CORRESPONDENCE.

ON "TEN YEAR NONFORFEITURE POLICIES."

To the Editor of the Journal of the Institute of Actuaries.

SIR,—Life policies under the above title are described in the July number of the *Journal* (p. 324) as having been largely issued in the United States.

The one point of novelty appertaining to them, is, that "if after two annual premiums have been paid, further payments are to be discontinued, the holder may, upon due surrender of the original policy in accordance with the rules of the Company, receive in lieu thereof a paid-up policy for as many tenth parts of the original sum insured as full annual premiums have been paid."

It may be worth while to investigate the formula for the annual premium necessary to provide for such a risk, and also to examine to what extent these assurances, in the absence of data as to the probability of surrender, are to be regarded as speculative.

Leaving out of view for a moment the restriction that two annual premiums must be paid before the policy can be surrendered, we will suppose that the surrender can take place at any time. Further, it is obvious that no person assured under this scheme would ask for a paid-up policy at any other time than when a premium became due. Let p_n be the probability at the time the nth renewal becomes payable, that the same will be paid :—then, x being the age at entry, the present value of the liability incurred by the Office at the commencement of each year will be as follows :—

$$\left.\begin{array}{ll}
\text{1st year,} & \dfrac{M_x-M_{x+1}}{D_x} \\
\text{2nd ,,} & p_1\dfrac{M_{x+1}-M_{x+2}}{D_x}+\dfrac{1}{10}(1-p_1)\dfrac{M_{x+1}}{D_x} \\
\text{3rd ,,} & p_1p_2\dfrac{M_{x+2}-M_{x+3}}{D_x}+\dfrac{2}{10}p_1(1-p_2)\dfrac{M_{x+2}}{D_x} \\
\vdots & \qquad\vdots \qquad\qquad\qquad \vdots \\
\text{9th ,,} & p_1p_2\ldots p_8\dfrac{M_{x+8}-M_{x+9}}{D_x}+\dfrac{8}{10}p_1p_2\ldots p_7(1-p_8)\dfrac{M_{x+8}}{D_x} \\
\text{10th ,,} & p_1p_2\ldots p_9\dfrac{M_{x+9}}{D_x}+\dfrac{9}{10}p_1p_2\ldots p_8(1-p_9)\dfrac{M_{x+9}}{D_x}
\end{array}\right\} \quad . . (1)$$

and if ϖ be the annual premium, the present value of all the premiums payable will be

$$\varpi\left(1+p_1\frac{D_{x+1}}{D_x}+p_1p_2\frac{D_{x+2}}{D_x}\ldots\ldots+p_1p_2\ldots p_9\frac{D_{x+9}}{D_x}\right) \quad . \quad . \quad . \quad (2)$$

The annual premium required will therefore be the sum of the expressions in (1) divided by the coefficient of ϖ in (2).

If in all that precedes we make $p_1=1$, the formulæ will then meet the case where two yearly premiums have to be paid before the privilege of surrender is allowed. In addition to making $p_1=1$ let us suppose $p_2=p_3=p_4\ldots\ldots=p_9$ and denote each of these by p, then (1) becomes

$$\frac{1}{D_x}\Big\{M_x-M_{x+2}+p(M_{x+2}-M_{x+3})+p^2(M_{x+3}-M_{x+4})\ldots+p^7(M_{x+8}-M_{x+9})+p^8M_{x+9}$$
$$+(1-p)\left(\frac{2}{10}M_{x+2}+\frac{3}{10}pM_{x+3}\ldots\frac{8}{10}p^6M_{x+8}+\frac{9}{10}p^7M_{x+9}\right)\Big\}$$
$$=\frac{1}{D_x}\left\{M_x-(1-p)\left(\frac{8}{10}M_{x+2}+\frac{7}{10}pM_{x+3}+\frac{6}{10}p^2M_{x+4}\ldots+\frac{1}{10}p^7M_{x+9}\right)\right\} \quad . \ (3)$$

and (2) becomes

$$\frac{\varpi}{D_x}(D_x+D_{x+1}+pD_{x+2}+p^2D_{x+3}\ldots+p^8D_{x+9}) \quad . \quad . \quad . \quad . \quad (4)$$

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The annual premium therefore in this case will be found by dividing the coefficient of $\frac{1}{D_x}$ in (3) by the coefficient of $\frac{\varpi}{D_x}$ in (4). The following table exhibits a few numerical values of ϖ, deduced from (3) and (4), corresponding to different assumed values of p. The Carlisle is the table of mortality used, and 3 per cent the rate of interest.

Age at entry.	Annual premium per cent.			
	$p=0$.	$p=\frac{1}{3}$.	$p=\frac{2}{3}$.	$p=1$.
30	4·881	4·867	4·834	4·769
40	5·856	5·848	5·808	5·697
50	6·738	6·749	6·760	6·745

It appears from these results, that when $p_1=1$, and the law of surrender is such that $p_2=p_3=p_4\ldots.=p_9$, the amount of the annual premium is affected very little notwithstanding any change that may be made in the value of p. If the assumed law were known to be true we might conclude with certainty that there is no speculation involved in these assurances but such as might be amply covered by a properly constructed table of premiums. We cannot however tell at what rate surrenders might take place and it will therefore be desirable to examine further into the subject by making some important alteration in the supposed law and comparing the numerical results with those obtained already. Now experience shows that when a surrender takes place it is usually during the earlier years of a policy's existence and but very seldom after it has been in force a lengthened term. In choosing a second hypothesis we will therefore make the following suppositions, namely $p_1=1$, $p_2=p_3=p_4=p_5(=p)$, and $p_6=p_7=p_8=p_9=1$. In this case the present value of the Society's risk is

$$\frac{1}{D_x}\left\{M_x-M_{x+2}+p(M_{x+2}-M_{x+3})+p^2(M_{x+3}-M_{x+4})+p^3(M_{x+4}-M_{x+5})+p^4M_{x+5}\right.$$
$$\left.+(1-p)\left(\frac{2}{10}M_{x+2}+\frac{3}{10}pM_{x+3}+\frac{4}{10}p^2M_{x+4}+\frac{5}{10}p^3M_{x+5}\right)\right\}$$
$$=\frac{1}{D_x}\left\{M_x-(1-p)\left(\frac{8}{10}M_{x+2}+\frac{7}{10}pM_{x+3}+\frac{6}{10}p^2M_{x+4}+\frac{5}{10}p^3M_{x+5}\right)\right\}\quad . \quad . \quad (5)$$

and the present value of the annual premiums is

$$\frac{\varpi}{D_x}\{D_x+D_{x+1}+pD_{x+2}+p^2D_{x+3}+p^3D_{x+4}+p^4(N_{x+4}-N_{x+9})\}\quad . \quad . \quad . \quad (6)$$

The premium, ϖ, is therefore equal to the coefficient of $\frac{1}{D_x}$ in (5) divided by the coefficient of $\frac{\varpi}{D_x}$ in (6).

The following are numerical illustrations of the values of ϖ for various assumed values of p, using the same table of mortality and rate of interest as before.

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Age at entry	Annual premium per cent.			
	$p=0.$	$p=\frac{1}{3}.$	$p=\frac{2}{3}.$	$p=1.$
30	4·881	4·865	4·820	4·769
40	5·856	5·843	5·782	5·697
50	6·738	6·748	6·752	6·745

We see from these figures that the variation in the amount of the yearly premium for different values of p is very trifling, and we see moreover, on comparing these figures with those before found that notwithstanding the considerable change made in the suppositions as to surrender, the annual premium is as nearly as possible the same. We may conclude from these results that it is at least highly probable that in such contracts as we have been considering, the speculative element under any circumstances is extremely small.

When $p_1=1$ and $p_2=0$ we get from (1) and (2) $\varpi=\frac{M_x-\frac{4}{5}M_{x+2}}{D_x+D_{x+1}}$, and when $p_1=p_2=p_3\ldots=p_9=1$ we find $\varpi=\frac{M_x}{N_{x-1}-N_{x+9}}$, the latter being the ordinary formula when a whole life assurance is paid for by ten equal annual premiums.

I am, Sir,
Your obedient servant,
SAMUEL YOUNGER.

316, *Regent Street,*
26th Sept., 1868.

*** We readily give insertion to the above letter on a subject which is not only of theoretical interest but may become of some practical importance. We should have preferred, however, to see the numerical examples worked out by the Experience Table, instead of the Carlisle, when probably some of the irregularities in the results would have disappeared. We should be glad now to see the question treated in another way, viz. by a comparison of the amount of the paid-up policy which the value of the policy would purchase, according to the office single premiums, with the amount of that granted under the regulations quoted above.—Ed. *J. I. A.*

ON A FORMULA IN THE CALCULUS OF FINITE DIFFERENCES.

To the Editor of the Journal of the Institute of Actuaries.

Sir,—I am not about to enter upon the consideration of a theory proposed by some writers, that all mathematical evidence resolves into a perception of identity, and that mathematical propositions are only diversified expressions of the simple formula, $a=a$.* It must however be admitted

* The following is quoted by Dugald Stewart ("Philosophy of the human mind," part 2, cap. 1) from a writer on the subject referred to. *Omnes Mathematicorum propositiones sunt identicæ et repræsentantur hac formulâ, $a=a$.* He adds, "This sentence, which I quote from a dissertation published at Berlin about 50 years ago" (1813), "expresses in a few words what seems to be now the prevailing opinion (more particularly on the Continent) concerning the nature of Mathematical evidence."