Shinzo Watanabe and Toitsu Watanabe
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# CONVERGENCE OF ISOTROPIC SCATTERING TRANSPORT PROCESS TO BROWNIAN MOTION

SHINZO WATANABE*) AND TOITSU WATANABE**)

## Introduction

Let us consider transporting particle in the $n$-dimensional Euclidian space $R^n$. It is assumed that a particle originating at a point $x \in R^n$ moves in a straight line with constant speed $c$ and continues to move until it suffers a collision. The probability that the particle has a collision between $t$ and $t + \Delta$ is $k\Delta + o(\Delta)$, where $k$ is constant. When a particle has a collision, say at $y$ in $R^n$, it moves afresh from $y$ with an isotropic choice of direction independent of past history.

It has been proved that, when $c$ and $k$ grows up indefinitely under the relation $k/c^2 = 2/n + o(1)$, the distribution of a particle converges weakly to that of Brownian motion for the one-dimensional case by N. Ikeda and H. Nomoto [2] (cf. M.A. Pinsky [4]), and for the two-dimensional case by To. Watanabe [6] (cf. A.S. Monin [3]).

The purpose of this paper is to show that the same result is also valid for the multi-dimensional case.

In section 1, we shall define the $n$-dimensional transport process with speed $c$. In section 2, we investigate the resolvent and its Fourier transform. In section 3, using the result of section 2, we shall show that the distribution of transport process with speed $c$ converges to that of the Brownian motion as $c \to \infty$ under the assumption: $k/c^2 = 2/n + o(1)$ (Theorem 1). In section 4, we shall show that the transport process with speed $c$ converges weakly to the Brownian motion, considering them as the measures on the space $\mathscr{C}$ of continuous functions.

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*) Department of Mathematics, Kyoto University.
**) Department of Applied Physics, Nagoya University.

## §1. Definition of transport process.

Let $S^{n-1}$ be the $(n-1)$-dimensional unit sphere in the $n$-dimensional Euclidian space $R^n$ and let $\sigma$ be the uniform probability measure on $(S^{n-1}, \boldsymbol{B}(S^{n-1}))$. We shall denote by $S$ the product space of $R^n$ and $S^{n-1}$, and $\boldsymbol{B}(S)$ the topological Borel field.

Let $\Theta = [\theta(t), +\infty, P_\theta, \theta \in S^{n-1}]$ be a right continuous strong Markov process of pure-jump type on $S^{n-1}$ defined by the following conditions:

(i) $P_\theta(\theta(t) = \theta,\ t < \tau \mid t < \tau) = 1;$

(ii) $P_\theta(\tau > t) = e^{-kt};$

(iii) $P_\theta(\theta(\tau) \in \Gamma) = \sigma(\Gamma),\ \Gamma \in \boldsymbol{B}(S^{n-1});$

where $\tau = \inf\{t : \theta(t) \neq \theta(0)\}$ and $k$ is a positive constant. The formula

$$A(t) = \int_0^t \theta(s)\, ds \tag{1.1}$$

defines $R^n$-valued continuous additive functional of $\Theta$. For each point $(x, \theta) \in S$, we define the following:

$$x^{(x,\theta)}(t) = (x + cA(t),\ \theta(t)); \tag{1.2}$$

$$P^{(x,\theta)}(x^{(x,\theta)}(t) \in B) = P_\theta(x^{(x,\theta)}(t) \in B),\quad B \in \boldsymbol{B}(S). \tag{1.3}$$

Then it is easy to see that $\boldsymbol{X}^{(x,\theta)} = [X^{(x,\theta)}(t), +\infty, P^{(x,\theta)}, (x,\theta) \in S]$ is a system of Markov family of random functions. Hence there corresponds the strong Markov process $\boldsymbol{X} = [X(t) = (x(t), \theta(t)), +\infty, P_{(x,\theta)}, (x,\theta) \in S]$ (cf. [1]).

DEFINITION 1.1. We call the Markov process $\boldsymbol{X}$ *the $n$-dimensional isotropic scattering transport process with speed $c$*, or simply, *the transport process with speed $c$*.

## §2. Fourier transform of resolvent operator.

Let us first introduce some spaces of functions defined on $S$ and $R^n$.

$C_0(S)$ be the space of continuous functions on $S$ such that $\lim_{|x| \to \infty} f(x, \theta) = 0$ with sup-norm $\|\cdot\|$.

$L^1(S)$ be the space of integrable functions on $S$ with norm $\|f\|_{L^1} = \int_S |f(x,\theta)|\, dx\, d\sigma(\theta)$.

$L^2(S)$ be the Hilbert space of square-integrable functions on $S$.

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$C_0(R^n)$ be the space of continuous functions on $R^n$ such that $\lim_{|x|\to\infty} F(x)=0$ with sup-norm $|||\cdot|||$.

$L^1(R^n)$ be the space of integrable functions on $R^n$ with norm $|||F|||_{L^1} = \int_{R^n} |F(x)|dx$.

$L^2(R^n)$ be the Hilbert space of square-integrable functions on $R^n$.

Let us denote by $T_t$, $t \geq 0$, the semigroup corresponding to the transport process $\boldsymbol{X}$, i.e.,

$$T_t f(x, \theta) = E_{(x,\theta)}[f(x(t))], \tag{2.1}$$

where $E_{(x,\theta)}$ is the expectation with respect to $P_{(x,\theta)}$-measure and by $R_\lambda$, $\lambda > 0$, the resolvent of $T_t$. Then we have

LEMMA 2.1. (i) *$T_t f$, $f \in C_0(S)$ or $L^1(S)$ or $L^2(S)$, is a solution of the following integral equation*:

$$u(t, x, \theta) = f(x + c\theta t, \theta)e^{-kt} + k\int_0^t e^{-ks}\, ds \int_{S^{n-1}} u(t-s,\ x + c\theta s, \vartheta) d\sigma(\vartheta). \tag{2.2}$$

*Moreover*, $\|T_t f\| \leq \|f\|$ *or* $\|T_t f\|_{L^1} \leq \|f\|_{L^1}$ *or* $\|T_t f\|_{L^2} \leq \|f\|_{L^2}$.

*Let $A$ with domain $D(A)$ be the infinitesimal generator of $T_t, t \geq 0$, in $L^2(S)$ and let $T_t^*$, $A^*$ be the adjoint of $A$ and $T_t$, respectively.*

(ii) $$D(A) = \{f \in L^2(S) : \langle\theta, \text{gard } f\rangle \in L^2(S)\} \text{ and}$$

$$A\, f(x, \theta) = c\langle\theta,\ \text{grad } f\rangle - k\ f(x, \theta) + k\int_{S^{n-1}} f(x, \vartheta)\, d\sigma(\vartheta), \tag{2.3}$$

*where* $\langle\theta,\ \text{grad } f\rangle = \sum_{i=1}^n \theta_i \frac{\partial}{\partial x_i} f(x, \theta)$, $\theta = (\theta_1, \cdots, \theta_n)$

(iii) $$D(A^*) = D(A) \text{ and}$$

$$A^* f(x, \theta) = -c\langle\theta,\ \text{grad } f\rangle - k\ f(x, \theta) + k\int_{S^{n-1}} f(x, \vartheta)\, d\sigma(\vartheta). \tag{2.4}$$

(iv) *If we put* $f^*(x, \theta) = f(x, -\theta)$, *then* $T_t^* f(x, \theta) = T_t f^*(x, -\theta)$.

*Proof.* It follows from the strong Markov property of $\boldsymbol{X}$ that $T_t f$ is a solution of (2.2) and the boundedness of $T_t f$ follows by solving the equation (2.2) with the method of succesive approximation. (ii) follows from (i), and (iii) follows from (ii) by integration by parts.

According to the general theory on semigroup, $T_t f$, $f \in D(A)$, is a unique solution of

$$\begin{cases} \dfrac{\partial}{\partial t} u = Au = c\langle \theta, \text{ grad } u\rangle - ku + k\displaystyle\int_{S^{n-1}} u(\vartheta)\ d\sigma(\vartheta) \\ u(t) \to f \quad \text{as} \quad t \to 0. \end{cases}$$

Putting $u^*(t, x, \theta) = T_t f^*(x, -\theta)$, we get

$$\begin{cases} \dfrac{\partial}{\partial t} u^* = -c\langle \theta, \text{ grad } u^*\rangle - ku^* + k\displaystyle\int u^*(\vartheta)\ d\sigma(\vartheta) \\ u^*(t) \to f \quad \text{as} \quad t \to 0. \end{cases}$$

On the other hand, $T_t^* f$, $f \in D(A^*)$, is a unique solution of

$$\begin{cases} \dfrac{\partial}{\partial t} v = A^* v = -c\langle \theta, \text{ grad } v\rangle - kv + k\displaystyle\int v(\vartheta)\ d\sigma(\vartheta) \\ v(t) \to f \quad \text{as} \quad t \to 0. \end{cases}$$

Hence $T_t^* f(x, \theta) = T_t f^*(x, -\theta)$. Thus we complete the proof.

REMARK. If $f(x, \theta)$ is a function independent of $\theta$, then $T_t^* f(x, \theta) = T_t f(x, -\theta)$.

Let $P$ be the mapping defined by

$$F \in L^1(R^n) \to (PF)(x, \theta) = F(x) \in L^1(S),$$

and let us denote the integration of $f$ over $S^{n-1}$ with respect to $\sigma$ by $\bar{f}$, i.e.,

$$\bar{f}(x) = \int_{S^{n-1}} f(x, \theta)\ d\sigma(\theta).$$

We now define the Fourier transform as a function of $x$ by the following:

$$\mathscr{F} f(\xi, \theta) = \int_{R^n} \exp\{-2\pi i \langle \xi, x\rangle\} f(x, \theta)\, dx, \quad f \in L^1(S), \tag{2.5}$$

where $\langle \xi, x\rangle = \xi_1 x_1 + \cdots + \xi_n x_n$ for $\xi = (\xi_1, \cdots, \xi_n) \in R^n$.

REMARK. $\overline{\mathscr{F} f}(\xi) = \mathscr{F} \bar{f}(\xi)$.

Putting

$$\varphi(\xi, \theta, t) = E_{(0, \theta)}[\exp\{-2\pi i \langle \xi, x(t)\rangle\}],$$

we have, because of the space-homogeneity of $\boldsymbol{X}$,

$$E_{(x, \theta)}[\exp\{-2\pi i \langle \xi, x(t)\rangle\}] = \exp\{-2\pi i \langle \xi, x\rangle\}\, \varphi(\xi, \theta, t).$$

Further we have

LEMMA 2.2. *Let* $F \in L^1(R^n) \cap L^2(R^n)$ *be a function such that* $\mathscr{F} F \in L^1(R^n, d\xi)$. *Then*

$$\mathscr{F}(\bar{R}_\lambda\ P\ F)(\xi) = \mathscr{F}F(\xi)\bar{\Phi}(\xi, \lambda), \tag{2.7}$$

*where* $\Phi(\xi, \theta, \lambda) = \int_0^\infty e^{-\lambda t}\varphi(\xi, \theta, t)\,dt$ and $\bar{\Phi}(\xi, \lambda) = \int_{S^{n-1}}\Phi(\xi, \theta, \lambda,)\,d\sigma(\theta)$.

*Proof.*

The right hand side of (2.7)

$$= \int_S \exp\{-2\pi i\langle\xi, x\rangle\}\ F(x)\,dx d\sigma(\theta) \int_0^\infty e^{-\lambda t}E_{(0,\theta)}[\exp\{-2\pi i\langle\xi, x(t)\rangle\}]\,dt$$

$$= \int_0^\infty e^{-\lambda t}\,dt\int_S (PF)(x,\theta)\,E_{(x,\theta)}[\exp\{-2\pi i\langle\xi, x(t)\rangle\}]\,dx\ d\sigma(\theta)$$

$$= \int_0^\infty e^{-\lambda t}\ dt\int_S T_t^*(PF)(x,\theta)\exp\{-2\pi i\langle\xi, x\rangle\}\,dx\ d\sigma(\theta)$$

$$= \int_S \exp\{-2\pi i\langle\xi, x\rangle\}\ dx\ d\sigma(\theta)\int_0^\infty e^{-\lambda t}\,T_t(PF)(x,\theta)\ dt$$

= the left hand side,

since $T_t^*(PF)(x,\theta) = T_t(PF)(x,\theta)$ by the remark of Lemma 2.1. Thus the proof is complete.

LEMMA 2.3. (i) $\varphi(\xi,\theta,t)$ *is the solution of the following integral equation:*

$$\psi(\xi,\theta,t) = \varphi_0(\xi,\theta,t) + k\int_0^t \varphi_0(\xi,\theta,s)\,\bar{\psi}(\xi, t-s)\ ds, \tag{2.8}$$

*where* $\varphi_0(\xi,\theta,t) = e^{-kt}\exp\{-2\pi i\langle\xi, c\theta t\rangle\}$.

(i′) $\bar{\varphi}(\xi, t)$ *is the solution of*

$$\psi(\xi, t) = \bar{\varphi}_0(\xi, t) + k\int_0^t \bar{\varphi}_0(\xi, s)\psi(\xi, t-s)\ ds. \tag{2.9}$$

(ii) $$\bar{\Phi}(\xi,\lambda) = (1 - k\bar{\Phi}_0(\xi,\lambda))^{-1}\bar{\Phi}_0(\xi,\lambda),$$

*where* $\bar{\Phi}_0(\xi,\lambda) = \int_0^\infty e^{-\lambda t}\bar{\varphi}_0(\xi, t)\ dt$.

*Proof.* Availing $K_{ac's}$ formula, we have

$$\varphi(\xi,\theta,t) = E_\theta\left[\exp\left\{-2\pi i\langle\xi, \int_0^t c\theta(s)\,ds\rangle\right\} : t < \tau\right] + E_\theta\left[\exp\left\{-2\pi i\langle\xi, \int_0^t c\theta(s)ds\rangle\right\} : t \geq \tau\right]$$

$$=e^{-kt}\exp\{-2\pi i\langle\xi, c\theta t\rangle\}+\int_0^t ke^{-ks}\,ds\int_{S^{n-1}} E\Big[\exp\Big\{-2\pi i\langle\xi, c\theta s+\int_0^{t-s}\theta(u)du\rangle\Big\}\Big]d\sigma(\vartheta)$$

$$=\varphi_0(\xi,\theta,t)+\int_0^t k\, e^{-ks}\exp\{-2\pi i\langle\xi, c\theta s\rangle\}\bar{\varphi}(\xi, t-s)\, ds$$

= the right hand side of (2.8), which implies (i).

(ii) is clear from the convolution rule on Laplace transform, and so we can complete the proof.

## § 3 Convergence in distribution.

Hereafter we assume:

$$k/c^2 = 2/n + o(1) \quad \text{when} \quad c\to\infty.$$

Since there is no essential difference in the following discussions, we shall assume always "$k/c^2 = 2/n$" for the sake of simplicity. Then we have

LEMMA 3.1. $$\lim_{c\to\infty}\bar{\Phi}(\xi,\lambda) = \frac{1}{\lambda+(|2\pi\xi|^2/2)}\,.$$

*Proof.*

$$\bar{\Phi}_0(\xi,\lambda)^{(*)} = \int_0^\infty e^{-\lambda t}e^{-kt}\int_{S^{n-1}}\exp\{-2\pi i\langle\xi, c\theta t\rangle\}\,d\sigma(\theta)\;dt \tag{3.1}$$

$$= \sum_{m=0}^\infty a_m^{(n)}|2c\pi\xi|^{2m}(\lambda+k)^{-(2m+1)}$$

$$= \frac{1}{\lambda+k}\Big[1-\frac{k}{2(\lambda+k)^2}|2\pi\xi|^2+\sum_{m=2}^\infty a_m^{(n)}\Big(\frac{c}{\lambda+k}\Big)^{2m}|2\pi\xi|^{2m}\Big]$$

where $a_0^{(n)}=1,\ a_1^{(n)}=-\dfrac{1}{n},\ a_2^{(n)}=\dfrac{3}{n(n+3)}$ and

$$a_m^{(n)} = \frac{(-1)^m\int_0^\pi\cos^{2m}\omega\sin^{n-2}\omega d\omega}{\int_0^\pi\sin^{n-2}\omega d\omega}\quad(m\geq 3),$$

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*) $$\bar{\varphi}_0(\xi,t)=\Gamma\Big(\frac{n}{2}\Big)e^{-kt}\Big(\frac{c|2\pi\xi|t}{2}\Big)^{-\frac{n-2}{2}}J_{\frac{n-2}{2}}(c|2\pi\xi|t),$$

$$\bar{\Phi}_0(\xi,\lambda)=\Gamma\Big(\frac{n}{2}\Big)\Big(\frac{c|2\pi\xi|}{2}\Big)^{-\frac{n-2}{2}}\{(\lambda+k)+c^2|2\pi\xi|^2\}^{\frac{n-4}{4}}\times$$

$$\times P^{-\frac{n-2}{2}}_{-\frac{n-2}{2}}\Big(\frac{\lambda+k}{\{(\lambda+k)^2+c^2|2\pi\xi|^2\}^{1/2}}\Big)$$

since

$$(3.2)\qquad \int_0^\infty e^{-\lambda t}\,dt\int_{S^{n-1}}\exp\{-i\langle\eta,\theta t\rangle\}\,d\sigma(\theta)$$

$$=\Big[\int_0^\pi \sin^{n-1}\omega\,d\omega\Big]^{-1}\int_0^\infty e^{-\lambda t}\,dt\int_0^\pi \exp\{i|\eta|t\cos\omega\}\sin^{n-2}\omega d\omega$$

$$=\Big[\int_0^\pi \sin^{n-2}\omega d\omega\Big]^{-1}\int_0^\infty e^{-\lambda t}\Big(\sum_{m=0}^\infty\int_0^\pi \frac{(i|\eta|t\cos\omega)^{2m}}{(2m)!}\sin^{n-2}\omega d\omega\Big)\,dt$$

$$=\sum_{m=0}^\infty a_m^{(n)}\frac{|\eta|^{2m}}{\lambda^{2m+1}}.$$

Hence

$$(3.3)\qquad \bar\Phi(\xi,\lambda)=\frac{\bar\Phi_0(\xi,\lambda)}{1-k\bar\Phi_0(\xi,\lambda)}$$

$$=\Big[\lambda+\frac{k^2}{2(\lambda+k)^2}|2\pi\xi|^2-a_2^{(n)}\frac{n^2k^3}{4(\lambda+k)^4}|2\pi\xi|^4$$

$$+k\sum_{m=3}^\infty a_m^{(n)}\frac{c^{2m}}{(\lambda+k)^{2m}}|2\pi\xi|^{2m}\Big]^{-1}$$

$$\Big[1-\frac{k}{2(\lambda+k)^2}|2\pi\xi|^2+a_2^{(n)}\frac{n^2k}{4(\lambda+k)^4}|2\pi\xi|^4+\sum_{m=3}^\infty a_m^{(n)}\frac{c^{2m}}{(\lambda+k)^{2m}}|2\pi\xi|^{2m}\Big]$$

$\rightarrow \dfrac{1}{\lambda+(|2\pi\xi|^2/2)}$, which completes the proof.

Let $\boldsymbol{B}=[B(t),\ +\infty,\ P_x^B,\ x\in R^n]$ be the $n$-dimensional Brownian motion, and $T_t^B$, $t\geq 0$; $R_\lambda^B$, $\lambda>0$, be the semigroup and resolvent of $\boldsymbol{B}$, respectively. Then we have

LEMMA 3.2. *Let $F\in L^1(R^n)\cap L^2(R^n)$ such that $\mathscr{F}F\in L^1(R^n,d\xi)$. Then*

$$|||\,\bar R_\lambda P\,F-R_\lambda^B F\,|||\rightarrow 0\ \text{ as }\ c\rightarrow\infty.$$

*Proof.* By Lemma 3.1 and the Lebesgue's convergence theorem, we get, noting $|\bar\Phi(\xi,\lambda)|\leq\dfrac{1}{\lambda}$,

$$|\bar R_\lambda P\,F(x)-R_\lambda^B F(x)|$$

$$=\Big|\int_{R^n}\exp\{2\pi i\langle x,\xi\rangle\}\,[\mathscr{F}(\bar R_\lambda PF)(\xi)-\mathscr{F}(R_\lambda^B F)(\xi)]\,d\xi\Big|$$

$$\leq\Big\|\mathscr{F}F(\xi)\Big[\bar\Phi(\xi,\lambda)-\frac{1}{\lambda+(|2\pi\xi|^2/2)}\Big]\Big\|_{L^1(d\xi)}.$$

$\rightarrow 0$, which implies the lemma.

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LEMMA 3.3. *Let $F$ be as in Lemma* 3.2. *Then*

$$\|R_\lambda P\, F - P\, \bar{R}_\lambda P\, F\| \to 0 \quad as \quad c \to \infty.$$

*Proof.* It follows from the strong Markov property of $\boldsymbol{X}$ that

$$\mathscr{F}(R_\lambda P\, F)\,(\xi,\theta) = \int_{R^n} \exp\{-2\pi i\langle \xi, x\rangle\}dx \int_0^\infty [e^{-\lambda t}dt\,[e^{-kt}(PF)\,(x + c\theta t, \theta)]$$

$$+\int_{R^n} \exp\{-2\pi i\langle \xi, x\rangle\}dx\int_0^\infty e^{-\lambda t}\,dt\Big[k\int_0^t e^{-ks}ds\int_{S^{n-1}} T_{t-s}\,(PF)\,(x + c\theta s, \vartheta)\,d\sigma(\vartheta)\Big].$$

The first term of right hand side

$$= \mathscr{F}F(\xi)\int_0^\infty e^{-\lambda t}\,e^{-kt}\exp\{2\pi i\langle \xi, c\theta t\rangle\}\,dt$$

$$= \mathscr{F}F(\xi)\,\frac{1}{\lambda + k - 2\pi i\langle \xi, c\theta\rangle}.$$

The second term

$$= \int_0^\infty e^{-\lambda t}\,dt\int_0^t ke^{-ks}\,ds\int_S \exp\{-2\pi i\langle \xi, x\rangle\}\,T_{t-s}(PF)\,(x + c\theta s, \vartheta)\;dx\;\,d\sigma(\vartheta)$$

$$= \int_0^\infty e^{-\lambda t}\,dt\int_0^t ke^{-ks}ds\,[\exp 2\pi i\langle \xi, c\theta s\rangle\}\,\mathscr{F}F(\xi)\;\;\bar{\varphi}(\xi, t - s)]$$

$$= \mathscr{F}F(\xi)\bar{\Phi}(\xi,\lambda)\int_0^\infty e^{-\lambda t}ke^{-kt}\exp\{2\pi i\langle \xi, c\theta t\rangle\}\;dt$$

$$= \mathscr{F}\,F(\xi)\,\bar{\Phi}(\xi,\lambda)\,\frac{k}{\lambda + k - 2\pi i\langle \xi, c\theta\rangle}.$$

Hence we have from Lemma 2.2

$$\begin{aligned}
&\|R_\lambda P\,F(x,\theta) - P\,\bar{R}_\lambda P\,F(x,\theta)\| \\
&\leq \|\!|\mathscr{F}(R_\lambda P\,F)\,(\xi,\theta) - \mathscr{F}(\bar{R}_\lambda P\,F)\,(\xi)\,\|\!|_{L^1(d\xi)} \\
&= \|\!|\mathscr{F}F(\xi)\,\bar{\Phi}(\xi,\lambda)\,[(\lambda + k - 2\pi i\langle \xi, c\theta\rangle)^{-1}(1 + k)] - \mathscr{F}F(\xi)\,\bar{\Phi}(\xi,\lambda)\,\|\!|_{L^1(d\xi)} \\
&\to 0 \quad \text{as} \quad c \to \infty,
\end{aligned}$$

completing the proof.

THEOREM 1. *Let* $F \in C_0(R^n)$. *Then*

$$\|T_t P\,F - P\;T_t^B F\| \to 0 \quad as \quad c \to \infty.$$

*Proof.* Since any $F \in C_0(R^n)$ can be approximated by the function as in Lemma 3.2 it follows from Lemma 3.2 and 3.3 that

$$\|R_\lambda P\ F - P\ R_\lambda^B F\| \to 0.$$

Hence, by the Trotter's theorem, we can conclude the theorem.

COROLLARY. *$P_{(x,\theta)}(x(t) \in E) \to P_x^B(B(t) \in E)$ as $c \to \infty$ for every $E \in \boldsymbol{B}(R^n)$ such that $\partial E$ has the Lebesgue measure 0.*

## § 4. Weak convergence.

Let $\mathscr{C}$ be the Fréchet-space of all continuous functions; $t \in [0, \infty) \to w(t) \in R^n$, with compact uniform topology and $\boldsymbol{B}(\mathscr{C})$ be the topological Borel field on $\mathscr{C}$. Since $x(t)$ are continuous in $t$, they induce the probability measure on $(\mathscr{C}, \boldsymbol{B}(C))$. We denote them by the same symbols $P_{(x,\theta)}$ and $P_x^B$, respectively. We assume, as in § 3, $k/c^2 = 2/n$[**].

DEFINITION 4.1. If, for any continuous function $\Psi$ on $\mathscr{C}$,

$$\lim_{c\to\infty} \int_{\mathscr{C}} \Psi(w)\ dP_{(x,\theta)}(w) = \int_{\mathscr{C}} \Psi(w)\ dP_x^B(w) \quad ((x,\theta) \in S),$$

then we call that $\boldsymbol{X} = [x(t), P_{(x,\theta)}]$ *converges weakly to* $\boldsymbol{B} = [B(t),\ P_x^B]$.

THEOREM 2. *The transport process $\boldsymbol{X}$ with speed $c$ converges weakly to Brownian motion $\boldsymbol{B}$ as $c \to \infty$.*

LEMMA 4.1. $\bar{E}_x[|x(t) - x(0)|^4] \leq K(n) t^2$

*Proof.* We have only to show that

$$\bar{E}_0[|x(t)|^4] \leq K(n)|t - s|^2,$$

because $\boldsymbol{X}$ is homogeneous with respect to $x \in R^n$. The coefficient of $(-2\pi i \xi)^4$ in

$$\frac{1}{4!}\int_0^\infty e^{-\lambda t} \bar{E}_0[\{-2\pi i \langle \xi, x(t) \rangle\}^4]\ dt$$

is equal to that of $(-2\pi i \xi)^4$ in

$$\bar{\Phi}(\xi, \lambda) = \int_0^\infty e^{-\lambda t} \bar{E}_0[\exp\{-2\pi i \langle \xi, x(t) \rangle\}]\ dt,$$

which is given by

$$\frac{3n\ k^2}{4(n+2)\lambda(\lambda+k)^4} + \frac{1}{4}\left(\frac{1}{(n+2)} + 1\right)\frac{k^4}{\lambda^2(\lambda+k)^4} + \frac{k^4}{4\lambda^3(\lambda+k)^4}$$

[**] Every following discussion remains valid if $k/c^2 = 2/n + o(1)$ $(c \to \infty)$.

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(cf. (3.3)). Hence we have

$$(4.1)\qquad \frac{1}{4!}\bar{E}_0[\langle\xi, x(t)\rangle^4]$$

$$=\Big[\frac{3n}{4(n+2)}\frac{1}{3!}k^2\int_0^t e^{-ks}s^3ds+\frac{1}{4}\Big(\frac{1}{n+2}+1\Big)\frac{1}{3!}k^3\int_0^t(t-s)e^{-ks}s^3ds$$

$$+\frac{1}{4}\frac{1}{2!3!}k^4\int_0^t(t-s)^2e^{-ks}s^3ds\Big]|\xi|^4,$$

since $\int_0^\infty e^{-\lambda t}\frac{1}{p!q!}dt\int_0^t(t-s)^p e^{-ks}s^q ds=\frac{1}{\lambda^{p+1}(\lambda+k)^{q+1}}$.

Therefore

$$\frac{1}{4!}\bar{E}_0[|x_i(t)|^4]\qquad (i=1,\cdots,n)$$

$$=\frac{3}{4!}\bar{E}_0[|x_i(t)x_j(t)|^2]\qquad (i\neq j)$$

$$=\text{the right hand side of (4.1)}$$

$$\leq K_0(n)t^2,$$

because

$$k^2\int_0^t e^{-ks}s^3ds\leq t^2\Big(k^2\int_0^s e^{-ks}sds\Big)$$

$$\leq t^2\Big(\int_0^\infty e^{-s}sds\Big)=\Gamma(2)t^2,$$

and similarly

$$k^3\int_0^t(t-s)e^{-ks}s^3ds\leq\Gamma(3)t^2,$$

$$k^4\int_0^t(t-s)^2e^{-ks}s^3ds\leq\Gamma(4)t^2.$$

Thus we get

$$\bar{E}_0[|x(t)|^4]=\sum_{1\leq i,j\leq n}\bar{E}_0[|x_i(t)x_j(t)|^2]$$

$$\leq 4!n^2K_0(n)t^2=K(n)t^2,$$

completing the proof.

LEMMA 4.2. $E_{(x,\theta)}[|x(t)-x(s)|^4]\leq K(n)|t-s|^2$.

*Proof.* It follows from the spatial homogeneity in $x \in R^n$ of $X$ and the equivalence of $[\gamma x(t), P_{(0,\theta)}]$ and $[x(t), P_{(0,\gamma\theta)}]$, where $\gamma$ be a rotation around the origin, that

$$\begin{aligned} &E_{(x,\theta)}[|x(t)-x(s)|^4] = E_{(0,\theta)}[|x(t)-x(s)|^4] \\ &= E_{(0,\gamma\theta)}[|\gamma^{-1}x(t)-\gamma^{-1}x(s)|^4] = E_{(0,\gamma\theta)}[|x(t)-x(s)|^4] \\ &= E_{(x,\gamma\theta)}[|x(t)-x(s)|^4], \end{aligned}$$

i.e., $E_{(x,\theta)}[|x(t)-x(s)|^4]$ is independent of $\theta \in S^{n-1}$. Hence $E_{(x,\theta)}[|x(t)-x(s)|^4]$, $\bar{E}_x[|x(t)-x(s)|^4]$, which concludes the proof by Lemma 4.1.

*Proof of Theorem* 2 is the direct consequence of the Prohorov's theorem by Theorem 1 and Lemma 4.2.

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