak{m}_\#)$ be the catapower of $(R, \mathfrak{m})$, and let $\Delta_\#$ be the flat pullback of Δ to $\operatorname{Spec} R_\#$, $I \subseteq R$ be an $\mathfrak{m}$-primary ideal, $\{\mathfrak{a}_m\}_{m \in \mathbb{N}}$ be a family of proper ideals and $\mathfrak{a}_\infty := [\mathfrak{a}_m]_m \subseteq R_\#$. Then we have*

$$\mathrm{sh}(\mathrm{ulim}_m\, \mathrm{fjn}^I(R, \Delta; \mathfrak{a}_m)) = \mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#, \mathfrak{a}_\infty) \in \mathbb{Q}.$$

In particular, if the limit $\lim_{m \longrightarrow \infty} \mathrm{fjn}^I(R, \Delta; \mathfrak{a}_m)$ exists, then we have

$$\lim_{m \longrightarrow \infty} \mathrm{fjn}^I(R, \Delta; \mathfrak{a}_m) = \mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#, \mathfrak{a}_\infty).$$

Proof. The proof is essentially the same as the proof of [BMS09, Theorem 1.2]. If $\tau(R, \Delta) \subseteq I$, then the assertion in the theorem is trivial. Therefore, we may assume that $\tau(R, \Delta) \not\subseteq I$.

For every integer $M > 0$, we set $\mathfrak{b}_{\infty, M} := \mathfrak{a}_\infty + (\mathfrak{m}_\#)^M$ and $\mathfrak{b}_{m, M} := \mathfrak{a}_m + \mathfrak{m}^M$ for every integer m. We write $s := \mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#; \mathfrak{m}_\#)$.

By Lemma 4.6, we have

$$|\mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#; \mathfrak{a}_\infty) - \mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#; \mathfrak{b}_{\infty, M})| \leqslant s/M \tag{1}$$

for every M.

By Proposition 2.10(iv), we have $s = \mathrm{fjn}^I(R, \Delta; \mathfrak{m})$. Therefore, it follows from Lemma 4.6 that

$$|\mathrm{fjn}^I(R, \Delta; \mathfrak{a}_m) - \mathrm{fjn}^I(R, \Delta; \mathfrak{b}_{m, M})| \leqslant s/M \tag{2}$$

for every m and M.

On the other hand, since $\mathfrak{b}_{\infty,M} = [\mathfrak{b}_{m,M}]_m$, it follows from Proposition 4.5 that there exists $T_M \in \mathfrak{U}$ such that

$$\mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#; \mathfrak{b}_{\infty,M}) = \mathrm{fjn}^{I}(R, \Delta; \mathfrak{b}_{m,M}) \tag{3}$$

for every $m \in T_M$.

By combining the equations (1)–(3), we have

$$|\mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#; \mathfrak{a}_\infty) - \mathrm{fjn}^{I}(R, \Delta; \mathfrak{a}_m)| \leqslant 2s/M$$

for every $m \in T_M$.

It follows from the definition of the shadow that

$$\mathrm{sh}(\mathrm{ulim}_m \, \mathrm{fjn}^{I}(R, \Delta; \mathfrak{a}_m)) = \mathrm{fjn}^{I \cdot R_\#}(R_\#, \Delta_\#; \mathfrak{a}_\infty),$$

which completes the proof. $\square$

5. Proof of main theorem

In this section, we introduce Condition $(\star)$ (Definition 5.2) which plays the key role in the proof of the main theorem and we prove some properties of Condition $(\star)$ (Propositions 5.4 and 5.6). By combining them with Proposition 3.17 and Theorem 4.7, we give the proof of the main theorem (Theorem 5.9).

Observation 5.1. Let X be a normal variety over a field k of characteristic zero, Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, $\mathfrak{a} \subseteq \mathcal{O}_X$ be a non-zero coherent ideal sheaf, $t \geqslant 0$ be a rational number, $x \in X$ be a closed point and $\mathfrak{m}_x \subseteq \mathcal{O}_X$ be the maximal ideal at x. We consider the *log canonical threshold*

$$\mathrm{lct}_x(X, \Delta, \mathfrak{a}^t; \mathfrak{m}) := \inf\{s \geqslant 0 \mid (X, \Delta, \mathfrak{a}^t \mathfrak{m}^s) \text{ is not log canonical at } x\}.$$

By considering a log resolution of (X, Δ), $\mathfrak{a}$ and $\mathfrak{m}$, we can show that there exist a real number $t' < t$ and rational numbers a, b such that

$$\mathrm{lct}_x(X, \Delta, \mathfrak{a}^s; \mathfrak{m}) = as + b \tag{4}$$

for every $t' < s < t$.

Choose integers $q \geqslant 2$ and $m \geqslant 0$ such that $q^m(q-1)t \in \mathbb{N}$. Then for every $n > m$, the nth digit of t in base q satisfies $t^{(n)} = l$ for some constant $l > 0$. Set $N := -al/q$. Then we have

$$\mathrm{lct}_x(X, \Delta, \mathfrak{a}^{\langle t \rangle_{n+1,q}}; \mathfrak{m}) = \mathrm{lct}_x(X, \Delta, \mathfrak{a}^{\langle t \rangle_{n,q}}; \mathfrak{m}) - N/q^n \tag{5}$$

for sufficiently large n.

Motivated by the observation above, we define the following condition.

Definition 5.2. Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple such that $t > 0$ and $(p^e - 1)(K_X + \Delta)$ is Cartier for some integer $e > 0$, let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal, and let $u, N \geqslant 0$ be integers. We say that $(R, \Delta, \mathfrak{a}^t, I, e, u, N)$ satisfies *Condition* $(\star)$ if for every $n \geqslant 0$ we have

$$\mathrm{fjn}_e^{I,n+1,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) \geqslant \mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) - N/p^{en}.$$

Remark 5.3. If $u \geqslant \widetilde{\mathrm{stab}}(R, \Delta, \mathfrak{a}, \mathfrak{m}; e)$, then we have

$$\mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) = \langle \mathrm{fjn}^I(R, \Delta, \mathfrak{a}^{\langle t \rangle_{n,q}}; \mathfrak{m}) \rangle^{n,q},$$

where we write $q := p^e$. Therefore, Condition $(\star)$ can be regarded as an analogue of the equation (5) in Observation 5.1. See also Corollary 5.5 below.

We also note that the equation (4) in Observation 5.1 may not hold for F-pure thresholds (cf. [Pér13, Example 5.3]).

We first give a sufficient condition for Condition $(\star)$.

Proposition 5.4. *Let $(X = \operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ be a triple such that $t > 0$ and $(p^e - 1)(K_X + \Delta)$ is Cartier for some $e > 0$, let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal, let $0 < l < p^e$ be a positive integer and let $n_0 \geqslant 0$ and $u \geqslant 2$ be integers. Set*

$$q = p^e, \quad N := q^{n_0+3}\mathrm{emb}(R), \quad t_0 := \frac{q^2}{q-1}, \quad \textit{and} \quad M_0 := \frac{(q^{n_0+6} - 1)\mathrm{emb}(R)}{q-1}.$$

Assume that:

(i) $q > \mu_R(\mathfrak{a})$;
(ii) $q > \ell\ell_R(R/I)$;
(iii) *the nth digit of t in base q satisfies $t^{(n)} = l$ for every $n \geqslant 2$; and*
(iv) $\tau_e^{n_0+1,u}(R, \Delta, \mathfrak{a}^{lt_0}) + \mathfrak{m}^{M_0} \cdot \tau(R, \Delta) \supseteq \tau_e^{n_0,u}(R, \Delta, \mathfrak{a}^{lt_0})$.

Then, $(R, \Delta, \mathfrak{a}^t, I, e, u, N)$ satisfies Condition $(\star)$.

Proof. By induction on $n \geqslant 0$, we will show the inequality

$$\mathrm{fjn}_e^{I,n+1,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) \geqslant \mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) - N/q^n. \tag{6}$$

Step 1. We consider the case when $n \leqslant n_0 + 2$. In this case, we have

$$N/q^n \geqslant q \cdot \mathrm{emb}(R) \geqslant \ell\ell_R(R/I) + \mathrm{emb}(R).$$

By Proposition 3.16(i), we have

$$\mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) \leqslant \ell\ell_R(R/I) + \mathrm{emb}(R).$$

Hence we have

$$\mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) - N/q^n \leqslant 0,$$

which implies the inequality (6).

Step 2. From now on, we assume $n \geqslant n_0 + 3$. Set $r := q^n \cdot \mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m})$. By Proposition 3.16(ii), we have $r \in \mathbb{Z}$. We first consider the case when

$$r \leqslant q^{n_0} \cdot \mathrm{emb}(R).$$

In this case, we have

$$\begin{aligned} \mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) - N/q^n &= \frac{r - q^{n_0+3}\mathrm{emb}(R)}{q^n} \\ &\leqslant 0, \end{aligned}$$

which shows the inequality (6). Therefore, we may assume $r > q^{n_0} \cdot \mathrm{emb}(R)$.

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

Step 3. Set $s := \lceil r/q^{n_0} \rceil - \mathrm{emb}(R) - 1$ and $s' := \lceil (s + M_0)/q^2 \rceil$. In this step, we will show the inclusion

$$\tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{r/q^n}) \subseteq \tau_e^{n+1,u}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{s/q^{n-n_0}}) + \tau_e^{n-n_0-2,2}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{s'/q^{n-n_0+2}}). \tag{7}$$

By the assumption (iii), $\alpha := tq^{n-n_0} - lt_0 = q^2 \lceil tq^{n-n_0-2} - 1 \rceil$ is an integer. It follows from Proposition 3.12(i), (v), and (vi) that

$$\begin{aligned}\tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{r/q^n}) &= \varphi_\Delta^{e(n-n_0)}(F^{e(n-n_0)}(\tau_e^{n_0,u}(R, \Delta, \mathfrak{a}^{tq^{n-n_0}} \mathfrak{m}^{r/q^{n_0}}))) \\ &\subseteq \varphi_\Delta^{e(n-n_0)}(F^{e(n-n_0)}(\mathfrak{a}^\alpha \mathfrak{m}^s \tau_e^{n_0,u}(R, \Delta, \mathfrak{a}^{lt_0}))).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\tau_e^{n+1,u}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{s/q^{n-n_0}}) &= \varphi_\Delta^{e(n-n_0)}(F^{e(n-n_0)}(\tau_e^{n_0+1,u}(R, \Delta, \mathfrak{a}^{tq^{n-n_0}} \mathfrak{m}^s))) \\ &\supseteq \varphi_\Delta^{e(n-n_0)}(F^{e(n-n_0)}(\mathfrak{a}^\alpha \mathfrak{m}^s \tau_e^{n_0+1,u}(R, \Delta, \mathfrak{a}^{lt_0}))).\end{aligned}$$

On the other hand, it follows from the definitions that

$$\begin{aligned}\tau_e^{n-n_0-2,2}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{s'/q^{n-n_0-2}}) &= \varphi_\Delta^{e(n-n_0)}(F^{e(n-n_0)}(\mathfrak{a}^\alpha \mathfrak{m}^{q^2(s'-1)} \tau(R, \Delta))) \\ &\supseteq \varphi_\Delta^{e(n-n_0)}(F^{e(n-n_0)}(\mathfrak{a}^\alpha \mathfrak{m}^{s+M_0} \tau(R, \Delta))).\end{aligned}$$

By combining them with the assumption (iv), we have the inclusion (7).

Step 4. In this step, we will show the inclusion

$$\tau_e^{n-n_0-2,2}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{s'/q^{n-n_0-2}}) \subseteq I. \tag{8}$$

It follows from the induction hypothesis that

$$\mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) \geqslant \mathrm{fjn}_e^{I,n-n_0-2,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) - \left(\sum_{i=n-n_0-2}^{n-1} \frac{N}{q^i} \right).$$

Therefore, we have the inequality

$$\begin{aligned}\frac{s'}{q^{n-n_0-2}} \geqslant \frac{s + M_0}{q^{n-n_0}} &\geqslant \frac{r/q^{n_0} - \mathrm{emb}(R) - 1 + M_0}{q^{n-n_0}} \\ &= \mathrm{fjn}_e^{I,n,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) + \frac{-\mathrm{emb}(R) - 1 + M_0}{q^{n-n_0}} \\ &\geqslant \mathrm{fjn}_e^{I,n-n_0-2,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}) - \left(\sum_{i=n-n_0-2}^{n-1} \frac{N}{q^i} \right) + \frac{-\mathrm{emb}(R) - 1 + M_0}{q^{n-n_0}} \\ &> \mathrm{fjn}_e^{I,n-n_0-2,u}(R, \Delta, \mathfrak{a}^t; \mathfrak{m}).\end{aligned}$$

Since we have $u \geqslant 2$, It follows from Proposition 3.12(vii) that

$$\tau_e^{n-n_0-2,2}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{s'/q^{n-n_0-2}}) \subseteq \tau_e^{n-n_0-2,u}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{s'/q^{n-n_0-2}}) \subseteq I.$$

Step 5. It follows from Proposition 3.12(i) that

$$\tau_e^{n,u}(R, \Delta, \mathfrak{a}^t \mathfrak{m}^{r/q^n}) \not\subseteq I.$$

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

Combining it with the inclusions (7) and (8), we have

$$\tau_e^{n+1,u}(R,\Delta,\mathfrak{a}^t\mathfrak{m}^{s/q^{n-n_0}})\not\subseteq I.$$

Hence, we have

$$\begin{aligned}\mathrm{fjn}_e^{I,n+1,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m}) &\geqslant \frac{s}{q^{n-n_0}}\\ &\geqslant \frac{r/q^{n_0}-\mathrm{emb}(R)-1}{q^{n-n_0}}\\ &= \mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m})-\frac{\mathrm{emb}(R)+1}{q^{n-n_0}}\\ &> \mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m})-\frac{N}{q^n},\end{aligned}$$

which completes the proof of the proposition. □

Corollary 5.5. *Suppose that $(X=\operatorname{Spec}R,\Delta,\mathfrak{a}^t)$ is a triple such that $t>0$ is a rational number and $(p^e-1)(K_X+\Delta)$ is Cartier for some integer $e>0$, and $I\subseteq R$ is an $\mathfrak{m}$-primary ideal. Then, there exist integers $e',u_0,N>0$ such that for every $u\geqslant u_0$, $(R,\Delta,\mathfrak{a}^t,I,e',u,N)$ satisfies Condition $(\star)$. In particular, there exists an integer $N'>0$ such that if we write $q:=p^{e'}$, then*

$$\mathrm{fjn}^I(R,\Delta,\mathfrak{a}^{\langle t\rangle_{n+1,q}};\mathfrak{m})\geqslant \mathrm{fjn}^I(R,\Delta,\mathfrak{a}^{\langle t\rangle_{n,q}};\mathfrak{m})-N'/q^n$$

for every integer $n\geqslant 0$.

Proof. Take an integer $m>0$ such that $q:=p^{em}$ satisfies the assumptions (i), (ii), and (iii) in Proposition 5.4. Set $l=t^{(2)}$ and $t_0:=q^2/(q-1)$. Then it follows from Proposition 2.6 that there exists an integer $n_0>0$ such that

$$\tau(R,\Delta,\mathfrak{a}^{\langle lt_0\rangle_{n_0,q}})=\tau(R,\Delta,\mathfrak{a}^{\langle lt_0\rangle_{(n_0+1),q}}).$$

Set $e':=em$, $u_0:=\widetilde{\mathrm{stab}}(R,\Delta,\mathfrak{a};e')$ and $N:=q^{n_0+3}\cdot\mathrm{emb}(R)$. Then the first assertion follows from Proposition 5.4.

Set $N':=N+1$. Then the second assertion follows from Remark 5.3. □

Proposition 5.6. *Suppose that $(R,\Delta,\mathfrak{a}^t)$ is a triple, $I\subseteq R$ is an $\mathfrak{m}$-primary ideal, and $q=p^e$, u and N are integers which satisfy all the conditions of Proposition 5.4. We further assume that $q>\ell\ell_R(R/I)+\mu_R(\mathfrak{a})+\mathrm{emb}(R)$. Then for every $n\geqslant 1$, we have*

$$\mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m})=\mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{b}^t;\mathfrak{m}),$$

where $\mathfrak{b}:=\mathfrak{a}+\mathfrak{m}^{q^{u+2}\cdot N}$. In particular, for every n, we have

$$\tau_e^{n,u}(R,\Delta,\mathfrak{a}^t)\subseteq I\quad\text{if and only if } \tau_e^{n,u}(R,\Delta,\mathfrak{b}^t)\subseteq I.$$

Proof. Set $M:=q^{u+2}\cdot N$, $M':=q^{u+1}\cdot N$, $s_n:=\mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m})$, and $\delta_n:=q^ns_n$ for every integer n. By Proposition 3.16(ii), we have $\delta_n\in\mathbb{N}$. It is enough to show the following claim.

Claim. *For every $n\geqslant 1$ and every ideal $\mathfrak{q}\subseteq\mathfrak{m}^{\max\{0,q^u\cdot\delta_n-M'\}}\cdot\tau(R,\Delta)$, we have*

$$\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}^t)\equiv\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{b}^t)\ (\mathrm{mod}\, I).$$

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

In fact, if the claim holds, then it follows from Proposition 3.12(iv) that

$$\begin{aligned}\tau_e^{n,u}(R,\Delta,\mathfrak{b}^t\mathfrak{m}^{s_n+\varepsilon}) &\equiv \tau_e^{n,u}(R,\Delta,\mathfrak{a}^t\mathfrak{m}^{s_n+\varepsilon}) \pmod I\\ &\equiv 0 \pmod I\end{aligned}$$

for every real number $0<\varepsilon\leqslant 1/q^n$. Therefore we have

$$\mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m}) \geqslant \mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{b}^t;\mathfrak{m}).$$

Similarly, if $s_n>0$, then we have

$$\begin{aligned}\tau_e^{n,u}(R,\Delta,\mathfrak{b}^t\mathfrak{m}^{s_n}) &\equiv \tau_e^{n,u}(R,\Delta,\mathfrak{a}^t\mathfrak{m}^{s_n}) \pmod I\\ &\not\equiv 0 \pmod I,\end{aligned}$$

which shows $\mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{a}^t;\mathfrak{m}) \leqslant \mathrm{fjn}_e^{I,n,u}(R,\Delta,\mathfrak{b}^t;\mathfrak{m})$. Since this inequality also holds when $s_n=0$, we complete the proof of the proposition. □

Proof of Claim. We use induction on n.

Step 1. We first consider the case when $n=1$. It follows from Proposition 3.12(iii) that

$$\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}^t) \equiv \tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{b}^t) \pmod{\tau_{e,\mathfrak{q}\cdot\mathfrak{m}^M}^{n,u}(R,\Delta)}.$$

Since we have $\mathfrak{q}\cdot\mathfrak{m}^M \subseteq \mathfrak{m}^{q^u\lceil q(\ell\ell_R(R/I)+\mathrm{emb}(R))-1\rceil}\cdot\tau(R,\Delta)$, it follows from Proposition 3.12(ii), (iv) and (v) that

$$\tau_{e,\mathfrak{q}\cdot\mathfrak{m}^M}^{n,u}(R,\Delta) \subseteq \mathfrak{m}^{\ell\ell_R(R/I)} \subseteq I.$$

Therefore, the assertion holds when $n=1$.

Step 2. From now on, we consider the case when $n\geqslant 2$. Set $\mathfrak{q}' := \varphi_\Delta^e(F_*^e(\mathfrak{a}^{t^{(n)}\cdot q^u}\mathfrak{q}))$ and $\mathfrak{q}'' := \varphi_\Delta^e(F_*^e(\mathfrak{b}^{t^{(n)}\cdot q^u}\mathfrak{q}))$. Then it follows from Proposition 3.12(ix) that

$$\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{a}^t) = \tau_{e,\mathfrak{q}'}^{n-1,u}(R,\Delta,\mathfrak{a}^t). \tag{9}$$

Similarly, by Proposition 3.13(ii), we have

$$\tau_{e,\mathfrak{q}}^{n,u}(R,\Delta,\mathfrak{b}^t) = \tau_{e,\mathfrak{q}''}^{n-1,u}(R,\Delta,\mathfrak{b}^t). \tag{10}$$

Step 3. In this step, we will show the equation

$$\tau_{e,\mathfrak{q}'}^{n-1,u}(R,\Delta,\mathfrak{a}^t) \equiv \tau_{e,\mathfrak{q}''}^{n-1,u}(R,\Delta,\mathfrak{a}^t) \pmod I. \tag{11}$$

Set $J := \varphi_\Delta^e(F_*^e(\mathfrak{m}^M\mathfrak{q}))$, then we have $\mathfrak{q}'\equiv\mathfrak{q}'' \pmod J$. By Proposition 3.12(iii), it is enough to show that

$$\tau_{e,J}^{n-1,u}(R,\Delta,\mathfrak{a}^t) \subseteq I.$$

Since we have $\delta_n \geqslant q\delta_{n-1}-qN$, it follows from Lemma 2.13 that

$$\begin{aligned}J &\subseteq \varphi_\Delta^e(\mathfrak{m}^{q^u\delta_n+M-M'}\cdot\tau(R,\Delta))\\ &\subseteq \mathfrak{m}^{(q^u\delta_n+M-M')/q-\mathrm{emb}(R)}\cdot\tau(R,\Delta)\\ &\subseteq \mathfrak{m}^{q^u\delta_{n-1}}\cdot\tau(R,\Delta).\end{aligned}$$

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

Therefore, it follows from Proposition 3.12(ii) and (iv) that

$$\tau_{e,J}^{n-1,u}(R,\Delta,\mathfrak{a}^t) \subseteq \tau_e^{n-1,u}(R,\Delta,\mathfrak{a}^t\mathfrak{m}^{s_{n-1}+(1/q^{n-1})}) \subseteq I,$$

which shows the equation (11).

Step 4. In this step, we will show the equation

$$\tau_{e,\mathfrak{q}''}^{n-1,u}(R,\Delta,\mathfrak{a}^t) \equiv \tau_{e,\mathfrak{q}''}^{n-1,u}(R,\Delta,\mathfrak{b}^t) \ (\mathrm{mod}\, I). \tag{12}$$

As in Step 3, we have

$$\begin{aligned}\mathfrak{q}'' &\subseteq \varphi_\Delta^e(F_*^e(\mathfrak{m}^{\max\{0,q^u\delta_n-M'\}}\cdot\tau(R,\Delta)))\\ &\subseteq \mathfrak{m}^{\max\{0,q^{u-1}\delta_n-(M'/q)-\mathrm{emb}(R)\}}\cdot\tau(R,\Delta)\\ &\subseteq \mathfrak{m}^{\max\{0,q^u\delta_{n-1}-M'\}}\cdot\tau(R,\Delta).\end{aligned}$$

By induction hypothesis, we get the equation (12).

By combining the equations (9)–(12), we complete the proof of the claim. □

Corollary 5.7 (Theorem 1.4). *Let $(X=\operatorname{Spec} R,\Delta)$ be a pair such that $(p^e-1)(K_X+\Delta)$ is Cartier for some integer $e>0$, let $I\subseteq R$ be an $\mathfrak{m}$-primary ideal, let $l,n_0\geqslant 0$ and $u\geqslant 2$ be integers, and let $t>0$ be a rational number such that $p^e(p^e-1)t\in\mathbb{N}$. We set $l:=t^{(2)}$, $t_0:=p^{2e}/(p^e-1)$ and $M_0=(p^{e(n_0+6)}-1)\cdot\mathrm{emb}(R)/(p^e-1)$. Then there exists an integer $n_1>0$ with the following property. For any ideal $\mathfrak{a}\subseteq R$ such that:*

(i) *$p^e>\mu_R(\mathfrak{a})+\ell\ell_R(R/I)+\mathrm{emb}(R)$; and*

(ii) *$\tau_e^{n_0+1,u}(R,\Delta,\mathfrak{a}^{lt_0})+\mathfrak{m}^{M_0}\cdot\tau(R,\Delta)\supseteq\tau_e^{n_0,u}(R,\Delta,\mathfrak{a}^{lt_0})$,*

we have

$$\tau_e^{n,u}(R,\Delta,\mathfrak{a}^t)\subseteq I \quad \text{if and only if } \tau_e^{n_1,u}(R,\Delta,\mathfrak{a}^t)\subseteq I$$

for every integer $n\geqslant n_1$.

Proof. By Propositions 5.4 and 5.6, $\mathfrak{b}:=\mathfrak{a}+\mathfrak{m}^{q^{u+n_0+5}\mathrm{emb}(R)}$ satisfies

$$\tau_e^{n,u}(R,\Delta,\mathfrak{a}^t)\subseteq I \quad \text{if and only if } \tau_e^{n,u}(R,\Delta,\mathfrak{b}^t)\subseteq I.$$

for every integer n.

On the other hand, it follows from Proposition 3.17 that there exists an integer $n_1>0$ which depends only on $\mu:=q-\mathrm{emb}(R)-1$, $M:=q^{u+n_0+5}\mathrm{emb}(R)$, e,u, and t such that for every integer $n>n_1$, we have

$$\tau_e^{n,u}(R,\Delta,\mathfrak{b}^t)\subseteq I \quad \text{if and only if } \tau_e^{n_1,u}(R,\Delta,\mathfrak{b}^t)\subseteq I,$$

which completes the proof. □

By using the method of ultraproduct, we can apply Corollary 5.7 to infinitely many ideals simultaneously.

Proposition 5.8. *Let $(X=\operatorname{Spec} R,\Delta)$ be a pair such that $(p^e-1)(K_X+\Delta)$ is Cartier for some integer $e>0$, let $I\subseteq R$ be an $\mathfrak{m}$-primary ideal, let $\{\mathfrak{a}_m\}_{m\in\mathbb{N}}$ be a family of ideals of R, let $t>0$ be a rational number, and let $\mathfrak{U}$ be a non-principal ultrafilter. Assume that:*

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

(i) $\tau(R, \Delta)$ *is* $\mathfrak{m}$*-primary or trivial;*
(ii) $p^e > \mu_R(\mathfrak{a}_m) + \ell\ell_R(R/I) + \mathrm{emb}(R)$ *for every* m*; and*
(iii) $p^e(p^e - 1)t \in \mathbb{N}$.

Then for any sufficiently large integer $u > 0$*, there exist an integer* n_1 *and* $T \in \mathfrak{U}$ *such that*

$$\tau_e^{n,u}(R, \Delta, \mathfrak{a}_m^t) \subseteq I \quad \text{if and only if } \tau_e^{n_1,u}(R, \Delta, \mathfrak{a}_m^t) \subseteq I$$

for every integer $n \geqslant n_1$ *and* $m \in T$.

Proof. Set $t_0 := p^{2e}/(p^e - 1)$. Since $p^e(p^e - 1)t \in \mathbb{N}$, there exists an integer $0 < l < p^e$ such that $t^{(n)} = l$ for every $n \geqslant 2$. By Corollary 5.7, it is enough to show that for any sufficiently large integer $u > 0$, there exist an integer n_0 and $T \in \mathfrak{U}$ such that for every $m \in T$, we have

$$\tau_e^{n_0+1,u}(R, \Delta, \mathfrak{a}^{lt_0}) + \mathfrak{m}^{M_0} \cdot \tau(R, \Delta) \supseteq \tau_e^{n_0,u}(R, \Delta, \mathfrak{a}^{lt_0}),$$

where $M_0 := (p^{e(n_0+6)} - 1)\mathrm{emb}(R)/(p^e - 1)$.

Let $(R_\#, \mathfrak{m}_\#)$ be the catapower of $(R, \mathfrak{m})$, $\Delta_\#$ be the flat pullback of Δ to $\operatorname{Spec} R_\#$ and $\mathfrak{a}_\infty$ be the ideal $[\mathfrak{a}_m]_m \subseteq R_\#$. It follows from Lemma 2.17 that for all integers $u, n \geqslant 0$ we have

$$\tau_e^{n,u}(R_\#, \Delta_\#, \mathfrak{a}_\infty^{l\cdot t_0}) = [\tau_e^{n,u}(R, \Delta, \mathfrak{a}_m^{l\cdot t_0})]_m.$$

By Proposition 2.6, there exists an integer $n_0 \geqslant 0$ such that

$$\tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^{\langle l\cdot t_0\rangle_{n_0,q}}) = \tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^{\langle l\cdot t_0\rangle_{(n_0+1),q}}).$$

On the other hand, by Proposition 3.12(viii), there exists an integer u_0 such that for all integers $u \geqslant u_0$ and $n \geqslant 0$, we have

$$\tau_e^{n,u}(R_\#, \Delta_\#, \mathfrak{a}_\infty^{l\cdot t_0}) = \tau(R_\#, \Delta_\#, \mathfrak{a}_\infty^{\langle l\cdot t_0\rangle_{n,q}}).$$

Therefore, we have

$$[\tau_e^{n_0,u}(R, \Delta, \mathfrak{a}_m^{l\cdot t_0})]_m = [\tau_e^{n_0+1,u}(R, \Delta, \mathfrak{a}_m^{l\cdot t_0})]_m \subseteq R_\#.$$

Since $\mathfrak{m}^{M_0} \cdot \tau(R, \Delta) \subseteq R$ is an $\mathfrak{m}$-primary ideal, it follows from Lemma 2.21 that there exists $T \in \mathfrak{U}$ such that for every $m \in T$, we have

$$\tau_e^{n_0,u}(R, \Delta, \mathfrak{a}_m^{l\cdot t_0}) \subseteq \tau_e^{n_0+1,u}(R, \Delta, \mathfrak{a}_m^{l\cdot t_0}) + \mathfrak{m}^{M_0} \cdot \tau(R, \Delta),$$

which completes the proof. $\square$

Theorem 5.9 (Main theorem). *Let* $(X = \operatorname{Spec} R, \Delta)$ *be a pair such that* $\tau(R, \Delta)$ *is* $\mathfrak{m}$*-primary or trivial and that* $(p^e - 1)(K_X + \Delta)$ *is Cartier for some integer* $e > 0$*, and let* $I \subseteq R$ *be an* $\mathfrak{m}$*-primary ideal. Then, the set*

$$\mathrm{FJN}^I(R, \Delta) := \{\mathrm{fjn}^I(R, \Delta; \mathfrak{a}) \mid \mathfrak{a} \subsetneq R\}$$

satisfies the ascending chain condition.

https://doi.org/10.1112/S0010437X19007358 Published online by Cambridge University Press

Proof. We assume the contrary. Then there exists a family of ideals $\{\mathfrak{a}_m\}_{m\in\mathbb{N}}$ such that the sequence $\{\mathrm{fjn}^I(R,\Delta;\mathfrak{a}_m)\}_{m\in\mathbb{N}}$ is a strictly ascending chain. Set $t := \lim_{m\longrightarrow\infty}\mathrm{fjn}^I(R,\Delta;\mathfrak{a}_m)$. It follows from Proposition 2.6 and Theorem 4.7 that $t \in \mathbb{Q}_{>0}$.

Let $\mathfrak{U}$ be a non-principal ultrafilter, let $R_\#$ be the catapower of R, let $\Delta_\#$ be the flat pullback of Δ to $\operatorname{Spec} R_\#$, and $\mathfrak{a}_\infty := [\mathfrak{a}_m]_m \subseteq R_\#$. Take elements $f_1,\dots,f_l \in R_\#$ such that $\mathfrak{a}_\infty = (f_1,\dots,f_l)$. Since the natural map $\prod_{m\in\mathbb{N}}\mathfrak{a}_m \longrightarrow [\mathfrak{a}_m]_m$ is surjective, there exists $f_{m,i}\in\mathfrak{a}_m$ for every $m\in\mathbb{N}$ such that $f_i = [f_{m,i}]_m$.

Set $\mathfrak{a}'_m := (f_{m,1},\dots,f_{m,l}) \subseteq \mathfrak{a}_m$. Since we have $[\mathfrak{a}'_m]_m = \mathfrak{a}_\infty$, it follows from Theorem 4.7 that $\mathrm{sh}(\mathrm{ulim}_m\,\mathrm{fjn}^I(R,\Delta;\mathfrak{a}'_m)) = t$. On the other hand, since we have $\mathrm{fjn}^I(R,\Delta;\mathfrak{a}'_m) \leqslant \mathrm{fjn}^I(R,\Delta;\mathfrak{a}_m) < t$, by replacing by a subsequence, we may assume that the sequence $\{\mathrm{fjn}^I(R,\Delta;\mathfrak{a}'_m)\}$ is a strictly ascending chain. By replacing $\mathfrak{a}_m$ by $\mathfrak{a}'_m$, we may assume $\mu_R(\mathfrak{a}_m)\leqslant l$ for every m.

By enlarging e, we may assume that $q = p^e$ satisfies the following properties:

(i) $q(q-1)t\in\mathbb{N}$; and

(ii) $q > \ell\ell_R(R/I) + l + \mathrm{emb}(R)$.

It follows from Proposition 5.8 that there exist integers $u, n_1 > 0$ and $T\in\mathfrak{U}$ such that

$$\tau_e^{n,u}(R,\Delta,\mathfrak{a}_m^t)\subseteq I \quad \text{if and only if } \tau_e^{n_1,u}(R,\Delta,\mathfrak{a}_m^t)\subseteq I$$

for every integer $n\geqslant n_1$ and $m\in T$. By enlarging u, we may further assume that $u \geqslant \widetilde{\mathrm{stab}}(R_\#, \Delta_\#,\mathfrak{a}_\infty;e)$. For every $m\in\mathbb{N}$ and for every sufficiently large $n\gg 0$ we have

$$\tau_e^{n,u}(R,\Delta,\mathfrak{a}_m^t)\subseteq \tau(R,\Delta,\mathfrak{a}_m^{\langle t\rangle_{n,q}})\subseteq I.$$

Therefore we have $\tau_e^{n_1,u}(R,\Delta,\mathfrak{a}_m^t)\subseteq I$ for every $m\in T$.

On the other hand, since $\langle t\rangle_{n_1,q} < t = \mathrm{fjn}^{I\cdot R_\#}(R_\#,\Delta_\#;\mathfrak{a}_\infty)$, we have

$$\begin{aligned}[\tau_e^{n_1,u}(R,\Delta,\mathfrak{a}_m^t)]_m &= \tau_e^{n_1,u}(R_\#,\Delta_\#,\mathfrak{a}_\infty^t)\\ &= \tau(R_\#,\Delta_\#,\mathfrak{a}_\infty^{\langle t\rangle_{n_1,q}})\\ &\nsubseteq I\cdot R_\#.\end{aligned}$$

Therefore, there exists a set $S\in\mathfrak{U}$ such that

$$\tau_e^{n_1,u}(R,\Delta,\mathfrak{a}_m^t)\nsubseteq I$$

for every $m\in S$. Since $S\cap T\neq\emptyset$, we have contradiction. □

Corollary 5.10 (Theorem 1.2). *Fix an integer $n\geqslant 1$, a prime number $p>0$ and a set $\mathcal{D}_{n,p}^{\mathrm{reg}}$ such that every element of $\mathcal{D}_{n,p}^{\mathrm{reg}}$ is an n-dimensional F-finite Noetherian regular local ring of characteristic p. The set*

$$\mathcal{T}_{n,p}^{\mathrm{reg}} := \{\mathrm{fpt}(A;\mathfrak{a}) \mid A\in\mathcal{D}_{n,p}^{\mathrm{reg}}, \mathfrak{a}\subsetneq A\},$$

satisfies the ascending chain condition.

Proof. We assume the contrary. Then there exists a sequence $\{A_m\}_{m\in\mathbb{N}}$ in $\mathcal{T}_{n,p}^{\mathrm{reg}}$ and ideals $\mathfrak{a}_m\subsetneq A_m$ such that the sequence $\{\mathrm{fpt}(A_m;\mathfrak{a}_m)\}$ is a strictly ascending chain.

Since test ideals commute with completion [HT04, Proposition 3.2], we may assume that $A_m = k_m[[x_1,\dots,x_n]]$ for some F-finite field k_m. Take an F-finite field k such that $k_m\subseteq k$ for every m. Let $(A,\mathfrak{m}_A)$ be the local ring $k[[x_1,\dots,x_n]]$. Then we have $\mathrm{fpt}(A;(\mathfrak{a}_mA)) = \mathrm{fpt}(A_m;\mathfrak{a}_m)$ by Lemma 2.11. Therefore, we have $\mathrm{fpt}(A_m;\mathfrak{a}_m)\in\mathrm{FJN}^{\mathfrak{m}_A}(A,0)$ for every m, which contradicts to Theorem 5.9. □

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of equicharacteristic. Then $(R, \mathfrak{m})$ is said to be a *quotient singularity* if there exist a regular affine variety $U = \operatorname{Spec} A$ over k, a finite group G with a group homomorphism $G \longrightarrow \operatorname{Aut}_k(U)$, and a point x of the quotient $V = U/G := \operatorname{Spec}(A^G)$ such that there exists an isomorphism $\widehat{R} \cong \widehat{\mathcal{O}_{V,x}}$ as rings. Moreover, if $|G|$ is coprime to $\operatorname{char}(k)$, then we say that $(R, \mathfrak{m})$ is a *tame quotient singularity.*

Lemma 5.11. *Let $(R, \mathfrak{m}, k)$ be a tame quotient singularity of dimension n. Then, there exists a finite group $G \subseteq \mathrm{GL}_n(k)$ with the following properties.*

(i) *The number $|G|$ is coprime to* $\operatorname{char}(k)$.

(ii) *The natural action of G on the affine space $\mathbb{A}^n_k$ has no fixed points in codimension* 1.

(iii) *Let $V := \mathbb{A}^n_k/G$ be the quotient and $x \in V$ be the image of the origin of $\mathbb{A}^n_k$. Then we have $\widehat{R} \cong \widehat{\mathcal{O}_{V,x}}$.*

Proof. The proof follows as in the case when $\operatorname{char}(k) = 0$ (see [dFEM10, p. 15]), but for the convenience of reader we sketch it here.

Since R is a tame quotient singularity, there exists a regular affine variety U, a finite group G which acts on U such that $|G|$ is coprime to $\operatorname{char}(k)$, and a point $x \in V$ such that $\widehat{R} \cong \widehat{\mathcal{O}_{V,x}}$. Take a point $y \in U$ with image x. By replacing G by the stabilizer subgroup $G_y \subseteq G$, we may assume that G acts on the regular local ring $(A, \mathfrak{m}_A) := (\mathcal{O}_{U,y}, \mathfrak{m}_y)$. Since $|G|$ is coprime to $\operatorname{char}(k)$, it follows from Maschke's theorem that the natural projection $\mathfrak{m}_A \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2$ has a section as $k[G]$-modules. This section induces $k[G]$-algebra homomorphism $\operatorname{Gr}_{\mathfrak{m}_A}(A) \longrightarrow A$, where $\operatorname{Gr}_{\mathfrak{m}_A}(A)$ is the associated graded ring of $(A, \mathfrak{m}_A)$. Therefore, by replacing U by $\operatorname{Spec}(\operatorname{Gr}_{\mathfrak{m}_A}(A))$, we may assume that $U = \mathbb{A}^n_k$ and $G \subseteq \mathrm{GL}_n(k)$.

Let $H \subseteq G$ be the subgroup generated by elements $g \in G$ which fixes some codimension one point of U. Since $|G|$ is coprime to $\operatorname{char}(k)$, it follows from the Chevalley–Shephard–Todd theorem (see for example [Ben93, Theorem 7.2.1]) that $U/H \cong \mathbb{A}^n_k$. By replacing U by U/H and G by G/H, we complete the proof of the lemma. ◻

Proposition 5.12 (Theorem 1.3). *Fix an integer $n \geqslant 1$, a prime number $p > 0$ and a set $\mathcal{D}^{\mathrm{quot}}_{n,p}$ such that every element of $\mathcal{D}^{\mathrm{quot}}_{n,p}$ is an n-dimensional F-finite Noetherian normal local ring of characteristic p with tame quotient singularities. The set*

$$\mathcal{T}^{\mathrm{quot}}_{n,p} := \{\operatorname{fpt}(R; \mathfrak{a}) \mid R \in \mathcal{D}^{\mathrm{quot}}_{n,p}, \mathfrak{a} \subsetneq R \textit{ is an ideal}\}$$

satisfies the ascending chain condition.

Proof. The proof is essentially the same as [dFEM10, Proposition 5.3]. Let $(R, \mathfrak{m}, k)$ be a local ring such that $R \in \mathcal{D}^{\mathrm{quot}}_{n,p}$, and let $\mathfrak{a} \subsetneq R$ be an ideal of R. Let G, V, and x be as in Lemma 5.11. Consider the natural morphism $\pi : U := \mathbb{A}^n_k \longrightarrow V$. Since G is a finite group, the morphism π is a finite surjective morphism with $\deg(\pi)$ coprime to $\operatorname{char}(k)$. Since G acts on U with no fixed points in codimension one, the morphism π is étale in codimension one.

Set $W := \operatorname{Spec}(\widehat{R})$ and $U' := U \times_V W$. Since U is a regular scheme and $W \longrightarrow V$ is a regular morphism, each connected component of U' is a regular scheme. Fix a connected component $U'' \subseteq U'$. Since the morphism $\widehat{\pi} : U'' \longrightarrow W$ is finite surjective, étale in codimension 1 and $\deg \widehat{\pi}$ is coprime to p, it follows from [HT04, Theorem 3.3] that

$$\operatorname{fpt}(W; \mathfrak{a}\mathcal{O}_W) = \operatorname{fpt}(U''; \mathfrak{a}\mathcal{O}_{U''}).$$

On the other hand, since the test ideals commute with completion [HT04, Proposition 3.2], we have

$$\operatorname{fpt}(R;\mathfrak{a}) = \operatorname{fpt}(W;\mathfrak{a}\mathcal{O}_W).$$

Therefore, it follows from Corollary 5.10 that the set $\mathcal{T}_{n,p}^{\mathrm{quot}}$ satisfies the ascending chain condition. □

We conclude with a natural question as below.

Question 5.13. Does Theorem 1.2 give an alternative proof of [dFEM10, Theorem 1.1]? Moreover, does Theorem 5.9 imply that the set of all jumping numbers of multiplier ideals with respect to a fixed $\mathfrak{m}$-primary ideal on a log $\mathbb{Q}$-Gorenstein pair over $\mathbb{C}$ satisfies the ascending chain condition?

We hope to consider this question at a later time.

Acknowledgements

The author wishes to express his gratitude to his supervisor Professor Shunsuke Takagi for his encouragement, valuable advice and suggestions. The author is also grateful to Professor Mircea Mustaţă for his helpful comments and suggestions. He would like to thank Doctor Sho Ejiri, Doctor Kentaro Ohno, Doctor Yohsuke Matsuzawa and Professor Hirom Tanaka for useful comments. He is also indebted to the referee for careful reading of the manuscript and thoughtful suggestions. A part of this work was carried out during his visit to University of Michigan with financial support from the Program for Leading Graduate Schools, MEXT, Japan. He was also supported by JSPS KAKENHI 17J04317.

References

And74 M. André, *Localisation de la lissité formelle*, Manuscripta Math. **13** (1974), 297–307.

Aoy83 Y. Aoyama, *Some basic results on canonical modules*, J. Math. Kyoto Univ. **23** (1983), 85–94.

Ben93 D. J. Benson, *Polynomial invariants of finite groups*, London Mathematical Society Lecture Note Series, vol. 190 (Cambridge University Press, Cambridge, 1993).

BMS08 M. Blickle, M. Mustaţă and K. E. Smith, *Discreteness and rationality of F-thresholds*, Michigan Math. J. **57** (2008), 43–61.

BMS09 M. Blickle, M. Mustaţă and K. E. Smith, *F-thresholds of hypersurfaces*, Trans. Amer. Math. Soc. **361** (2009), 6549–6565.

BSTZ10 M. Blickle, K. Schwede, S. Takagi and W. Zhang, *Discreteness and rationality of F-jumping numbers on singular varieties*, Math. Ann. **347** (2010), 917–949.

dFEM10 T. de Fernex, L. Ein and M. Mustaţă, *Shokurov's ACC conjecture for log canonical thresholds on smooth varieties*, Duke Math. J. **152** (2010), 93–114.

dFEM11 T. de Fernex, L. Ein and M. Mustaţă, *Log canonical thresholds on varieties with bounded singularities*, in *Classification of algebraic varieties*, EMS Series of Congress Reports, vol. 3 (European Mathematical Society, Zürich, 2011), 221–257.

Gol98 R. Goldblatt, *Lectures on the hyperreals*, in *An introduction to nonstandard analysis*, Graduate Texts in Mathematics, vol. 188 (Springer, New York, NY, 1998).

HMX14 C. D. Hacon, J. McKernan and C. Xu, *ACC for log canonical thresholds*, Ann. of Math. (2) **180** (2014), 523–571.

HT04 N. Hara and S. Takagi, *On a generalization of test ideals*, Nagoya Math. J. **175** (2004), 59–74.

HNW17 D. J. Hernández, L. Núñez-Betancourt and E. E. Witt, *Local $\mathfrak{m}$-adic constancy of F-pure thresholds and test ideals*, Math. Proc. Cambridge Philos. Soc. **64** (2017), 1–11.

HNWZ16 D. J. Hernández, L. Núñez-Betancourt, E. E. Witt and W. Zhang, *F-pure thresholds of homogeneous polynomials*, Michigan Math. J. **65** (2016), 57–87.

Kun76 E. Kunz, *On Noetherian rings of characteristic p*, Amer. J. Math. **98** (1976), 999–1013.

Mat89 H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, second edition (Cambridge University Press, Cambridge, 1989).

Pér13 F. Pérez, *On the constancy regions for mixed test ideals*, J. Algebra **396** (2013), 82–97.

ST18 K. Sato and S. Takagi, *General hyperplane sections of threefolds in positive characteristic*, J. Inst. Math. Jussieu, doi:10.1017/S1474748018000166.

Sch10 H. Schoutens, *The use of ultraproducts in commutative algebra*, Lecture Notes in Mathematics, vol. 1999 (Springer, Berlin, 2010).

Sch09 K. Schwede, *F-adjunction*, Algebra Number Theory **3** (2009), 907–950.

Sch10 K. Schwede, *Centers of F-purity*, Math. Z. **265** (2010), 687–714.

ST14 K. Schwede and K. Tucker, *Test ideals of non-principal ideals: computations, jumping numbers, alterations and division theorems*, J. Math. Pure Appl. **102** (2014), 891–929.

Sho92 V. Shokurov, *Three-dimensional log perestroikas*, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), 105–203. With an appendix by Yujiro Kawamata.

Sta18 The Stacks Project Authors, Stacks Project, https://stacks.math.columbia.edu (2018).

Tak06 S. Takagi, *Formulas for multiplier ideals on singular varieties*, Amer. J. Math. **128** (2006), 1345–1362.

Kenta Sato ktsato@ms.u-tokyo.ac.jp

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan