Canad. Math. Bull. Vol. **48** (4), 2005 pp. 500–504

# Extension of Holomorphic Functions From One Side of a Hypersurface

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*Abstract.* We give a new proof of former results by G. Zampieri and the author on extension of holomorphic functions from one side $\Omega$ of a real hypersurface $M$ of $\mathbb{C}^n$ in the presence of an analytic disc tangent to $M$, attached to $\bar{\Omega}$ but not to $M$. Our method enables us to weaken the regularity assumptions both for the hypersurface and the disc.

## 1 Introduction

Let $M$ be a $C^{1,\alpha}$ hypersurface of $\mathbb{C}^n$ for $0 < \alpha < 1$, and $\Omega$ a domain of $\mathbb{C}^n$ with boundary $M$. We prove in Theorem 2.3 that the existence of an analytic disc $A$ tangent to $M$ at a point $z^o \in M \cap \partial A$, $C^1$ up to the boundary, attached to $\bar{\Omega}$ but not to $M$, that is satisfying $\partial A \subset \bar{\Omega}$ but $\partial A \not\subset M$, implies extension of holomorphic functions from $\Omega$ to a full neighborhood of $z^o$. Also, if $\bar{A}$ is contained in $\bar{\Omega}$ but not in $M$ in any neighborhood of $z^o$, then the above result yields extension of germs at $z^o$ of holomorphic functions on $\Omega$. In fact, let $z_k$ be a sequence of points of $\bar{A}$ which approach $z^o$ and belong to $\Omega$ and not to $M$, and let $\Delta_k \subset \Delta$ be a sequence of (smooth and small) discs contained in $\Delta$ with $z_k \in A(\partial\Delta_k)$, and which coincide with $\Delta$ in a neighborhood of $\tau = 1$. Define $A_k$ as $A$ restricted to $\Delta_k$ (which implies $A_k \subset \bar{\Omega}$); then our subsequent Theorem 2.3 applies, in particular, to this sequence of discs $A_k$.

We observe now that if $M$ contains a complex hypersurface, say $h = 0$, then $\frac{1}{h}$ does not extend. In particular, one-sided discs through $z^o$ tangent to $M$ are in fact contained in $M$ in a neighborhood of $z^o$.

We can restate our theorem in terms of propagation of extendibility of holomorphic functions from one side of $M$ to $\mathbb{C}^n$ along a disc $A$ whose boundary is contained in $M$. In fact, let $A$ be tangent to $M$ at $z^o \in \partial A$, and $f$ be holomorphic in $\Omega$ and extend holomorphically to a full neighborhood of another point $z^1 \in \partial A$. By a small perturbation $\tilde{\Omega}$ of $\Omega$ which keeps $\Omega$ unchanged in a neighborhood of $z^o$ and such that $z^1$ becomes a point of the interior of $\tilde{\Omega}$, we enter in the assumptions of the subsequent Theorem 2.3. Thus $f$ extends holomorphically also to a neighborhood of $z^o$. If $A$ is a "defective" disc, the above propagation principle is already contained in [1] and [10]. If $A \subset M$, and $z^o$ belongs to the interior of $A$, it is the main result of [6] which is also valid for submanifolds of any codimension, not necessarily for hypersurfaces. Note that in this case $A$ does not need to be small.

Our theorem is closely related to the results of [2] and [11] where the technique of the infinitesimal deformation of the disc $A$ is used. Instead, in the present paper, we

Received by the editors November 17, 2003; revised February 20, 2004.
AMS subject classification: 32D10 32V25.
Keywords: analytic discs, Poisson integral, holomorphic extension.




https://doi.org/10.4153/CMB-2005-046-6 Published online by Cambridge University Press

use a method which is the "boundary version" of that in [7]. We only deal with the disc $A$ and its translations inward $\Omega$, and therefore avoid use of the implicit function Theorem. This allows us to weaken the assumption of regularity of $M$ (resp. $A$) from $C^{2,\alpha}$ (resp. $C^{1,\alpha}$) to $C^{1,\alpha}$ (resp. $C^1$). Also, this yields a simple and geometric proof.

## 2 Statement and Proof of the Main Theorem

Let $\Omega$ be a domain of $\mathbb{C}^n$, $z$ a point of $\Omega$, $\{B\}$ the system of balls with center $z$, $\nu$ a unit vector in $\mathbb{C}^n$, $f$ a holomorphic function on $\Omega$. We denote by $\Delta = \{\tau \in \mathbb{C} : |\tau| < 1\}$ the standard disc in $\mathbb{C}$, and set $\Delta_r = \{r\tau : \tau \in \Delta\}$.

***Definition 2.1*** (i) We set

$$r_f^{\nu,B} = \sup\Big\{ r : f \text{ extends holomorphically to a neighborhood of } \tilde{z} + \nu\Delta_r \text{ for any } \tilde{z} \in B\Big\}.$$

(ii) We also set

$$r_f^{\nu}(z) = \sup_B r_f^{\nu,B}.$$

It is clear that $r_f^\nu(z)$, $z \in \Omega$, is a lower semicontinuous function of $z$. We will make an essential use of the following elementary remark. Let $z \in \Omega$ and $z^o \in \partial\Omega$ be a pair of points with the property that the vector $z^o - z$ is normal to $\partial\Omega$ at $z^o$. We write $\nu = \frac{z^o - z}{|z^o - z|}$ and $\delta(z) = |z^o - z|$; thus $\delta(z)$ is the distance of $z$ from $\partial\Omega$. In this situation, for a holomorphic function $f$ on $\Omega$:

if $r_f^\nu(z) > \delta(z)$, then $f$ extends holomorphically to a full neighborhood of $z^o$.

The proof is a consequence of the definition itself of $r_f^\nu(z)$. We discuss now in more detail the properties of $r_f^\nu$. We first show that it describes the convergence radius of the Taylor expansion of $f$ in the $\nu$-direction. In other terms we claim that

$$r_f^\nu(\xi) = \sup\{r : |\partial_\nu^k f(z)| \leq ck!\, r^{-k} \text{ for some } B, \forall z \in B, \forall k \in \mathbb{N}\}, \tag{2.1}$$

where $\partial_\nu$ denotes the holomorphic derivative along the $\nu$-direction. In fact "$\leq$" is clear by Cauchy's inequalities. As for "$\geq$", we denote by $(z_1, z')$ the variables in $\mathbb{C}^n$, and suppose that the direction of $\nu$ is that of the $z_1$-axis. In a polydisc $\xi + (\Delta_\epsilon \times \Delta_\epsilon \times \cdots)$, $f$ is the sum of a "double" series in $z_1 - \xi_1$ and $z' - \xi'$ that we may rearrange as $\sum_k a_k(z')(z_1 - \xi_1)^k$ the coefficients $a_k(z')$ being holomorphic. If $r$ is a number as in the right side of (2.1), then the above series converges for $z_1 \in \xi_1 + \Delta_r$ and therefore defines a holomorphic function on $\xi + (\Delta_r \times \Delta_\epsilon \times \cdots)$. This proves that $f$ is holomorphic in a neighborhood of $z + (\Delta_r \times \{0\} \times \{0\} \cdots)$ $\forall z \in B$ for a ball $B$ with center $\xi$; in particular $r_f^\nu(\xi) > r$ which proves our claim. We prove next the following central statement (*cf.* also [7]):

***Proposition 2.2*** *Let $f$ be holomorphic in $\Omega$; then $\log r_f^\nu$ is plurisuperharmonic in $\Omega$, that is, over any 1-dimensional disc contained in $\Omega$, it stands above the harmonic extension from the boundary.*

**Proof** Fix a point $\xi_o$, consider nearby points $\xi$, and denote by $S_\xi$ discs with center $\xi$ contained in $\Omega$ approaching a *limit* disc $S_{\xi_o}$. We will use the notation "m.v.$_{\partial S_\xi}$" to denote the mean value along $\partial S_\xi$; we have for any $r < r^\nu_f$ along $\partial S_\xi$:

$$\begin{aligned} \log|\partial^k_\nu f(\xi)| &\le \text{m.v.}_{\partial S_\xi} \log|\partial^k_\nu f| \\ &\le \log ck! - k\,\text{m.v.}_{\partial S_\xi} \log r, \end{aligned} \tag{2.2}$$

where the first inequality is clear because $\log|\partial^k_\nu f|_{S_\xi}$ is subharmonic, and the second is a consequence of (2.1). With the notation $t := \log r$ we then have

$$\begin{aligned} \log r^{\nu,B}_f &= \sup\{t : \log|\partial^k_\nu f(\xi)| < \log ck! - kt \ \forall \xi \in B\} \\ &\ge \sup\{t : t < \text{m.v.}_{\partial S_\xi} \log r^\nu_f \ \forall \xi \in B\} \\ &= \inf_{\forall \xi \in B} \text{m.v.}_{\partial S_\xi} \log r^\nu_f, \end{aligned} \tag{2.3}$$

where the central inequality follows from (2.2). It follows

$$\begin{aligned} \log r^\nu_f(\xi_o) = \sup_B \log r^{\nu,B}_f &\ge \liminf_\xi \text{m.v.}_{\partial S_\xi} \log r^\nu_f \\ &\ge \text{m.v.}_{\partial S_{\xi_o}} \log r^\nu_f, \end{aligned} \tag{2.4}$$

where the first inequality follows from (2.3) and the second from Fatou's Theorem. ■

Let $A = A(\tau)$, $\tau \in \bar{\Delta}$, be a small analytic disc in $\mathbb{C}^n$, $C^1$ up to the boundary. This means that $A$ extends as a $C^1$ embedding of a neighborhood of $\bar{\Delta}$ into $\mathbb{C}^n$.

***Theorem 2.3*** *Let $\Omega$ be a domain of $\mathbb{C}^n$ with a $C^{1,\alpha}$ boundary $M = \partial\Omega$ in a neighborhood of a point $z^o$ of $M$, let $A$ be a small disc $C^1$ up to the boundary, with $z^o \in \partial A$, which satisfies*

$$\begin{cases} T_{z^o}A \subset T_{z^o}M, \\ \partial A \subset \bar{\Omega}, \\ \partial A \cap \Omega \neq \varnothing \end{cases} \tag{2.5}$$

*Let $B$ be a ball with center $z^o$ which contains $\bar{A}$; then holomorphic functions on $\Omega \cap B$ extend holomorphically to a fixed neighborhood of $z^o$.*

**Proof** We select a point $z^1 \in \partial A \cap \Omega$ and fix our notation with $z^o = A(1)$, $z^1 = A(-1)$. We also choose complex coordinates $z = (z_1, z')$, $z = x + iy$ in $\mathbb{C}^n$ so that $z^o = 0$, $M$ is defined by the equation

$$y_1 = h(x_1, z') \text{ with } h(0,0) = 0 \text{ and } \partial h(0,0) = 0,$$

https://doi.org/10.4153/CMB-2005-046-6 Published online by Cambridge University Press

and $\Omega$ is the side of $M$ defined by $y_1 < h$. For $0 \le r \le 1$, $0 \le \theta \le 2\pi$, denote by $\tau = re^{i\theta}$ the point in the standard disc $\Delta$. Let $\nu := (i, 0, \dots)$ be the unit exterior normal to $\Omega$ at $z^o$, and $\mu$ the unit tangent vector parallel to $\partial_r A(1)$. Choose a $(2n-2)$-dimensional plane $L \subset T_{z^o}M$ transversal to $\mu$, decompose $T_{z^o}M = \mathbb{R}\mu \oplus L$, and, for small parameters $\eta$ and $\beta$, and for a vector $\lambda \in L$ with $|\lambda| \le 1$, define

$$A_{\eta,\beta,\lambda} = -\eta\nu + \beta\eta(\mu + \lambda) + A.$$

Denote by $\epsilon$ the diameter of $A$, and take a holomorphic function $f$ on $\Omega \cap B$. Let $c$ be a local bound for the $C^{1,\alpha}$-norm of $M$. For a constant $\sigma$ which depends on the distance of $z^1$ to $\partial\Omega$ and on neither $\eta$ nor $\beta$, we have

(2.6)
$$\begin{cases} r_f^\nu \circ A_{\eta,\beta,\lambda}(e^{i\theta}) \ge \eta(1 - c\beta^{1+\alpha}\eta^\alpha)\big(1 - (\epsilon\theta)^2\big) & \text{for any } \theta \text{ in } [0, 2\pi], \\ r_f^\nu \circ A_{\eta,\beta,\lambda}(e^{i\theta}) > \sigma & \text{for } \theta \text{ in a neighborhood of } \pi. \end{cases}$$

Write $\eta(\theta) := \eta(1 - c\beta^{1+\alpha}\eta^\alpha)(1 - (\epsilon\theta)^2)$ and, for $\tau \in \Delta$, $\theta \in [0, 2\pi]$, denote by $P(\tau, \theta)$ the Poisson kernel. Evaluation at $z = A_{\eta,\beta,\lambda}(r)$ with $r \in [-1, 1]$ close to 1 yields

$$\begin{aligned} \log r_f^\nu(z) &\ge \int_0^{2\pi} P(r,\theta) \log \eta(\theta)\, d\theta + \int_{\pi-\epsilon}^{\pi+\epsilon} P(r,\theta) \log\Big(\frac{\sigma}{\eta}\Big)\, d\theta \\ &\ge \log\big(\eta(1 - c\beta^{1+\alpha}\eta^\alpha)\big) + \int_o^{2\pi} P(r,\theta) \log\big(1 - (\epsilon\theta)^2\big)\, d\theta \\ &\quad + \int_{\pi-\epsilon}^{\pi+\epsilon} P(r,\theta) \log\Big(\frac{\sigma}{\eta}\Big)\, d\theta. \end{aligned} \tag{2.7}$$

Denote by $\mathcal{I}_1$ and $\mathcal{I}_2$ the first and second integral respectively in the second and third lines of (2.7). Note that $P(r,\theta) = \frac{1-r^2}{1+r^2-2r\cos\theta} \le c(\theta^{-2})(1-r)$. It follows that $\mathcal{I}_1 \ge -\epsilon(1-r)$ and $\mathcal{I}_2 \ge c_1 \log(\frac{1}{\eta})(1-r)$ for a suitable $c_1 > 0$. By this, (2.7) implies for $z = A_{\eta,\beta,\lambda}(r)$

$$r_f^\nu(z) \ge \eta(1 - c\beta^{1+\alpha}\eta^\alpha)\big(1 - \epsilon(1-r)\big)\left(1 + c_1 \log\Big(\frac{1}{\eta}\Big)(1-r)\right), \tag{2.8}$$

(provided that $c_1 \log(\frac{1}{\eta})(1-r)$ is small). Recall that $A$ is $C^1$ up to the boundary and tangent to $M$, and that $M$ itself is $C^{1,\alpha}$. We use the notation $I_{\eta,\beta,\lambda} = \{A_{\eta,\beta,\lambda}(r)\ \forall r \in [-1, 1]\}$. It is clear that for any $\eta$, $\beta$, there is $\lambda$ such that $I_{\eta,\beta,\lambda}$ contains a point $z_{\eta,\beta,\lambda} = I_{\eta,\beta,\lambda}(r_{\eta,\beta,\lambda})$ having all coordinates, but $y_1$, which are 0. Now, for this point $r_{\eta,\beta,\lambda}$ must be proportional to $\beta\eta$. By this fact we can easily check that

$$\delta(z_{\eta,\beta,\lambda}) \le \eta(1 - c_2\beta^2), \tag{2.9}$$

for a suitable $c_2 > 0$. We notice that in order to get extension at $z^o = 0$ it will suffice to show that

$$r_f^\nu(z_{\eta,\beta,\lambda}) > \delta(z_{\eta,\beta,\lambda}). \tag{2.10}$$

In turn, on account of (2.8), (2.9), it will suffice for (2.10) to show that

$$\frac{c_1}{2}\eta\log\Big(\frac{1}{\eta}\Big)(1-r) > c\eta^{1+\alpha}\beta^{1+\alpha} + \epsilon\eta(1-r) + c_2\eta\beta^2. \tag{2.11}$$

Now, it is clear that by choosing $\beta = \eta^{\frac{1}{\alpha}-1}$, (2.11) will be satisfied for sufficiently small $\eta$ (which also implies, among other things, that $c_1\log(\frac{1}{\eta})(1-r)$ is small). This proves (2.10) and implies holomorphic extension of $f$ at $z^o$. ■

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https://doi.org/10.4153/CMB-2005-046-6 Published online by Cambridge University Press