STABLE MODEL OF $X_0(125)$

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Abstract

In this paper, the components in the stable model of $X_0(125)$ over $\mathbb{C}_5$ are determined by constructing (in the language of R. Coleman's 'Stable maps of curves', to appear in the Kato Volume of *Doc. Math.*) an explicit semi-stable covering. Empirical data is then offered regarding the placement of certain CM j-invariants in the supersingular disk of $X(1)$ over $\mathbb{C}_5$, which suggests a moduli-theoretic interpretation for the components of the stable model. The paper then concludes with a conjecture regarding the stable model of $X_0(p^3)$ for $p > 3$, which is as yet unknown.

1. *Introduction*

The purpose of this paper is to begin to advance the work of Deligne–Rapoport, Katz–Mazur, Edixhoven, and others regarding the stable model of $X_0(p^n)$. In the simplest case of $X_0(p)$ ($p \neq 2, 3$), the minimal resolution over $\mathbb{Z}_p$ is always semi-stable, and was explicitly described in [**5**, VI.6.16]. Edixhoven later worked out the minimal resolution of the Katz–Mazur model for $X_0(p^n)$ over $\mathbb{Z}_p$ (a special case of [**6**, 1.4]), but this model is never semi-stable when $n \geqslant 2$. However, in the $n = 2$ case, Edixhoven did go on to work out a semi-stable model for $X_0(p^2)$ over a finite extension of $\mathbb{Z}_p^{\mathrm{unr}}$ (see [**6**, 2.1, 2.5]). Unfortunately, as Edixhoven states in his introduction, his methods do not generalize sufficiently to calculate the stable model when $n > 2$ because of 'wild ramification'.

Recently, more progress toward understanding the stable model of $X_0(p^n)$ has been made by Coleman, using a moduli-theoretic approach. In [**3**], Coleman shows that for $p > 3$ the ordinary region of $X_0(p^n)$ has exactly $2n$ connected components (over $\mathbb{C}_p$), whose reductions can be described using Igusa curves. He was also able to give a moduli-theoretic interpretation of the horizontal components of Edixhoven's model for $X_0(p^2)$. In particular, the points on these components correspond to pairs (E, C) such that $E/C[p]$ is (in the language of [**2**, 3.3]) 'too-supersingular'; that is, such that $E/C[p]$ has no canonical subgroup.

In the first half of this paper we begin with the model for $X_0(125)$ constructed in [**9**, 4], and we apply techniques of rigid analysis to construct a semi-stable covering. From this point on, all such statements will always mean that the work takes place over R_p, the ring of integers in $\mathbb{C}_p$. Paraphrasing [**4**, 2], this means that we will find a finite set of disjoint affinoids $\{A_i\}$ with good reduction, such that the complement in $X_0(125)$ of $\bigcup A_i$ is simply the disjoint union of annuli. By [**4**, 2.1], this is equivalent to determining a semi-stable model for the curve. The goal of the remaining sections, then, is to provide a moduli-theoretic interpretation of the components, which leads to a conjecture regarding the stable model of $X_0(p^3)$. To do this, we first map the affinoids in the semi-stable covering

down to $X(1)$ via an appropriate moduli-theoretic map (using formulas from [9]). Then we show by explicit calculations that the images contain the j-invariants of curves with a certain type of complex multiplication (CM). Along with similar data for $p = 7$ and $p = 13$, this leads to two conjectures that could be considered the main results of the paper. Conjecture 3.4.1 concerns the distribution of CM curves inside a given 'ss disk' (supersingular disk) of $X(1)$, and Conjecture 3.4.2 generalizes the stable model description of $X_0(125)$ to a description of $X_0(p^3)$. These conjectures will be at least partially proven in an upcoming joint work with Robert Coleman.

2. Stable model of $X_0(125)$

In this section, we determine the stable model for $X_0(125)$ (genus 8) by essentially constructing in the language of [4] a semi-stable covering for the curve. The initial model will be the one determined in [9, 4], namely the following system of equations:

$$f_{125}^+(x, y) = y^4 - x^5 + 5xy^3 + 15x^2y^2 + 25x^3y + 25x^4 + 5y^3 \tag{1}$$
$$+ 5xy^2 - 25x^3 + 15y^2 + 25x^2 + 25y - 25x + 25 = 0;$$
$$xu^2 - yu + 5 = 0. \tag{2}$$

As modular functions, the q-expansions at infinity of x, y, and u can be expressed in terms of the Dedekind eta-function as follows:

$$u(q) = \frac{\eta(q)}{\eta(q^{25})}; \qquad x(q) = \frac{u(q^5)}{u(q)}; \qquad y(q) = u(q^5) + \frac{5}{u(q)}.$$

In [9, 4] it is shown that x and y are functions on $X_0(125)$ which are fixed by the Atkin–Lehner involution, w_{125}. Therefore, Equation (1) actually describes the genus-2 quotient curve,

$$X_0(125)^+ = X_0(125)/w_{125}.$$

Then u is a third function on $X_0(125)$ (actually, a pullback of a function on $X_0(25)$), which generates, by Equation (2), the degree-2 extension from $X_0(125)^+$ up to $X_0(125)$.

In later sections we will want to interpret the various components of our model in moduli-theoretic terms, in the hope of conjecturing what happens in general (for $X_0(p^3)$, $p \neq 5$). For this section, however, we will stick to bare-bones rigid analysis. First we show that $X_0(125)^+$ has good reduction, by finding an explicit good-reduction model. Then we show that the ten ramification points in the degree-2 extension lie in two equidistant sets of five, all within a wide-open annulus. The result of this information is that the stable model of $X_0(125)$ has five components: four of genus 2, and one of genus 0. In particular, two components with the same (stable) reduction as $X_0(125)^+$ will be switched under the Atkin–Lehner involution. The other two genus-2 affinoids are fixed by Atkin–Lehner, and have trivial quotients.

2.1. Good reduction of $X_0(125)^+$

At the end of [9] there is a proof that $X_0(125)^+$ has good reduction; this is based on a hyperelliptic model for the genus-2 curve. If we are to understand the extension up to $X_0(125)$, however, this hyperelliptic model is not optimal. For this reason we will now offer a proof using a different model – one that will be more appropriate for the full analysis of $X_0(125)$. Essentially, we will show that with respect to the parameter y, the affinoid $v_5(y) = 3/4$ is a genus-2 affinoid (in the sense of [9, 1]) with good reduction. Then, in the section that follows, we will see that the ten ramification points in the degree-2 extension

up to $X_0(125)$ are actually separated from this affinoid by a wide-open annulus. This is a key point in understanding the stable model of $X_0(125)$.

PROPOSITION 2.1.1. *The curve $X_0(125)^+$ has good reduction. More specifically, the affinoid described by $v_5(y) = 3/4$ is a genus-2 affinoid with good reduction.*

Proof. We begin by choosing any $r \in \mathbb{C}_5$ satisfying $r^5 + 25r - 25$, and making the change of variables $x_0 = x - r$. Plugging this into $f^+(x, y)$, we then obtain a polynomial $g^+(x_0, y)$ whose coefficients are polynomials in r, as given in Table 1. The entry in the x_0^ith row and the y^jth column is the coefficient of $x_0^i y^j$.

Table 1: Coefficients of $g^+(x_0, y) = f^+(x_0 + r, y)$.

	y^4	y^3	y^2	y^1	y^0
x_0^5					-1
x_0^4					$-5r + 25$
x_0^3				25	$-10r^2 + 100r - 25$
x_0^2			15	$75r$	$-10r^3 + 150r^2 - 75r + 25$
x_0^1		5	$30r + 5$	$75r^2$	$-5r^4 + 100r^3 - 75r^2 + 50r - 25$
x_0^0	1	$5r + 5$	$15r^2 + 5r + 15$	$25r^3 + 25$	$25r^4 - 25r^3 + 25r^2$

When $v_5(y) = 3/4$, the Newton polygon for g^+, considered as a polynomial in x_0, shows that $v_5(x_0) = 1/2$. Consequently, there are three terms with minimal valuation, that on their own (ignoring other terms) would form the equation

$$x_0^5 + 25x_0 = 15y^2. \tag{3}$$

This motivates a second change of variables, which will result in a good reduction model. We first choose any α and β in $\mathbb{C}_5$ with $v_5(\alpha) = 1/2$ and $v_5(\beta) = 3/4$. Then we make the following change of variables: $\alpha x_1 = x_0$ and $\beta y_1 = y$. To be completely precise, then, the final choice of model for the curve $X_0(125)^+$ in terms of the parameters x_1 and y_1 is the equation

$$\frac{1}{15\beta^2} f_{125}^+(\alpha x_1 + r, \beta y_1) = 0.$$

This equation has integral coefficients, and reduces modulo the maximal ideal of R_5 to the following equation over $\bar{\mathbb{F}}_5$:

$$y_1^2 = \frac{\alpha^5}{15\beta^2} x_1 \left(x_1^4 + \frac{25}{\alpha^4} \right). \tag{4}$$

From Equation 4 we see that (as claimed) the affinoid $v_5(y) = 3/4$ (or, equivalently, $v_5(y_1) = 0$) is a genus-2 affinoid with good reduction. In rigid terms, we have shown that the entire region $v_5(y) < 3/4$ is simply one residue disk, while $v_5(y) > 3/4$ describes five residue disks. $\square$

2.2. *Rigid distribution of ramified points*

We now turn our attention to understanding what happens to the curve $X_0(125)^+$ in the quadratic extension up to $X_0(125)$. In the light of the previous section, it is crucial that we

understand where the ramification points lie in relation to the affinoid $v_5(y) = 3/4$. Equation (2) tells us that the ramification points satisfy $y^2 = 20x$. Substituting this into Equation (1), we obtain a polynomial in y satisfied by the y-coordinates of these ten ramification points. The valuations of the coefficients of this degree-10 polynomial, say $p_{\text{ram}}(y) = \sum_{i=0}^{10} a_i y^i$, are as shown below.

i	10	9	8	7	6	5	4	3	2	1	0
$v_5(a_i)$	0	∞	3	4	4	5	5	6	6	7	7

The Newton polygon for $p_{\text{ram}}(y)$ then tells us that $v_5(y) = 7/10$ at each root. As an immediate consequence, all ten ramification points lie outside of the affinoid $v_5(y) = 3/4$. We will now show that the region described by $1/2 < v_5(y) < 3/4$ is in fact a wide-open annulus (not parameterized by y, though), and that these ten points lie in two equidistant sets of five within the circle $v_5(y) = 7/10$. The stable model of $X_0(125)$ will then follow from an argument similar to the one used in [9, 4.6].

PROPOSITION 2.2.1. *The region of $X_0(125)^+$ described by $1/2 < v_5(y) < 3/4$ is a wide-open annulus.*

Proof. The main idea here is that (on this region) the terms x_0^5 and $15y^2$ of $g^+(x_0, y)$ have minimal valuation (by Newton polygons). In other words, the curve is well approximated by the equation $x_0^5 = 15y^2$. This suggests that we should be able to parameterize the region using the annulus

$$A_s = \{s \mid 1/5 < v_5(s) < 1/4\}$$

and a map close to $x_0 = s^2$, $y = s^5/\sqrt{15}$ (for a fixed square root of 15). This can be made precise by applying a souped-up version of Hensel's lemma (see [9, 2.3]). First, set $x_0 = s^2$ exactly, and then consider the polynomial $h(y) = s^{-10}g^+(s^2, s^5 y/\sqrt{15})$. Note that the coefficients of $h(y)$ are integral-valued functions on A_s. Also, it is straightforward to check that $v_5(h(1)) > 0$ and $v_5(h'(1)) = 0$ everywhere on A_s. Therefore, there is a unique integral-valued function on A_s which is a root of $h(y)$ close to $y = 1$. To parameterize the region explicitly, then, we simply take y to be $s^5/\sqrt{15}$ times this root:

$$x_0 = s^2; \qquad y = \frac{s^5}{\sqrt{15}}(1 + \text{smaller terms on } A_s). \tag{5}$$

It is immediate that this defines a map from A_s to $X_0(125)^+$, that it is an injection, and that the image is contained in the region $1/2 < v_5(y) < 3/4$. The onto argument is a little more subtle, though. Let (x_0, y) be any point satisfying $1/2 < v_5(y) < 3/4$. The Newton polygon of g^+ as a polynomial in x_0 shows that $2/5 < v_5(x_0) < 1/2$. But then the Newton polygon of g^+ as a polynomial in y shows that only two of the four points with this x_0 coordinate satisfy the condition $1/2 < v_5(y) < 3/4$. On the other hand, there are two points with this x_0 coordinate in the image of A_s, identified simply by taking s to be either square root of x_0. Therefore the map must be onto, by this simple counting argument. $\square$

We had previously shown that the ten ramified points in the degree-2 extension up to $X_0(125)$ lay in the region $v_5(y) = 7/10$. Now we know from Equation (5) that this region is in fact a circle, parameterized by $v_5(s) = 6/25$. What we would like to do now is analyze the geometry of these ten points with respect to the parameter s. One way to do this would be to work out explicitly enough terms in the power series satisfied by the ten s-coordinates

24

to determine the relative distances. It is possible (and much easier), however, to infer the desired information from the relative distances with respect to x- and y-coordinates. This is the approach that we will take in proving the following proposition.

PROPOSITION 2.2.2. *The ten ramification points of $X_0(125)^+$ lie in two equidistant sets of five, all within the circle $v_5(s) = 6/25$, such that for all $i \neq j$,*

$$v_5(s_i - s_j) = 6/25 \quad or \quad v_5(s_i - s_j) = 13/50.$$

Proof. First move one root of $p_{\mathrm{ram}}(y)$ to 0 by a simple translation, and then look at the Newton polygon of the resulting polynomial. This polynomial has four roots of valuation $8/10$ and five of valuation $7/10$. From this, we learn that for a fixed y-coordinate of a ramified point, say y_0, we have $v_5(y - y_0) = 7/10$ at five of the ramified points, and $v_5(y - y_0) = 8/10$ at the remaining four. Actually, we know a lot more, because $y \approx s^5/\sqrt{15}$, and we are working 5-adically. Therefore, when $v_5(s_1) = v_5(s_2) = 6/25$, we have

$$v_5(s_1 - s_2) > 6/25 \iff v_5(s_1^5 - s_2^5) > 6/5 \iff v_5(y_1 - y_2) > 7/10.$$

It follows that the ten points at least break up into two sets of five under the relation $v_5(s_i - s_j) > 6/25$, but it does *not* follow that within each subset the points are equidistant. To determine this fact, we now look at the geometry of the x-coordinates.

Similar to what was done for the y-coordinates of the ten ramified points, it is straightforward to calculate the polynomial satisfied by the x-coordinates. Then, moving one root to 0, we find that the difference of any two x-coordinates has valuation precisely $1/2$. Of course, the same can then be said about the x_0-coordinates, since x and x_0 differ by a constant. So now choose any two ramified points, say with s-coordinates s_1 and s_2, satisfying $v_5(s_1 - s_2) > 6/25$. We can determine exactly how close these points are from the following formula:

$$\tfrac{1}{2} = v_5(x_0(s_1) - x_0(s_2)) = v_5(s_1^2 - s_2^2) = v_5(s_1 - s_2) + v_5(s_1 + s_2).$$

The point is that

$$v_5(s_1 - s_2) > \tfrac{6}{25} \implies v_5(s_1 + s_2) = \tfrac{6}{25} \implies v_5(s_1 - s_2) = \tfrac{13}{50}.$$

This proves the remaining part of the proposition. □

2.3. *Stable model*

In this section we finally give a complete description of a semi-stable model of $X_0(125)$. Essentially, this will be done in two steps. First, we show that lying over the two minimal affinoid disks of Proposition 2.2.2 (containing the ramification points) there are two genus-2 affinoids in $X_0(125)$ with good reduction. Since $X_0(125)^+$ has good reduction at $v_5(y) = 3/4$, and the extension is a trivial two-sheeted cover over this region, there will then be two more genus-2 affinoids in $X_0(125)$ that are isomorphic copies of this one. So just from the fact that the genus of $X_0(125)$ is 8, there can be no other nontrivial components in the stable model. However, in the main part of Proposition 2.3.2 we show that there is one component of genus 0 that meets the four genus-2 affinoids in distinct residue classes. This determines the stable model, and a figure summarizing this data along with the results of [9, 4] then concludes the section.

PROPOSITION 2.3.1. *Let s_1 be the s-coordinate of any of the ten ramification points in the extension from $X_0(125)^+$ up to $X_0(125)$. The region lying over the affinoid disk $v_5(s - s_1) \geqslant 13/50$ is a genus-2 affinoid with good reduction.*

Proof. For convenience, we reparameterize this disk with the disk described by $v_5(t) \geqslant 0$, by letting $t = \beta(s - s_1)$, where $\beta \in \mathbb{C}_5$ is anything satisfying $v_5(\beta) = -13/50$. This places one ramification point at $t = 0$, and the other four on the circle $v_5(t) = 0$. To fix notation, say the t-coordinates of the other four ramification points are $\alpha_1, \alpha_2, \alpha_3$ and α_4, where $v_5(\alpha_i) = 0$ and $v_5(\alpha_i - \alpha_j) = 0$ for $i \neq j$.

Now, we know that the quadratic extension can be obtained by taking the square root of an appropriate analytic function. For example, if we let $z = 2xu - y$, Equation (2) simply becomes $z^2 = y^2 - 20x$. The function $y^2 - 20x$ has been shown to have simple roots at the five ramification points, and no other roots (on this disk). Therefore we may rewrite Equation (2) in terms of t and z as

$$z^2 = t(t - \alpha_1)(t - \alpha_2)(t - \alpha_3)(t - \alpha_4)P(t),$$

where $P(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ is an analytic and non-vanishing function on the disk $v_5(t) \geqslant 0$. But this means that $v_5(a_0) < v_5(a_i)$ for all $i > 0$ (see, for example, [11], 6.2.2]). Therefore, by making the subsitution $z_1 = z/\sqrt{a_0}$, we arrive at a model for the quadratic extension of the disk that has the following reduction:

$$z_1^2 = t(t - \alpha_1)(t - \alpha_2)(t - \alpha_3)(t - \alpha_4). \tag{6}$$

Since the roots of the right-hand side are distinct, this is the equation for a genus-2 curve over $\bar{\mathbb{F}}_5$, which proves the proposition. $\qquad \square$

PROPOSITION 2.3.2. *The stable model of $X_0(125)$ has five components: four genus-2 components that do not intersect each other, and one genus-0 component, which intersects each of the others in exactly one place.*

Proof. By using the quadratic formula and an appropriate expansion for the square root, it is straightforward to show that the regions $v_5(y) < 7/10$ and $v_5(y) > 7/10$ of $X_0(125)^+$ split trivially in the degree-2 extension (as in [9], 4.3]). Therefore, over the affinoid $v_5(y) \geqslant 3/4$, which was shown in Section 2.1 to have good reduction and genus-2, we have two isomorphic copies in $X_0(125)$. In addition, we have just shown in Proposition 2.3.1 that there are two more genus-2 affinoids with good reduction lying over two disks within the circle $v_5(s) = 6/25$. Since the genus of the whole curve $X_0(125)$ is only 8, there can be no other non-trivial components in the stable model. All that remains, then, is to understand how these components fit together, and to see whether or not there are also genus-0 components in the stable model.

To answer this question, it suffices to look at the reduction of Equation (2) over an appropriate affinoid. So we choose ramification points s_1 and s_2 as before, with $v_5(s_1 - s_2) = 6/25$, and we define an affinoid $B \subseteq A_s$ by

$$B = \{s \in A_s \mid v_5((s - s_1)(s - s_2)) = 12/25\}.$$

In other words, B is just the circle $v_5(s) = 6/25$ minus the two residue disks containing the ramification points in the extension up to $X_0(125)$. Let $\hat{B}$ denote the affinoid of $X_0(125)$ lying over B.

Recall that on this region, $v_5(x) = 2/5$ and $v_5(x - r) > 2/5$ for a particular $r \in \mathbb{C}_5$ that satisfies $r^5 + 25r - 25 = 0$. It follows then that if we make the substitutions $s_0 = s/\alpha$ and $u_0 = u/\beta$, where $v_5(\alpha) = 6/25$ and $v_5(\beta) = 3/10$, Equation (2) reduces to the following:

$$u_0^2 - \frac{\alpha^5}{\sqrt{15}\beta r} s_0^5 u_0 + \frac{5}{\beta^2 r} = 0. \tag{7}$$
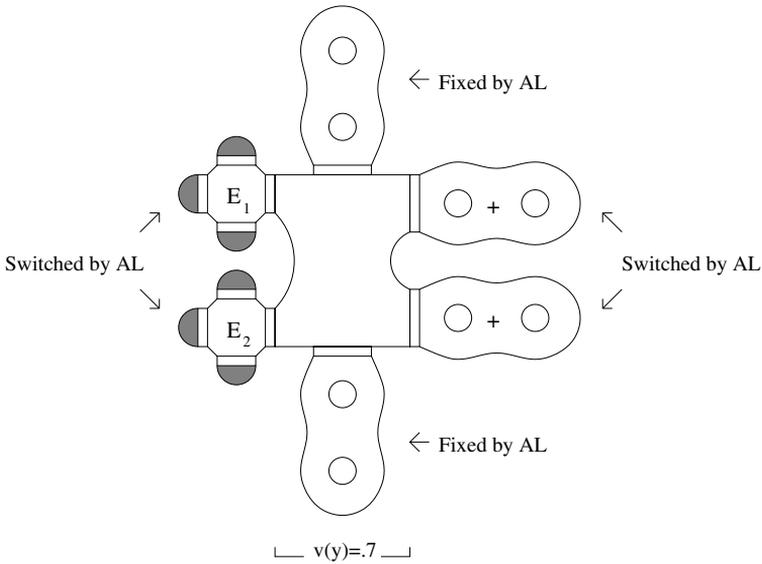
Figure 1: $X_0(125)/\mathbb{C}_5$, Semi-stable covering

Equation (7) is nonsingular over the reduction of $\hat{B}$. Indeed, the two residue disks that were removed from A_s correspond precisely to the s_0-coordinates of the two (finite) singular points over $\bar{\mathbb{F}}_5$. Therefore, $\hat{B}$ has good reduction. Furthermore, the four genus-2 affinoids meet the reduction of $\hat{B}$ in four distinct residue classes: the two singular points and the two distinct $s_0 = 0$ points. Therefore this genus-0 component cannot be blown down, and the stable model can only be as claimed. □

REMARK. It would be interesting to determine a precise field extension over which the stable model is defined, as well as the resulting action of the Galois group on the special fiber. Unfortunately, this does *not* follow from our calculations. Specifically, we do not have explicit equations for the two genus-2 components determined in Proposition 2.3.1. What we *do* know is that the ramification index of any such field must be divisible by 100, since the width of each of the four annuli bounding the genus-2 components is in fact 1/100.

NOTE 2.3.1. Figure 1 reflects all of the information from the preceding propositions regarding the stable model of $X_0(125)$. In addition, while the entire region described by $v_5(y) < 7/10$ consists of two residue disks, these two residue disks were shown in [9, 4] to contain components of great moduli-theoretic import. In particular, this region contains six trivial ordinary components, which are shaded in the figure. We will also show in the following section that the two components marked 'E_1' and 'E_2' (that is, the two components described by $v_5(y^2 - 5) = 1$) map down via appropriate moduli-theoretic maps to the unique horizontal component of Edixhoven's model for $X_0(25)$. Therefore, while not essential for a discussion of the stable model of $X_0(125)$, these components *are* essential to an understanding of the general conjecture regarding $X_0(p^3)$ that concludes the paper. For this reason, these features have also been included in the figure.

3. *Moduli-theoretic interpretation*

In this section, we begin to formulate a moduli-theoretic interpretation of the components in the semi-stable model of $X_0(125)$ shown in Figure 1. This interpretation will then provide

the basis for a conjecture regarding the stable model of $X_0(p^3)$. Philosophically, the main idea is to determine the image of each component in $X(1)$ via an appropriate map, and then to ask what special moduli-theoretic properties are held by the elliptic curves corresponding to points in that region.

3.1. *Image in $X(1)$ of $X_0(125)$ components*

To begin mapping the components of $X_0(125)$ down to $X(1)$, we first need to be very precise about which maps we are using.

DEFINITION 3.1.1. Let $M, N, d \in \mathbb{N}$ be such that $dM \mid N$. Then we define a map π_d from $X_0(N)$ to $X_0(M)$ in moduli-theoretic terms by

$$\pi_d(E, C) = (E/C[d], C[Md]/C[d]).$$

NOTE 3.1.1. The map π_d satisfies the compatibility condition, $\pi_{d_1} \circ \pi_{d_2} = \pi_{d_1 d_2}$, wherever applicable. For example, one could calculate $\pi_5 : X_0(25) \longrightarrow X(1)$ by way of either factorization: $\pi_5 = \pi_5 \circ \pi_1$, or $\pi_5 = \pi_1 \circ \pi_5$.

Up until now we have simply used u to refer to a parameter on $X_0(125)$, but in reality our u is $\pi_1^* u$, where u is a certain parameter on $X_0(25)$ and π_1 is the 'forgetful map' as above. So applying $\pi_1 : X_0(125) \longrightarrow X_0(25)$ amounts to just taking the u coordinate. We will not *need* a formula for $\pi_5 : X_0(125) \longrightarrow X_0(25)$. For lower-level moduli-theoretic maps, we will simply borrow the relevant formulas from [9]; we reproduce them for convenience in Table 2.

Table 2: Moduli-theoretic maps for analysis of $X_0(5^n)$

Moduli-theoretic map	Equation
$\pi_1 : X_0(5) \longrightarrow X(1)$	$\pi_1^*(j) = \dfrac{(t^2 + 2 \cdot 5^3 t + 5^5)^3}{t^5}$
$\pi_5 : X_0(5) \longrightarrow X(1)$	$\pi_5^*(j) = \dfrac{(t^2 + 10t + 5)^3}{t}$
$w_5 : X_0(5) \longrightarrow X_0(5)$	$w_5^*(t) = \dfrac{125}{t}$
$\pi_1 : X_0(25) \longrightarrow X_0(5)$	$\pi_1^*(t) = \dfrac{u^5}{u^4 + 5u^3 + 15u^2 + 25u + 25}$
$\pi_5 : X_0(25) \longrightarrow X_0(5)$	$\pi_5^*(t) = u(u^4 + 5u^3 + 15u^2 + 25u + 25)$
$w_{25} : X_0(25) \longrightarrow X_0(25)$	$w_{25}^*(u) = 5/u$

PROPOSITION 3.1.1. *The two components marked E_1 and E_2 in Figure 1, specifically the regions described by $v_5(y^2 - 5) = 1$, are exactly the inverse images via π_5 and π_{25} of the disk $v_5(j) \geqslant 5/2$ in $X(1)$.*

Proof. From the proof of [9, 4.4], it follows that one component, say E_1, is precisely the inverse image via π_1 of the affinoid in $X_0(25)$ described by $v_5(u^2 - 5) = 1$. The

other component, say E_2, is the inverse image via π_1 of the circle $v_5(u) = 1/10$. To interpret E_1, we first apply $\pi_5 : X_0(25) \longrightarrow X_0(5)$ (using the formula from Table 2) to see that E_1 is precisely the inverse image of the circle $v_5(t) = 5/2$. Then, by composing with $\pi_1 : X_0(5) \longrightarrow X(1)$, we find that E_1 is actually the inverse image via π_5 of the disk $v_5(j) \geqslant 5/2$ inside $X(1)$. Similarly, we interpret E_2 by applying π_{25}. First, E_2 is the inverse image via π_5 of the circle $v_5(t) = 1/2$ inside $X_0(5)$. Then, by composing with π_5 again, we conclude that E_2 is actually the inverse image via π_{25} of that same disk, $v_5(j) \geqslant 5/2$. □

PROPOSITION 3.1.2. *The region $v_5(y) \geqslant 7/10$ in $X_0(125)$ maps via π_5 onto the circle $v_5(j) = 3/2$ in $X(1)$. The images of the four genus-2 affinoids lie in four distinct residue classes within this circle, specifically described by $v_5(j^4 - 5^6) > 6$.*

Proof. From Equations (1) and (2) it follows that $v_5(y) \geqslant 7/10$ is exactly the inverse image in $X_0(125)$ via π_1 of the region $v_5(u) = 3/10$. Then, from the formula for $\pi_5 : X_0(25) \longrightarrow X_0(5)$, this is the entire inverse image of the circle $v_5(t) = 3/2$ in $X_0(5)$. By the formula for $\pi_1 : X_0(5) \longrightarrow X(1)$, this in turn maps onto the circle $v_5(j) = 3/2$ inside $X(1)$. Furthermore, since the maps are approximately $\pi_5^* t = u^5$ and $\pi_1^* j = t$ when restricted to the circles $v_5(u) = 3/10$ and $v_5(t) = 3/2$, the composition map is one-to-one on residue classes. Now, in the proof of Proposition 2.3.2 (see Equation (7)), it was shown that π_1 maps the four genus-2 components into four distinct residue disks within the circle $v_5(u) = 3/10$. Therefore, the only issue remaining is *which* four residue disks inside the circle $v_5(j) = 3/2$ contain the images.

Recall that two of the affinoids contain the ten ramification points in the extension up from $X_0(125)^+$. This set of points maps via π_5 onto the two points of $X_0(5)$ satisfying the equation $t^2 = 125$. Therefore the images of these two affinoids in $X_0(5)$ lie in the two residue disks described by $v_5(t^2 - 125) > 3$. Of course this means that the affinoids map via π_5 into the residue disks $v_5(j^2 - 125) > 3$ inside of $X(1)$. Now we look at the other two affinoids, which by Equation (7) land via π_1 inside the disks described by $v_5(u^2 + 5/r) > 3/5$, where r was a root of the equation $r^5 + 25r - 25 = 0$. Doing some quick arithmetic, we find that this is equivalent to

$$v_5(u^{10} + 5^5/r^5) > 3 \Longleftrightarrow v_5(u^{10} + 5^3) > 3.$$

Then, applying first $\pi_5 : X_0(25) \longrightarrow X_0(5)$ and then $\pi_1 : X_0(5) \longrightarrow X(1)$, we see that these disks map onto the two disks of $X(1)$ described by $v_5(j^2 + 125) > 3$. Therefore, all four residue disks within the circle $v_5(j) = 3/2$ can now be described by $v_5(j^4 - 5^6) > 6$, as claimed. □

NOTE 3.1.2. From the formulas for w_5 and $\pi_1 : X_0(5) \longrightarrow X(1)$, the circle $v_5(t) = 3/2$ is the Atkin–Lehner circle of $X_0(5)$ (fixed by w_5), and $v_5(j) = 3/2$ is its image via the forgetful map.

3.2. *Computing the too-supersingular region of $X(1)$*

In the previous section we defined two affinoids of $X_0(125)$, E_1 and E_2, which mapped via π_5 and π_{25} respectively onto the disk $v_5(j) \geqslant 5/2$ inside of $X(1)$. It is fairly straightforward to provide a moduli-theoretic description for the points lying in this disk. In particular, we will now show that this disk consists of all points corresponding to an elliptic curve E which is in the language of [2] too-supersingular – that is, a curve that has no canonical subgroup.

PROPOSITION 3.2.1. *The curve $E/\mathbb{C}_5$ is too-supersingular if and only if $v_5(j(E)) \geqslant 5/2$.*

Proof. Parameterize the supersingular disk using the disk $v_5(t) > 0$ and the map that takes each t to the j-invariant of the following curve:

$$E_t : y^2 = x^3 + tx + 1; \qquad j(t) = \frac{6912t^3}{4t^3 + 27}.$$

Strictly speaking, this is a degree-3 covering of the supersingular disk, ramified only at $j = 0$.

   Working out the (degree-12) polynomial for the x-coordinates of $E_t[5]$, we see that the Newton polygon has vertices $\{(0, 0), (10, v_5(t)), (12, 1)\}$ when $v_5(t) < 5/6$, and $\{(0, 0), (12, 1)\}$ otherwise. If we take $z = x/y$ to be a parameter at infinity (containing all of $E_t[5]$ since E_t is supersingular), this translates to the following information. When $v_5(t) < 5/6$ we have $v_5(z) = v_5(t)/20$ for twenty of the points in $E_t[5]$, and

$$v_5(z) = \frac{1 - v_5(t)}{4} > \frac{v_5(t)}{20}$$

for the other four points (the canonical subgroup, along with $z = 0$). When $v_5(t) \geqslant 5/6$, we have $v_5(z) = 1/24$ for all nonzero points of $E_t[5]$, and therefore there is no canonical subgroup. This proves the claim, since $v_5(j) = 3v_5(t)$. $\qquad \square$

NOTE 3.2.1. Robert Coleman has shown in [3] that the unique horizontal component (for a given supersingular curve mod p) of Edixhoven's model for $X_0(p^2)$ is exactly the inverse image via π_5 of the corresponding too-supersingular disk. This justifies the choice of the names E_1 and E_2, since it now follows from Propositions 3.1.1 and 3.2.1 that these two components are simply π_1^{-1} and π_5^{-1} of Edixhoven's horizontal component for $X_0(25)$.

3.3. *Placement data for CM elliptic curves in $X(1)$*

   From Propositions 3.1.1 and 3.2.1, we have a clear way to interpret the components E_1 and E_2 in the semi-stable model for $X_0(125)$. In particular, the component E_1 simply contains all points corresponding to pairs (E, C) such that $E/C[5]$ is too-supersingular. Likewise, the component E_2 simply contains all points corresponding to pairs (E, C) such that $E/C[25]$ is so. Similarly, there is a clear moduli-theoretic interpretation for the union of the components of Proposition 3.1.2, as the entire region $v_5(y) \geqslant 7/10$ has been shown to be precisely π_5^{-1} of the Atkin–Lehner circle of $X_0(5)$. What is still unclear at this time is how to interpret the four special disks inside of the circle $v_5(j) = 3/2$, described by $v_5(j^4 - 5^6) > 6$, which contain the images of the four genus-2 affinoids of $X_0(125)$. In this section, we attempt to answer that question with empirical data placing certain types of 'CM points' in those disks. We begin by making a conjecture regarding the placement of the j-invariants of CM curves $E/\mathbb{C}_5$ such that $\text{End}(E) \otimes \mathbb{Z}_5$ is the maximal order in a ramified (quadratic) extension of $\mathbb{Q}_5$.

CONJECTURE 3.3.1. *Suppose that $E/\mathbb{C}_5$ is an elliptic curve with CM.*
   (1) *If $\text{End}(E) \otimes \mathbb{Z}_5 \cong \mathbb{Z}_5[\sqrt{-5}]$, then $v_5(j(E)^2 - 125) > 3$.*
   (2) *If $\text{End}(E) \otimes \mathbb{Z}_5 \cong \mathbb{Z}_5[\sqrt{-10}]$, then $v_5(j(E)^2 + 125) > 3$.*

   To provide empirical evidence in support of the conjecture, we have worked out explicitly the j-invariants of various curves with these two types of CM. For a given endomorphism ring R, this is done by first determining explicit representatives for the ideal class group of R (using [8, 8 Section 1] for non-maximal orders). Then, writing each representative as $c(\mathbb{Z} + \tau\mathbb{Z})$, we use the usual q-expansion formula to approximate $j(\tau)$ sufficiently well.

From the theory of CM curves, these j-invariants (for a fixed R) are conjugate algebraic integers, so that the polynomial with these roots is monic and has integer coefficients. The conjecture is verified, then, if this polynomial in j is sufficiently close 5-adically to a power of $j^2 - 125$ (Case 1) or $j^2 + 125$ (Case 2). For easy verification of a few of the most basic examples (and to make the conjecture more concrete), we have included some complete sets of τ values in Tables 3 and 4.

Table 3: Examples for Case 1 of Conjecture 3.3.1

End(E)	Values of τ
$\mathbb{Z}[\sqrt{-5}]$	$\tau = \sqrt{-5}, \frac{1+\sqrt{-5}}{2}$
$\mathbb{Z}[2\sqrt{-5}]$	$\tau = 2\sqrt{-5}, \frac{2(1+\sqrt{-5})}{3}, \frac{2(2+\sqrt{-5})}{3}, \frac{2\sqrt{-5}}{5}$
$\mathbb{Z}[3\sqrt{-5}]$	$\tau = 3\sqrt{-5}, \frac{3(1+\sqrt{-5})}{2}, \frac{3\sqrt{-5}}{5}, \frac{3(3+\sqrt{-5})}{7}$
$\mathbb{Z}[\sqrt{-30}]$	$\tau = \sqrt{-30}, \frac{\sqrt{-30}}{2}, \frac{\sqrt{-30}}{3}, \frac{\sqrt{-30}}{5}$
$\mathbb{Z}\left[\frac{1+\sqrt{-55}}{2}\right]$	$\tau = \frac{1+\sqrt{-55}}{2}, \frac{1+\sqrt{-55}}{4}, \frac{-1+\sqrt{-55}}{4}, \frac{5+\sqrt{-55}}{10}$
$\mathbb{Z}[\sqrt{-70}]$	$\tau = \sqrt{-70}, \frac{\sqrt{-70}}{2}, \frac{\sqrt{-70}}{5}, \frac{\sqrt{-70}}{7}$

Table 4: Examples for Case 2 of Conjecture 3.3.1

End(E)	Values of τ
$\mathbb{Z}[\sqrt{-10}]$	$\tau = \sqrt{-10}, \frac{1}{2}\sqrt{-10}$
$\mathbb{Z}[2\sqrt{-10}]$	$\tau = 2\sqrt{-10}, \frac{2\sqrt{-10}}{5}, \frac{2(2+\sqrt{-10})}{7}, \frac{2(1+\sqrt{-10})}{11}$
$\mathbb{Z}\left[\frac{1+\sqrt{-15}}{2}\right]$	$\tau = \frac{1+\sqrt{-15}}{2}, \frac{1+\sqrt{-15}}{4}$
$\mathbb{Z}[\sqrt{-15}]$	$\tau = \sqrt{-15}, \frac{\sqrt{-15}}{3}$
$\mathbb{Z}\left[\frac{1+\sqrt{-35}}{2}\right]$	$\tau = \frac{1+\sqrt{-35}}{2}, \frac{5+\sqrt{-35}}{6}$
$\mathbb{Z}[\sqrt{-65}]$	$\tau = \sqrt{-65}, \frac{1+\sqrt{-65}}{2}, \frac{1+\sqrt{-65}}{3}, \frac{-1+\sqrt{-65}}{3},$ $\frac{\sqrt{-65}}{5}, \frac{1+\sqrt{-65}}{6}, \frac{-1+\sqrt{-65}}{6}, \frac{5+\sqrt{-65}}{10}$

If true, the preceding conjecture would provide at least the beginning of a moduli-theoretic description of the components in the stable model for $X_0(125)$. However, at the end of the paper we will seek to propose a general conjecture for the stable model of $X_0(p^3)$. Therefore, we will want to know how Conjecture 3.3.1 generalizes to other primes. For this reason, we have also checked a number of examples for $p = 7$ and $p = 13$, and have found that the data consistently supports Conjectures 3.3.2 and 3.3.3. All three conjectures will be tied together with a general conjecture in the next and final section of the paper.

CONJECTURE 3.3.2. *Suppose that $E/\mathbb{C}_7$ is an elliptic curve with CM. Let*

$$j_0(E) = j(E) - 1728.$$

(1) *If $End(E) \otimes \mathbb{Z}_7 \cong \mathbb{Z}_7[\sqrt{-7}]$, then $v_7(j_0(E)^4 - 7^4) > 4$.*
(2) *If $End(E) \otimes \mathbb{Z}_7 \cong \mathbb{Z}_7[\sqrt{-21}]$, then $v_7(j_0(E)^4 + 7^4) > 4$.*

CONJECTURE 3.3.3. *Suppose that $E/\mathbb{C}_{13}$ is an elliptic curve with CM. Let*

$$j_0(E) = j(E) - 5.$$

(1) *If $End(E) \otimes \mathbb{Z}_{13} \cong \mathbb{Z}_{13}[\sqrt{-13}]$, then $v_{13}(j_0(E)^{14} - 13^7) > 7$.*
(2) *If $End(E) \otimes \mathbb{Z}_{13} \cong \mathbb{Z}_{13}[\sqrt{-26}]$, then $v_{13}(j_0(E)^{14} + 13^7) > 7$.*

3.4. *Conjectural moduli-theoretic interpretation*

For a conjecture regarding the components in the stable model of $X_0(p^3)$, it is at least clear what 'types' of components are suggested by the $X_0(125)$ data. We will naturally conjecture that in general, for each supersingular elliptic curve (at least defined over $\mathbb{F}_p$), we have components that look like E_1, E_2, the four genus-2 components, and the trivial component that intersects the others. In other words, we will conjecture that there are components that have the same moduli-theoretic properties that the components in our semi-stable covering of $X_0(125)$ have been either proven or conjectured to have. Of the various ways in which to improve such a conjecture, one way would certainly be to conjecture the number of such components, and to give an explanation of the number. This we will do with the help of Lemma 3.4.1. Conjecture 3.4.1 uses the lemma to generalize the conjectures of the previous section regarding the placement of CM j-invariants. Conjecture 3.4.2 then takes this into account in describing the stable model of $X_0(p^3)$. Finally, we conclude the paper with a guess that goes one step farther and predicts the genera of the components.

DEFINITION 3.4.1. Choose $\alpha \in \mathbb{F}_p$ a quadratic non-residue, and $i \in \mathbb{F}_{p^2}$ with $i^2 = \alpha$. Then we define a finite $\mathbb{F}_p$-algebra

$$\bar{A}_p = \mathbb{F}_p[i, \varepsilon_j, \varepsilon_k],$$

where $\varepsilon_j \varepsilon_k = \varepsilon_k \varepsilon_j = \varepsilon_j^2 = \varepsilon_k^2 = 0$ and $i\varepsilon_j = \varepsilon_k = -\varepsilon_j i$.

LEMMA 3.4.1. $\mathbb{F}_{p^2}^*$ *acts on the nilradical $\mathcal{N}(\bar{A}_p) = \{c\varepsilon_j + d\varepsilon_k\}$ so that:*

(1) $Sta(x) = \mathbb{F}_p^*$, *and consequently $|\operatorname{Orb}(x)| = p + 1$ for all $x \neq 0$;*
(2) *for all $x_1 = c_1 \varepsilon_j + d_1 \varepsilon_k$ and $x_2 = c_2 \varepsilon_j + d_2 \varepsilon_k$,*

$$x_1 \sim x_2 \iff c_1^2 - \alpha d_1^2 = c_2^2 - \alpha d_2^2.$$

Proof. The action is simply conjugation, after identifying $\mathbb{F}_{p^2}$ with $\mathbb{F}_p[i]$ of course. Now, for part (1), let $x = c\varepsilon_j + d\varepsilon_k$ and $a + bi \in Sta(x)$. Then, by definition,

$$(a + bi)(c\varepsilon_j + d\varepsilon_k) \frac{a - bi}{a^2 - \alpha b^2} = c\varepsilon_j + d\varepsilon_k.$$

Solving and setting ε_j and ε_k coefficients equal, this leads to the following system of linear equations in c and d:

$$(2b^2\alpha)c + (2ab\alpha)d = 0;$$
$$(2ab)c + (2b^2\alpha)d = 0.$$

When $b = 0$ (that is, when $a + bi \in \mathbb{F}_p^*$), the system is trivially satisfied for all (c, d). When $b \neq 0$, however, the determinant is $4b^2\alpha(b^2\alpha - a^2) \neq 0$ and therefore $x = 0$ is the only solution.

Similarly, it is a straightforward exercise in congruences to verify half of Part (2), namely that $x_1 \sim x_2$ implies that $c_1^2 - \alpha d_1^2 = c_2^2 - \alpha d_2^2$. This means that as soon as $c^2 - \alpha d^2 = k \neq 0$

has one solution, it must have at least $p + 1$ solutions, by part (1). But $c^2 - \alpha d^2 = k$ always has a solution, since the set $\{\alpha d^2 + k\}$ has $(p + 1)/2$ elements and therefore must contain a quadratic residue. Therefore we are done, since there are $p - 1$ choices for $k \neq 0$, $p + 1$ solutions to $c^2 - \alpha d^2 = k$, and only $p^2 - 1$ nonzero elements of $\mathcal{N}(\bar{A}_p) - \{0\}$ to begin with. □

The reason that this lemma is relevant is as follows. Let A be a supersingular curve so that $\mathrm{End}(A) \otimes \mathbb{F}_p \cong \bar{A}_p$. The isomorphism is non-canonical, but it suffices to choose an isomorphism once and for all. Now suppose that $E/\mathbb{C}_p$ is a CM curve with $p \,||\, \mathrm{Disc}(\mathrm{End}(E))$ and such that $\bar{E} \cong A$. Identify $\mathrm{End}(E)$ with a subring of $\mathbb{C}_p$ via the canonical embedding coming from the action of $\mathrm{End}(E)$ on holomorphic differentials. Then, for every isomorphism $\lambda : \bar{E} \longrightarrow A$, we obtain an embedding $\sigma_\lambda : \mathrm{End}(E) \longrightarrow \mathrm{End}(A)$. Furthermore, part (2) of Lemma 3.4.1 implies that for any uniformizer $u \in \mathrm{End}(E)$, the conjugacy class of $\sigma_\lambda(u)$ inside $\mathrm{End}(A) \otimes \mathbb{F}_p \cong \bar{A}_p$ is independent of λ. So without taking λ into account, there are only $p + 1$ options for the image of $\sigma_\lambda(u)$ inside $\bar{A}_p$. To illustrate this point, consider the following example.

EXAMPLE 3.4.1. Let $E/\mathbb{C}_7$ be a curve with $\mathrm{End}(E) = \mathbb{Z}[\sqrt{-7}]$, and let A be the unique supersingular elliptic curve in characteristic 7. From [10, 5.1] we may take $\mathrm{End}(A)$ to be a maximal order in the quaternion algebra $\mathbb{Q}[i, j, k]$ with $i^2 = -1$, $j^2 = -7$ and $ij = -ji = k$. Let u be the uniformizer $2\sqrt{-7} \in \mathrm{End}(E)$, and suppose that for a particular isomorphism $\lambda : \bar{E} \longrightarrow A$ we have $\sigma_\lambda(u) = a + bi + cj + dk$. Since

$$(a + bi + cj + dk)^2 = a^2 - b^2 - 7c^2 - 7d^2 + 2a(bi + cj + dk) = -4 \cdot 7,$$

we must immediately have $a = 0$ and $b \equiv 0 \pmod{7}$. Therefore we have $c^2 + d^2 \equiv 4$, which means that for this example the image of $\sigma_\lambda(u)$ inside $\bar{A}_7$ must lie in the conjugacy class (of eight elements) containing

$$\{\pm 2\varepsilon_j, \pm 2\varepsilon_k, \pm 3\varepsilon_j \pm 3\varepsilon_k\}.$$

However, the precise image of $\sigma_\lambda(u)$ is *not* independent of λ. So we cannot yet use this image to put an equivalence relation on the set of curves with a given endomorphism ring. To obtain an invariant that is independent of λ, we first note that $\mathrm{Aut}(A)$ acts transitively on the set of isomorphisms from $\mathrm{End}(\bar{E})$ to $\mathrm{End}(A)$ via conjugation, with ± 1 as the kernel in all cases. So, letting $i(A) = |\mathrm{Aut}(A)|/2$, we see that the conjugacy class defined by u inside of $\bar{A}_p$ can be broken down into $(p + 1)/i$ subsets, so that the *subset* containing the image of $\sigma_\lambda(u)$ is now independent of λ. Again, we illustrate this point by revisiting the previous example.

EXAMPLE 3.4.2. The unique supersingular curve A in characteristic 7 has $|\mathrm{Aut}(A)| = 4$. In terms of the above identification, the automorphism group is generated by the (invertible) element i. So, for example, if $\sigma_\lambda(u) \equiv 2j$ for some λ, then $\sigma_{i\lambda}(u) \equiv i(2j)i^{-1} = -2j$. This means that if we want to associate to u an element of $\bar{A}_7$ that is *independent* of λ, we *cannot* distinguish between $\pm 2\varepsilon_j$. Similarly, conjugation by i breaks down the entire set of eight elements into four subsets of order 2, namely

$$\{\pm 2\varepsilon_j\}, \{\pm 2\varepsilon_k\}, \{\pm(3\varepsilon_j - 3\varepsilon_k)\}, \text{ and } \{\pm(3\varepsilon_j + 3\varepsilon_k)\}.$$

So, independent of λ, it makes sense to say that there are $4 = (7 + 1)/2$ options for the image of $\sigma_\lambda(u)$ in $\bar{A}_7$.

Six ordinary components

$\pi_p^{-1}(\mathrm{AL})$

E_1

E_2

π_p^{-1} of discs in AL
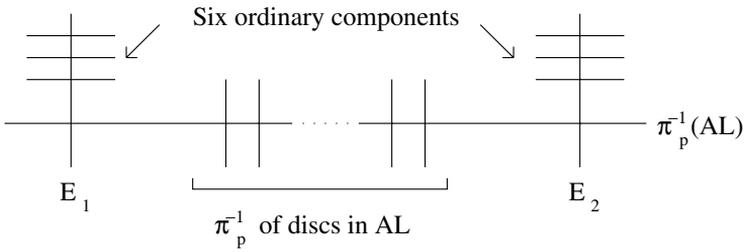
Figure 2: Conjectural partial graph of $X_0(p^3)$ stable reduction

The result of the preceding argument is that we have shown how to put an equivalence relation with $(p+1)/i$ (possibly empty) classes on the set of CM curves E, reducing to a fixed supersingular curve A and with a fixed endomorphism ring such that $p\,||\,\mathrm{Disc}(\mathrm{End}(E))$. Indeed, with the same argument we may even generalize the relation by requiring only that $\mathrm{End}(E)\otimes\mathbb{Z}_p$ be fixed. Since there are only two ramified quadratic extensions of $\mathbb{Q}_p$, this makes a total of $2(p+1)/i$ classes into which all such CM curves must fall. What we have *not* shown, however, is that this equivalence relation is reflected somehow in the geometric placement of the j-invariants of these CM curves inside of $X(1)$. So we have provided evidence for Conjecture 3.4.1, which would explain Conjectures 3.3.1, 3.3.2 and 3.3.3 with a general theory. However, we do not yet have a proof of the result at this time.

CONJECTURE 3.4.1. *Let A be a supersingular elliptic curve defined over $\mathbb{F}_p$, and let $i = i(A) = |Aut(A)|/2$.*

(1) *The j-invariants of all CM curves $E/\mathbb{C}_p$ such that $\bar{E} = A$ and*

$$p\,||\,\mathrm{Disc}(\mathrm{End}(E))$$

lie in $2(p+1)/i$ residue disks inside the corresponding Atkin–Lehner circle, $(p+1)/i$ for each ramified quadratic extension of $\mathbb{Q}_p$.

(2) *Two such curves, E_1 and E_2, lie in the same residue disk if and only if*

$$\mathrm{End}(E_1)\otimes\mathbb{Z}_p = \mathrm{End}(E_2)\otimes\mathbb{Z}_p$$

and for any uniformizer $u \in \mathbb{C}_p$ of the common image we have

$$\sigma_{\lambda_1}(u) \equiv \sigma_{\lambda_2}(u) \in \bar{A}_p, \quad \text{for some } \lambda_i : \bar{E}_i \longrightarrow A, \ i = 1, 2.$$

CONJECTURE 3.4.2. *There is a semi-stable covering of $X_0(p^3)$ defined over R_p such that for each supersingular elliptic curve A defined over $\mathbb{F}_p$ there is a connected component of the supersingular locus containing (only) the following:*

(1) *one component lying via π_p over the Atkin–Lehner circle (of A);*

(2) *two components, E_1 and E_2, which are π_p^{-1} and $\pi_{p^2}^{-1}$ of the too-supersingular disk (of A); and*

(3) *$2(p+1)/i$ components lying via π_p over the CM disks of Conjecture 3.4.1.*

The intersections of these components with each other, and with the six ordinary components of $X_0(p^3)$ are as pictured in Figure 2.

It is important to acknowledge that Conjecture 3.4.2 is based on just one very well-understood example. Ironically, however, if we make a more precise conjecture, it is easy to
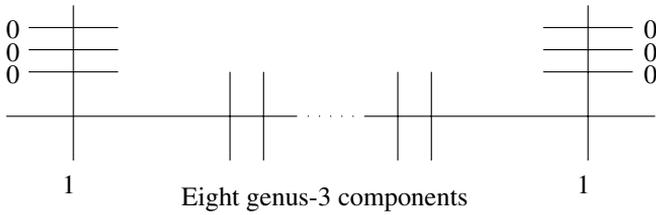
Figure 3: Conjectural graph of $X_0(7^3)$ semi-stable model, $g = 26$.
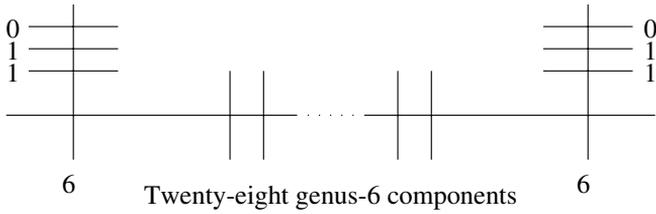
Figure 4: Conjectural graph of $X_0(13^3)$ semi-stable model, $g = 184$.
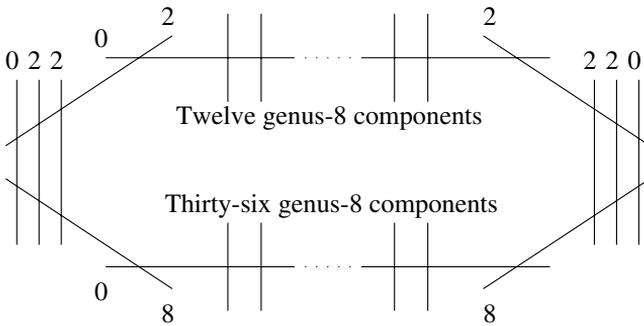
Figure 5: Conjectural graph of $X_0(17^3)$ semi-stable model, $g = 417$.

obtain far more corroborating data. In particular, from looking more closely at the $X_0(125)$ example, one might go so far as to hazard the following guess.

GUESS. The component lying over AL is trivial, E_1 and E_2 are analytically isomorphic copies of Edixhoven's horizontal component, and each component lying over a CM disk has genus $(p - 1)/2$ (and is hyperelliptic).

These statements also hold for $X_0(125)$, and with the guess it becomes possible to generate complete graphs of conjectural semi-stable models for $X_0(p^3)$. The genera of E_1 and E_2 each come directly from [6, 2.5]. The genera of the ordinary components come from [6, 2.5] and the fact (from [3, Section 1, the '*coup de grâce*']) that the four nontrivial ordinary components of $X_0(p^3)$ are isomorphic copies of the two nontrivial ordinary components of $X_0(p^2)$. Using this approach, we have generated graphs for $X_0(7^3)$, $X_0(13^3)$, and $X_0(17^3)$ (see Figures 3, 4, and 5). In each of these cases, and in fact even in the *general* case, it is

easy to show that the total genus of the curve is at least correct. While it is perhaps too early to make the guess an official conjecture, the corroborating data of this genus calculation seems to be very promising.

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