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Quantum Multiple Construction of Subfactors

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Abstract. We construct the quantum s-tuple subfactors for an AFD II_1 subfactor with finite index and depth, for an arbitrary natural number s. This is a generalization of the quantum multiple subfactors by Erlijman and Wenzl, which in turn generalized the quantum double construction of a subfactor for the case that the original subfactor gives rise to a braided tensor category. In this paper we give a multiple construction for a subfactor with a weaker condition than braidedness of the bimodule system.

1 Introduction

The asymptotic subfactors of AFD II_1 subfactors with finite index and depth were constructed by Ocneanu [10,11] and Popa [12], and are regarded as Drinfel'd's quantum double construction in the language of subfactor theory. J. Erlijman gave a multiple construction for subfactors arising from braid group representations that generalizes the double construction for a certain class of subfactors [1–4]. Furthermore, she and H. Wenzl gave the multiple construction for braided categories, which includes the cases for subfactors that give rise to braided categories, and obtained the dual principal graphs for several cases [5]. In this paper we construct the quantum multiple subfactors for subfactors whose paragroup satisfies the generalized Yang–Baxter equation [7]. The class of subfactors with this condition includes the ones with non-braided bimodule systems, such as type E_6, E_8 subfactors. It also includes subfactors with non-commutative bimodule systems, such as $M \subset M \rtimes \mathfrak{S}_3$. It is expected that the quantum multiple subfactors constructed in this paper are of finite depth. It is easily observed that the subfactors constructed in this paper include the ones given in [5].

Throughout this paper all the von Neumann algebras are of type AFD, and all the subfactors are assumed to be of finite index and finite depth, except the ones that we are about to construct, for which these properties need to be proved. For the definitions of terms such as paragroups, connections, flatness, string algebras, see [6, Ch. 9–11].

2 The Commuting Square and Biunitary Connection

Let $N \subset M$ be a subfactor and (G, H, β, W) be its paragroup, as in the following picture, where G (resp. H) is the (dual) principal graph, and β^2 is the index of the

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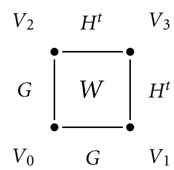
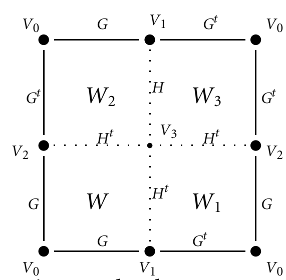
subfactor. Note that the notation here is upside down from the usual notation. When a graph is laid out so that the even vertices are on the bottom (resp. left) and the odd vertices are on the top (resp. right), we consider it to be in the "right position", and if it is the other way around, we consider it to be a renormalized one and call them W_1, W_2, W_3 for horizontal renormalization, vertical renormalization, and the sequence of the two, respectively.

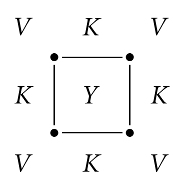


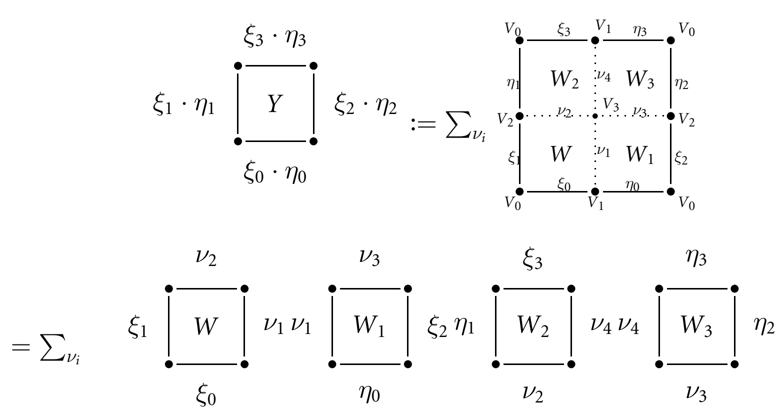
Let ω be its global index. We construct the s-dimensional nested graphs in the first 2^s-ant of $\mathbb{R}^s$ as follows.

First we construct an enlarged biunitary connection obtained as a product of W and its renormalizations.



Let $K = G \cdot G^t$, *i.e.*, a bipartite graph whose even and odd vertices are both $V := V_0$, and whose edges are given by concatenation of the edges in G and those in G^t. We define a new biunitary connection Y as follows:





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By construction Y is a flat connection, and its renormalizations are identical to itself. This connection produces $N \subset M_1$, where M_1 is the basic construction of $N \subset M$ and thus the asymptotic inclusion is the same.

2.1 Nested Algebras on Higher Dimensional Lattices

Now we construct the high-dimensional nested algebras using the connection Y. For the two dimensional case, see [6, §11.3].

For $\mathbf{n} := (n_0 \cdots n_{s-1}) \in \mathbb{Z}^s_{\geq 0}$, let $v_\mathbf{n}$ be the lattice point located at $\mathbf{n}$. Sometimes $v_\mathbf{n}$ may be simply denoted by $\mathbf{n}$, abusing the notation. Let $E_{\mathbf{n},i}$ be the lattice edge connecting $\mathbf{n}$ and $\mathbf{n} + e_i$, where e_i is the unit i-th vector. (Note that we number the coordinates from zero.) We define a nested graph $\mathcal{K}$ as follows: let $V_\mathbf{n}$ be equal to V as a set, located at $\mathbf{n}$. Let $K_{\mathbf{n},i}$ be identical to the graph K, lying along the lattice edge $E_{\mathbf{n},i}$. The vertices of $K_{\mathbf{n},i}$ are identified with $V_\mathbf{n} \cup V_{\mathbf{n}+e_i}$ in an obvious manner. We define a nested graph by $\mathcal{K} = \bigcup_{\mathbf{n},i} K_{\mathbf{n},i}$. A path of $\mathcal{K}$ is a concatenation of edges in $\mathcal{K}$. We call a path consisting of lattice edges a *lattice path* in order to distinguish it from a path of the graph. For a path ξ we denote its length by $|\xi|$, the origin and the end by $s(\xi), r(\xi)$ respectively. The same notions for a lattice path are denoted similarly. For a path ξ, we denote by $[\xi]$ the lattice path that ξ lies along. For $\mathbf{n} \in \mathbb{Z}^s_{\geq 0}$ we denote $|\mathbf{n}| := n_0 + n_1 + \cdots + n_{s-1}$.

Let $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^s_{\geq 0}$ be so that $\mathbf{n} - \mathbf{m} \in \mathbb{Z}^s_{\geq 0}$, and L be a lattice path with $s(L) = \mathbf{m}$, $r(L) = \mathbf{n}$, $|L| = |\mathbf{n} - \mathbf{m}|$. Let $p \in V_\mathbf{m}$, $q \in V_\mathbf{n}$. We define

$$\mathrm{Path}_{p,q;L} := \mathrm{span}_\mathbb{C}\{\xi \mid \xi \in \mathcal{K}, s(\xi) = p, r(\xi) = q, [\xi] = L\}.$$

Note that given another lattice path L' with the same condition, we have

$$\begin{aligned}\mathrm{Path}_{p,q;L} &\cong \mathrm{Path}_{p,q;L'} \\ &\cong \mathrm{Path}^{|\mathbf{n}-\mathbf{m}|}_{p,q} K \\ &:= \mathrm{span}_\mathbb{C}\{\xi \mid \xi \text{ is a path in } K, s(\xi) = p, r(\xi) = q, |\xi| = |\mathbf{n} - \mathbf{m}|\}.\end{aligned}$$

We give an isomorphism $\mathrm{Path}_{p,q;L} \cong \mathrm{Path}_{p,q;L'}$ by identifying the basis, using the flat connection Y as follows.

Definition 2.1 Let $\xi \in \mathrm{Path}_{p,q;L}$, $\eta \in \mathrm{Path}_{p,q;L'}$. For simplicity we assume that $L \cap L' = \{v_\mathbf{m}, v_\mathbf{n}\}$. Let S be a union of squares with the edges in $\bigcup_{\mathbf{n},i} E_{\mathbf{n},i}$, so that $\partial S = L \cup L'$. Assume that S is taken so that the area is minimum. We define a conjugate-linear form by

$$\langle \xi, \eta \rangle = \sum_\sigma \prod_k Y(\sigma_k),$$

where $\sigma = \bigcup_k \sigma_k$ is a surface that lies along with S so that $\partial\sigma = \xi \cup \eta$, and σ_k's are distinct squares with edges in $\mathcal{K}$, $Y(\sigma_k)$ is the evaluation of the flat connection Y on σ_k. Namely, $\langle \xi, \eta \rangle$ is given as a state sum of Y taken over all possible surfaces that lie along S with the boundary $\xi \cup \eta$.

Definition 2.2 If $\langle\,,\,\rangle$ as above is well defined (*i.e.,* if it does not depend on the choice of S) and non-degenerate, we define an isomorphism of $\mathrm{Path}_{p,q;L} \to \mathrm{Path}_{p,q;L'}$ by

$$\xi \to \sum_{\eta \in \mathrm{Path}_{p,q;L'}} \langle \xi, \eta \rangle \eta.$$

We define a path space by $\mathrm{Path}_{p,q} := \mathrm{Path}_{p,q;L}$, where the space does not depend on the choice of L under the given isomorphism. We define an algebra at $\mathbf{n}$ by

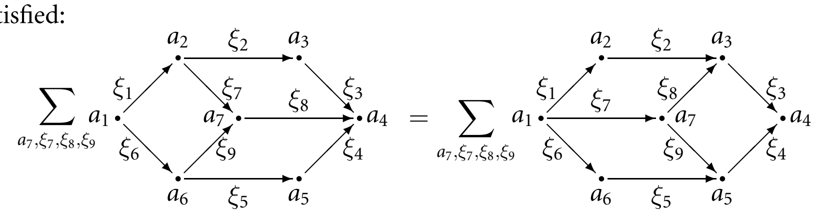
$$A_{\mathbf{n}} := \mathrm{Path}_{*,\mathbf{n}} \otimes \mathrm{Path}^*_{*,\mathbf{n}},$$

where we consider $* \in V_{\mathbf{0}}$, $\mathrm{Path}_{*,\mathbf{n}} := \bigoplus_{q \in V_{\mathbf{n}}} \mathrm{Path}_{*,q}$, and that the dual space is given with respect to $\langle\,,\,\rangle$. We denote an element in $A_{\mathbf{n}}$ by $(\xi, \eta) = \xi \otimes \eta^*$. Note that $(c\xi, \eta) = c(\xi, \eta) = (\xi, \bar{c}\eta)$ for $c \in \mathbb{C}$. The $*$-algebra structure is given by $(\xi, \eta) \cdot (\xi', \eta') := \delta_{\eta,\xi'}(\xi, \eta')$ and $\overline{c(\xi, \eta)} := \bar{c}(\eta, \xi)$ for $c \in \mathbb{C}$

As it is, the conjugate-linear form is ill defined since it depends on the choice of S. And it is not obvious if $\langle,\rangle$ is indeed non-degenerate. In the following we address those issues.

In order to have well-definedness, the flat connection Y needs to satisfy an additional condition.

Assumption **(Generalized Yang–Baxter equation)** A biunitary connection is said to satisfy the generalized Yang–Baxter equation (GYBE) if the following equality is satisfied:



We assume that our flat connection Y satisfies GYBE. We call the geometric move in the equation a "GYBE move".

The paragroups that correspond to subfactors with braided bimodule systems satisfy GYBE, using the translation of the language of flat connection into rational conformal field theory as in [13, §2], and applying Reidemeister move III. One needs to construct 2×2 connections with braided system for the vertices. Note that the biunitary connections for ADE Dynkin diagrams, including non-flat ones, satisfy the relation [6, §11.9]. It is also a straightforward arithmetic computation of cells to check that the flat connection for $\mathfrak{S}_3$ group subfactor with the bimodule system corresponding to the group elements also satisfy GYBE. The data for the cells are obtained from matrix entries of representations of the group, see [6, §10.6]. Note that the choice of the basis of a representation only amounts to gauge equivalence of the connections (checked by the author using Mathematica). So it does not have to be braided nor even flat, though (sub)equivalent to the braided ones.)

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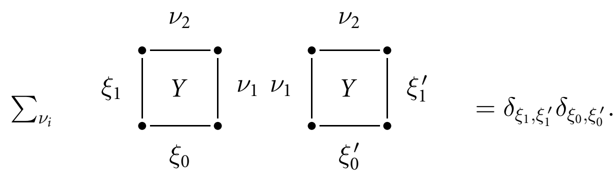
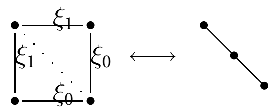
In the following we prove that GYBE is indeed a sufficient condition for well-definedness of the conjugate-linear form $\langle \xi, \eta \rangle$. For future use we restate the axiom of biunitarity, and define a geometric move associated to it.

Definition 2.3 Recall the biunitarity of a connection implies



The corresponding geometric move is as follows:



We call this a unitary move.

For the time being our focus is on spacial geometry. We do not think about the actual graph $\mathcal{K}$ lying in the space. We omit the word "lattice" when we discuss paths, edges, vertices, etc.

Definition 2.4 Let $v_{\mathbf{n}}$, $E_{\mathbf{n},i}$ be as before. We abuse notation and sometimes consider e_i to be also an edge parallel to e_i, located possibly anywhere, *i.e.*, $E_{\mathbf{n},i}$ for any $\mathbf{n}$. Consider paths ξ and η in $\bigcup_{\mathbf{n},i} E_{\mathbf{n},i}$ with $s(\xi), s(\eta) = v_{\mathbf{n}}$, $r(\xi), r(\eta) = v_{\mathbf{n+e1}}$, and $|\xi| = |\eta| = s$, where $\mathbf{e} = (e_0, \ldots, e_{s-1})$ and $\mathbf{1} = (1)_i$ (*i.e.*, $\mathbf{e1} = e_0 + \cdots + e_{s-1}$). Notice that any such path has one-to-one correspondence with the elements of the symmetric group $\mathfrak{S}_s$ by $\sigma \in \mathfrak{S}_s \leftrightarrow e_{\sigma(0)} \cdot e_{\sigma(1)} \cdot e_{\sigma(2)} \cdots e_{\sigma(s-1)}$ up to the initial point.

We consider an embedded surface in $\mathbb{R}^s_{\geq}$ to be always a union of squares with the edges in $\bigcup_{\mathbf{n},i} E_{\mathbf{n},i}$. We call a surface a *minimal surface* if there is no surface with a smaller area with a given boundary loop. Note that the area is the same as the number of squares that make up the surface.

We define a surface S to be *spanned by* ξ *and* η if $\partial S = \overline{\xi \cup \eta - \xi \cap \eta}$ and the boundary of each component of S is connected. We denote the set of the surfaces spanned by ξ, η by $\mathcal{F}_{\xi,\eta}$, and the minimal surfaces by $\mathcal{MF}_{\xi,\eta}$. Note that the set $\mathcal{F}_{\xi,\eta}$ is not empty for $\xi \neq \eta$. An element is given by the union of squares corresponding to transpositions needed to transform ξ to η, as described in the proof of Proposition 2.6.

Lemma 2.5 *Any minimal surface with one boundary component is contractible.*

Proof Any minimal surface with area 1 is contractible. Let n be the minimum of the area for which there exists a minimal surface S with a non-zero genus, with respect to its boundary ρ. Let A be a square in S so that $A \cap \partial S$ has only one component. Such a square should exist: if all the squares with non-trivial intersection with ∂S have more than one component, *i.e.*, two opposite edges, then S must be an annulus. Consider $S \backslash A =: S'$. The area of S' is smaller than that of S, and has the same genus. And

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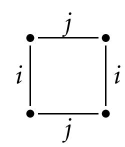
S' is a minimal surface with respect to its boundary: if there is a surface S'' with the same boundary with smaller area, then $S'' \cup A$ (or $S'' \setminus A$, if $S'' \subset A$) would have a smaller area than S, which is contradiction. Since S' has a smaller area than n, it leads to contradiction. ■

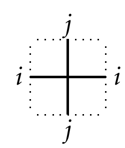
From now on we only consider the contractible surfaces and restrict the elements of $\mathcal{F}_{\xi,\eta}$ to be contractible.

Proposition 2.6 *Any two minimal surfaces spanned by ξ and η are deformed to each other by GYBE and unitary moves.*

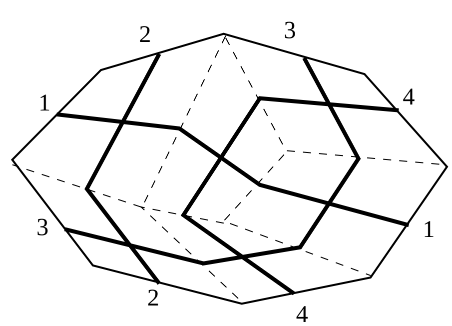
Proof For simplicity assume that $\mathbf{n} = \mathbf{0}$, $\xi = e_0 \cdot e_1 \cdot e_2 \cdots e_{s-1}$. and that ξ and η do not intersect except at the beginning and the end. Let $S \in \mathcal{F}_{\xi,\eta}$. In this case S is homeomorphic to a disk. We label all the edges in S parallel to e_i by i. Then S is a union of squares looking like this:



We draw a crossing on the square like this:



We label the strand across e_i also by i. The picture of S with these decorations, for $s = 4$, $\eta = e_3 \cdot e_2 \cdot e_4 \cdot e_1$, appears as follows:



From now on we consider any surface as so decorated. We observe that each $S \in \mathcal{F}_{\xi,\eta}$ gives rise to a degenerate tangle:

$$T_S \in \mathcal{T}_\sigma = \{ \text{ tangles with } s \text{ input and output, sending } i \text{ to } \sigma(i) \text{ without a self-crossing } \}.$$

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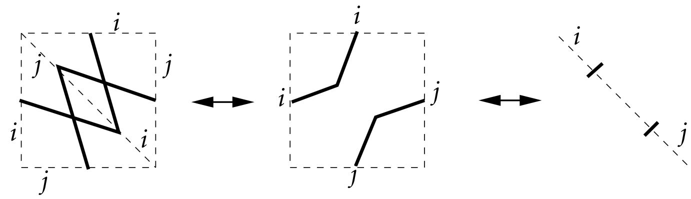
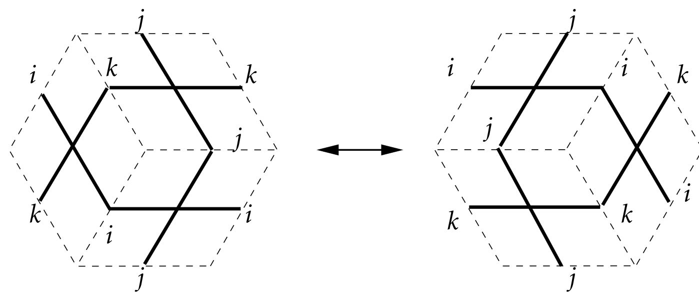
We introduce the following moves.

Reidemeister II (R-II)



Reidemeister III (R-III)



Note that those moves correspond to the Reidemeister moves in knot theory. (For the definitions of Reidemiser moves as well as elementary knowledge of knot theory, see [9].) The middle picture in an R-II move actually does not appear in our situation but is drawn just to have an association with knot theory. It immediately degenerates into the right-most picture. Note also that the Reidemeister I move does not appear in our situation since it requires a square with all four edges labeled by the same name.

Lemma 2.7 *For $T \in \mathfrak{T}_\sigma$, let $c(T)$ be the number of crossings, and*

$$c(\mathfrak{T}_\sigma) := \min_{T \in \mathfrak{T}_\sigma} c(T).$$

Then $c(\mathfrak{T}_\sigma) = w(\sigma)$ equals the length of σ as a word written in transpositions f$\sigma_i = (i, i+1)$g. Any $T \in \mathfrak{T}_\sigma$ is deformed to some $T' \in \mathfrak{T}_\sigma$, so that $c(T') = c(\mathfrak{T}_\sigma)$ by performing R-II *and* R-III *moves finitely many times.*

Proof The first statement is clear, as a transposition corresponds to a crossing. The second statement follows from the fact that $\mathfrak{S}_s$ is generated by σ_i's with relations $\sigma_i^2 = 1$, $\sigma_i\sigma_{i\pm1}\sigma_i = \sigma_{i\pm1}\sigma_i\sigma_{i\pm1}$, and any element $\sigma \in \mathfrak{S}_s$ is reduced to minimum word length expression by applying those relations finitely many times. ∎

The next lemma follows in a similar manner.

Lemma 2.8 *Let $T, T' \in \mathfrak{T}_\sigma$ so that $c(T) = c(T') = c(\mathfrak{T}_\sigma)$. Then T is deformed to T' by a sequence of* R-II *and* R-III *moves.*

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This lemma implies Proposition 2.6. ∎

We now come back to Definition 2.1. For paths $\xi \in \mathrm{Path}_{p,q;L}$ and $\eta \in \mathrm{Path}_{p,q;L'}$, consider two minimal surfaces S_1 and S_2 spanned by $[\xi]$ and $[\eta]$. Let $\langle \xi, \eta \rangle_{S_i}$ be the conjugate-linear form given in Definition 2.1, using S_i. By the above discussion, there is a sequence of intermediate surfaces $\{S_\pi\}$ that connect S_1 and S_2 where adjacent surfaces differ by R-II and R-III moves. By the assumption of GYBE and biunitarity of the connection, $\langle \xi, \eta \rangle_{S_\pi}$ is constant. Thus we proved that $\langle \xi, \eta \rangle_{S_1} = \langle \xi, \eta \rangle_{S_2}$, *i.e.*, $\langle \xi, \eta \rangle$ is well defined.

We prove the non-degeneracy of $\langle \xi, \eta \rangle$ as follows: by Lemma 2.5, a minimal surface S spanned by L, L' is a disk. For simplicity let us assume that $s(L) = s(L') = \mathbf{0}$, $r(L) = r(L') = \mathbf{e1}$, and that $L = e_0 \cdot e_1 \cdots e_{s-1}$, $L' = e_{\tau(0)} \cdot e_{\tau(1)} \cdots e_{\tau(s-1)}$ for $\tau \in \mathfrak{S}_s$. The surface S determines a minimal expression $\tau = \tau_{k+1}\tau_k \cdots \tau_1$, where τ_j's are transpositions. For $1 \leq j \leq k$, let $L_j = e_{\tau^j(0)} \cdot e_{\tau^j(1)} \cdots e_{\tau^j(s-1)}$, where $\tau^j = \tau_j \tau_{j-1} \cdots \tau_1$. Let $L_0 = L$, $L_{k+1} = L'$. Then we have a conjugate linear form on $\mathrm{Path}_{p,q,L_j} \times \mathrm{Path}_{p,q,L_{j+1}}$:

$$\langle \xi_j, \xi_{j+1} \rangle = \begin{cases} 0 & \text{if } \xi_j \text{ and } \xi_{j+1} \text{ disagree on } L_j \cap L_{j+1}, \\ Y(\sigma) & \text{otherwise,} \end{cases}$$

where ξ_j, ξ_{j+1} are paths in Path_{p,q,L_j} and $\mathrm{Path}_{p,q,L_{j+1}}$ respectively, and σ is a square bounded by ξ_j, ξ_{j+1} (which corresponds to τ_{j+1}). Since Y is a biunitary connection, this linear form is non-degenerate. Noticing that the state sum given in Definition 2.2 implies

$$\langle \xi, \eta \rangle = \sum_{\xi_1, \ldots, \xi_k} \langle \xi, \xi_1 \rangle \langle \xi_1, \xi_2 \rangle \cdots \langle \xi_k, \eta \rangle,$$

we conclude that $\langle \xi, \eta \rangle$ is non-degenerate. Note that we did not need flatness of Y nor GYBE for this proof.

Hence we have well-defined path spaces $\mathrm{Path}_{p,q}$ and algebras $A_{\mathbf{n}}$.

2.2 Construction of the Commuting Square for the Multiple Subfactor

Now we give a nested structure of algebras $\{A_{\mathbf{n}}\}$ and commuting squares arising from it.

We define the embedding $A_{\mathbf{n}} \subset A_{\mathbf{n}+e_i}$ by $(\xi, \eta) \mapsto \sum_{\gamma \in K_{\mathbf{n},i}} (\xi \cdot \gamma, \eta \cdot \gamma)$ (*i.e.*, γ is parallel to e_i). We define a trace on $A_{\mathbf{n}}$ by $\mathrm{tr}(\xi, \eta) := \delta_{\xi,\eta} \beta^{-2|\mathbf{n}|} \mu(r(\xi))$, where μ is the Perron–Frobenius eigenvector of the original connection W. This is compatible with the embedding. One can check that the conditional expectation $E_{\mathbf{n},i} \colon A_{\mathbf{n}+e_i} \to A_{\mathbf{n}}$ will be given by

$$(\xi \cdot \xi', \eta \cdot \eta') \mapsto \delta_{\xi',\eta'} \frac{\mu(r(\xi'))}{\beta^2 \mu(r(\xi))} (\xi, \eta),$$

see [6, Lemma 11.7].

Proposition 2.9 *Consider the following diagram.*

$$\begin{array}{ccc} A_{\mathbf{n}+e_j} & \hookrightarrow & A_{\mathbf{n}+e_i+e_j} \\ \uparrow & & \uparrow \\ A_{\mathbf{n}} & \hookrightarrow & A_{\mathbf{n}+e_i} \end{array}$$

where $i \neq j$. The identification of the bases of $A_{\mathbf{n}+e_i+e_j}$ via $A_{\mathbf{n}+e_i}$ and via $A_{\mathbf{n}+e_j}$ is given by the connection Y. Then this is a commuting square, with conditional expectations defined as above.

Proof It is proved by straightforward computation: take $x = (\xi \cdot \gamma, \eta \cdot \gamma) \in A_{\mathbf{n}+e_i}$, where $(\xi, \eta) \in A_{\mathbf{n}}$. $E_{\mathbf{n}}(x) = \frac{\mu(r(\gamma))}{\beta^2 \mu(r(\xi))}(\xi, \eta)$. Embed x into $A_{\mathbf{n}+e_i+e_j}$, change basis using Y and apply $E_{\mathbf{n}+e_j}$; the result is equal to $E_{\mathbf{n}}(x)$, using the unitarity of the connection. The coefficients are adjusted by the constant coming from renormalization. ■

Corollary 2.10 *The following diagram is a commuting square.*

$$\begin{array}{ccc} A_{\mathbf{n}+m_j e_j} & \hookrightarrow & A_{\mathbf{n}+m_i e_i+m_j e_j} \\ \uparrow & & \uparrow \\ A_{\mathbf{n}} & \hookrightarrow & A_{\mathbf{n}+m_i e_i} \end{array}$$

where $i \neq j$, and $m_i, m_j \in \mathbb{Z}$.

Theorem 2.11 *The following is a commuting square:*

$$\begin{array}{ccc} A_{n\mathbf{e1}} & \hookrightarrow & A_{(n+1)\mathbf{e1}} \\ \uparrow & & \uparrow \\ \bigvee_i A_{ne_i} & \hookrightarrow & \bigvee_i A_{(n+1)e_i} \end{array}$$

where the trace is defined as before, and conditional expectation is determined uniquely by the trace.

The proof is given by successive applications of the lemma below.

Lemma 2.12 *The following is a commuting square for all j:*

$$\begin{array}{ccc} A_{\mathbf{n}} & \xhookrightarrow{E_{\mathbf{n}}} & A_{\mathbf{n}+e_j} \\ \uparrow & & \uparrow \\ \bigvee_i A_{n_i e_i} & \xhookrightarrow{E_j} & \bigvee_{i \neq j} A_{n_i e_i} \vee A_{(n_j+1)e_j} \end{array}$$

Proof Using Proposition 2.9 we have

$$(**)\qquad \begin{array}{ccc} A_{\mathbf{n}} & \overset{E_{\mathbf{n}}}{\hookrightarrow} & A_{\mathbf{n}+e_j} \\ \uparrow & & \uparrow \\ A_{n_j e_j} & \overset{E_j}{\hookrightarrow} & A_{(n_j+1)e_j} \end{array}$$

Take $a \in A_{n_j e_j}$ and $b \in A_{(n_j+1)e_j}$. Then $\tilde{E}_j(ab) = a\tilde{E}_j(b)$. Thus by the uniqueness of the conditional expectation we have $\tilde{E}_j|_{A_{(n_j+1)e_j}} = E_j$. Now the elements in $\bigvee_{i \neq j} A_{n_i e_i}$ and those in $A_{(n_j+1)e_j}$ commute each other since Y is a flat connection. So it suffies to show that $\tilde{E}_j(ab) = E_{\mathbf{n}}(ab)$ for $a \in \bigvee_{i\neq j} A_{n_i e_i}$, $b \in A_{(n_j+1)e_j}$. Observe that

$$\begin{aligned} \tilde{E}_j(ab) = a\tilde{E}_j(b) = aE_{\mathbf{n}}(b) & \qquad \text{by } (**) \\ = E_{\mathbf{n}}(ab) & \qquad \text{because } a \in A_{\mathbf{n}}. \end{aligned}$$

■

We obtain the quantum multiple inclusion $P \subset Q$ of the subfactor $N \subset M$ using the periodic commuting square as in Theorem 2.11, where

$$P := \overline{\bigcup_n \bigvee_i A_{ne_i}}^{w}, \quad Q := \overline{\bigcup_n A_{n\mathbf{e1}}}^{w},$$

with the inclusion given by $\bigvee_i A_{ne_i} \subset A_{n\mathbf{e1}}$ which is compatible with the union due to the commuting square condition. Note that when $s = 2$, it coincides with the asymptotic inclusion $N \vee (N' \cap M_\infty) \subset M_\infty$. By comparing the commuting square given in Theorem 2.11 and the one in [5, §3.2], one may observe that our result here gives a generalization of [5]. The correspondence is given by translation of the notions such as string algebras and endomorphism algebras on bimodules, embedding of an element of a string algebra and tensor product of an endomorphism with an identity map, etc. We do not discuss the details here, but note that X in [5] is given by ${}_N M_N$ in this paper, and the relation between the embedding $A_{n\mathbf{e1}} \overset{i}{\subset} A_{(n+1)\mathbf{e1}}$ in this paper and $A_{ns} \overset{j}{\subset} A_{(n+1)s}$ in [5] is $i = u_{n+1} j u_n{}^*$, where u_n is as defined in [5, §3.2]. Thus the embedding $\bigvee_i A_{ne_i} \subset A_{n\mathbf{e1}}$ in this paper does not need a unitary conjugation u_n as seen in [5]. Therefore the commuting squares in both settings are equivalent. For the correspondence between two languages, see [6, Ch. 11-12].

3 Intermediate Subfactors and the Bratteli Digarams of the Commuting Square

The Bratteli diagram L of the inclusion $\bigvee_i A_{ne_i} \subset A_{n\mathbf{e1}}$ is determined by the fusion structure of N-N bimodules $\{X_k\}$. We show it by constructing intermediate subfactors.

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Proposition 3.1 *Consider the following commuting squares:*

$$
\begin{array}{ccc}
A_{ne_0} \vee A_{ne_1} \cdots \vee A_{ne_{s-1}} & \hookrightarrow & A_{(n+1)e_0} \vee A_{(n+1)e_1} \cdots \vee A_{(n+1)e_{s-1}} \\
\downarrow & & \downarrow \\
A_{n(e_0+e_1)} \vee A_{ne_2} \vee \cdots \vee A_{ne_{s-1}} & \hookrightarrow & A_{(n+1)(e_0+e_1)} \vee \cdots \vee A_{(n+1)e_{s-1}} \\
\downarrow & & \downarrow \\
A_{n(e_0+e_1+e_2)} \vee A_{ne_3} \vee \cdots \vee A_{ne_{s-1}} & \hookrightarrow & A_{(n+1)(e_0+e_1+e_2)} \vee \cdots \vee A_{(n+1)e_{s-1}} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
A_{n(e_0+\cdots+e_{s-2})} \vee A_{ne_{s-1}} & \hookrightarrow & A_{(n+1)(e_0+\cdots+e_{s-2})} \vee A_{(n+1)e_{s-1}} \\
\downarrow & & \downarrow \\
A_{n\mathbf{1}\cdot\mathbf{e}} & \hookrightarrow & A_{(n+1)\mathbf{1}\cdot\mathbf{e}}
\end{array}
$$

The commuting square on the j-th floor (in the European way) gives the subfactor

$$N \vee (N' \cap M_\infty) \otimes \underbrace{N \otimes \cdots \otimes N}_{j \text{ times}} \subset M_\infty \otimes \underbrace{N \otimes \cdots \otimes N}_{j \text{ times}},$$

where the embedding is given by the asymptotic inclusion of $N \subset M$ tensored with the identities of N.

This follows from the next lemma.

Lemma 3.2 *For each j, the commuting square*

$$
\begin{array}{ccc}
A_{n(e_0+\cdots+e_{j-1})} \vee A_{ne_j} & \hookrightarrow & A_{(n+1)(e_0+....+e_{j-1})} \\
\downarrow & & \downarrow \\
A_{n(e_0+\cdots+e_j)} & \hookrightarrow & A_{(n+1)(e_0+\cdots+e_j)}
\end{array}
$$

gives the asymptotic inclusion.

Proof Let $B_{m,n} := A_{m(e_0+\ldots+e_{j-1}),ne_j}$ Then the above commuting square is written as follows.

$$
\begin{array}{ccc}
B_{n,0} \vee B_{0,n} & \hookrightarrow & B_{n,0} \vee B_{0,n} \\
\downarrow & & \downarrow \\
B_{n,n} & \hookrightarrow & B_{n+1,n+1}
\end{array}
$$

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This commuting square gives the asymptotic subfactor of $B_{0,\infty} \subset B_{1,\infty}$. Since the commuting net of algebras $\{B_{n,m}\}$ is given by the biunitary connection

$$\underbrace{YYY\cdots Y}_{m\text{ times}}$$

(composed horizontally) which gives the same subfactor $N \subset M_1$ as Y does, we obtain the asymptotic inclusion. ■

Thus the Bratteli diagram in each step of the left column of the diagram in Proposition 3.1 is given by $\{\text{fusion graph } \times (\times^j \text{ trivial graph})\}$. Connecting all of this, the Bratteli diagram L of the inclusion $\bigvee_i A_{ne_i} \subset A_{n\mathbf{e1}}$ is given by the s-fusion graph of N-N bimodules $\{X_k\}$, that is, the set of vertices corresponding to the simple components in $\bigvee_i A_{ne_i}$ is given by $\{(X_0,\dots,X_{s-1})\}_{X_j\in\mathcal{X}_{N-N}}$, the set of vertices corresponding to the simple components in $A_{n\mathbf{1}\cdot\mathbf{e}}$ is given by $\mathcal{X}_{N\text{-}N}$, and the number of edges between $(X_0,\dots,X_{s-1})$ and Y is given by $N^Y_{X_0,\dots,X_{s-1}} := \dim \operatorname{Hom}(X_0\otimes_N\cdots\otimes_N X_{s-1}, Y)$. In particular this implies that $P \subset Q$ is irreducible if $N \subset M$ is irreducible.

The following proposition is obtained directly from the construction.

Proposition 3.3 *The Perron–Frobenius eigenvalue β_L of L is given by $\omega^{\frac{s-1}{2}}$, and the Perron–Frobenius eigenvector μ_L is given by*

$$\mu_L(i_0,i_1,\dots,i_{s-1}) = \mu(0)\mu(1)\cdots\mu(s-1), \quad \mu_L(j) = \beta_L\mu(j),$$

where each number is an index of V_0 thus corresponds to each vertex. Recall that μ was the Perron–Frobenius eigenvector of the original connection, and that $\omega = [\![M:N]\!]$.

The above proposition implies that $[Q:P] = \omega^{s-1}$.

The following lemma is not necessary but noteworthy. For simplicity, we omit $\otimes_N$ as long as there is no confusion.

Lemma 3.4 *For the set of N-N bimodules $\mathcal{X} := \{X_k\}$ and any $Y \in \mathcal{X}$, $s \in \mathbb{N}$, the following equality holds:*

$$\sum_{X_i\in\mathcal{X}} N^Y_{X_1,\dots,X_s}\mu_1\cdots\mu_s = \omega^{s-1}\mu_Y,$$

where $N^Y_{X_1,\dots,X_n} := \dim \operatorname{Hom}(X_1\cdots X_n, Y)$, and $\mu_i = \mu(X_i)$.

Proof We proceed by induction. The case $s = 2$ is shown in [6, Lemma 12.10]. Suppose it holds for $s-1$. Note that $N^Y_{X_1,\dots,X_s} = \sum_Z N^Z_{X_1,\dots,X_{s-1}} N^Y_{Z,X_s}$. Thus

$$\begin{aligned}\sum_{X_i\in\mathcal{X}} N^Y_{X_1,\dots,X_s}\mu_1\cdots\mu_s &= \sum_{X_s,Z\in\mathcal{X}} N^Y_{Z,X_s}\mu_s \sum_{X_i\in\mathcal{X}} N^Z_{X_1,\dots,X_{s-1}}\mu_1\cdots\mu_{s-1}\\ &= \sum_{X_s,Z\in\mathcal{X}} N^Y_{Z,X_s}\mu_s\mu(Z)\omega^{s-2} \quad \text{(by the inductive hypothesis)}\\ &= \omega^{s-1}\mu(Y) \quad \text{(using the case } s=2)\end{aligned}$$

■

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