ON SEMI-PERFECT GROUP RINGS

W.D. Burgess

(received January 14, 1969)

1. Introduction. In what follows the notation and terminology of [7] are used and all rings are assumed to have a unity element.

The purpose of this note is to give some partial answers to the question: under which conditions on a ring A and a group G is the group ring AG semi-perfect?

For the convenience of the reader a few definitions and results will be reviewed. A ring R is called semi-perfect if R/RadR (Jacobson radical) is completely reducible and idempotents can be lifted modulo RadR (i.e., if x is an idempotent of R/RadR there is an idempotent e of R so that e + RadR = x). A homomorphic image of a semi-perfect ring is again semi-perfect [2, Lemma 2.2]; and R_n, the ring of $n \times n$ - matrices over a ring R, is semi-perfect if and only if R is semi-perfect [6, Theorem 3]. The commutative semi-perfect rings are the finite direct products of local rings [2].

If A is a ring and G a group, AG denotes the discrete group ring. If H is a subgroup of G, ωH is the right ideal of AG generated by $\{1 - h | h \in H\}$; if H is normal, this is an ideal and $AG/\omega H \simeq A(G/H)$. If I is a right ideal of A then IG denotes the elements

Canad. Math. Bull. vol. 12, no. 5, 1969

https://doi.org/10.4153/CMB-1969-083-9 Published online by Cambridge University Press

of AG with coefficients in I; when I is an ideal so is IG and $AG/IG \simeq (A/I)G$. A group ring AG is regular if and only if A is regular, G is locally finite and the order of every finite sub-group of G is a unit in A [3, Theorem 3].

2. Necessary Conditions. Since $A \simeq AG/\omega G$ it follows that A is semi-perfect if AG is and, assuming $A/\mathrm{Rad}A \simeq D^{(1)}_{n(1)} \times \ldots \times D^{(k)}_{n(k)}$, $D^{(i)}$ a division ring, $D^{(i)}_{n(i)}G$ is semi-perfect. This last is because $AG/(\mathrm{Rad}A)G$ is semi-perfect as is

$$\frac{(A/\mathrm{Rad}A)G}{IG} \simeq D^{(i)}_{n(i)}G$$

where

$$I \simeq D^{(1)}_{n(1)} \times \cdots \times D^{(i-1)}_{n(i-1)} \times D^{(i+1)}_{n(i+1)} \times \cdots \times D^{(k)}_{n(k)} .$$

It is clear that for any ring A, $A_nG \simeq (AG)_n$ (assign to $B_1g_1 + \ldots + B_sg_s \in A_nG$ the matrix with ij entry $a^{(1)}_{ij}\, g_1 + \ldots + a^{(s)}_{ij}\, g_s$ where $a^{(m)}_{ij}$ is the ij entry of B_m). By the result quoted above, $D^{(i)}G$ is semi-perfect for $i = 1, \ldots, s$.

PROPOSITION 1. *If* AG *is semi-perfect so is* A *and so is* DG *for each division ring appearing in the factors of* A/RadA.

Definition 2 [8]. A group G is called an ID group (integral domain group) if for each ring A with no zero divisors except zero AG has no zero divisors except zero.

It is easily seen [8, Theorem 3.2] that a non-trivial ID group is torsion free and that any ordered group (such as a torsion free Abelian group) is ID. Clearly, if A has no zero divisors and G is an

ID group, 0 and 1 are the only idempotents of AG.

PROPOSITION 3. *If* $G \neq \{1\}$ *is an* ID *group*, AG *is not semi-perfect*.

Proof. If AG is semi-perfect, DG is semi-perfect for some division ring D. Hence, if e + RadDG is an idempotent of DG/RadDG, either $e \in$ RadDG or $1 - e \in$ RadDG. Since DG/RadDG is completely reducible, it follows that DG/RadDG is a division ring. Also $DG/\omega G \simeq D$ so ωG is a primitive ideal and, thus, RadDG $\subseteq \omega G$. But there can be no proper ideals of DG properly containing RadDG, so $\omega G =$ RadDG. This implies [3, Proposition 15] that G is a p-group for some prime p. This is a contradiction.

COROLLARY 4. *If* G *is an extension of a group by a non-trivial* ID *group then* AG *is not semi-perfect for any ring* A.

Proof. By factoring out an ideal of AG one gets a group ring of an ID group which cannot be semi-perfect.

As a special case, if G is Abelian and AG is semi-perfect then G is torsion, since every non-torsion Abelian group is an extension of group by a non-trivial ID group. However, if G is Abelian, a more detailed statement can be made.

PROPOSITION 5. *If* AG *is semi-perfect*, G *an Abelian group*, *then either* G *is finite or* $G \simeq H \times G_p$, G_p *an infinite p-group*, H *finite*, $p \nmid |H|$ *and each of the division rings associated with the completely reducible ring* A/RadA *is of characteristic* p.

https://doi.org/10.4153/CMB-1969-083-9 Published online by Cambridge University Press

Proof. As we have seen, if AG is semi-perfect so is DG where D is a division ring from A/RadA. If D has characteristic zero then DG is regular (and, hence RadDG = 0). This means that DG is completely reducible and, by the Maschke Theorem [3, p. 660], G is finite. If D has characteristic $p, G \simeq H \times G_p$ where G_p is a p-group and H has no elements of order p. Then $DH \simeq DG/\omega G_p$ is semi-perfect and regular. As above, H is finite.

Corollary 4 and Proposition 5 lead one to conjecture that AG semi-perfect implies that G is torsion. The following example shows that G need not be locally finite. In [5, Chapter 8] there is an exposition of the Golod-Šafarevič Theorem which gives a p-group G which is not locally finite. In the particular example given in [5], A is taken to be a field of characteristic p and, in AG, ωG is a nil ideal. Hence $RadAG = \omega G$, $AG/\omega G \simeq A$ is a field, so AG is semi-perfect.

3. *Sufficient Conditions*. It was shown in [3, Proposition 9] that if A is Artinian or if G is locally finite then

$RadA = A \cap RadAG$.

LEMMA 6 ([3, Proposition 16 (iii) and (iv)]). *If* G *is Abelian then* $\omega G = RadAG$ *if and only if* G *is a p-group*, $p = 0$ *in* A, *and* A *is semi-primitive*.

COROLLARY 6. *If* G *is an Abelian p-group and* A *a finite direct product of commutative local rings whose factor fields are of characteristic* p *then* AG *is semi-perfect*.

Proof. Let $A \simeq L_1 \times \ldots \times L_n$, L_i local, $L_i/\text{Rad}L_i \simeq F_i$ a field of characteristic p. Then $AG \simeq L_1G \times \ldots \times L_nG$ and for each i,

$$L_iG/\text{Rad}L_iG \simeq \frac{L_iG/(\text{Rad}L_i)G}{\text{Rad}L_iG/(\text{Rad}L_i)G} \simeq F_iG/\text{Rad}F_iG \simeq F_i$$

by the proposition. Hence, each L_iG is local.

A computation, which appears for example in [8, Theorem 1.4], shows that if $G \simeq H \times K$ then $AG \simeq (AH)K$. This yields a converse to Proposition 5.

COROLLARY 7. *If* A *is commutative,* $G \simeq G_p \times H$ *where* G_p *is a p-group,* H *is finite and* $p \nmid |H|$, *then* AG *is semi-perfect if* AH *is a finite direct product of local rings whose factor fields are of characteristic* p.

PROPOSITION 8. *If* A *is semi-perfect,* G *finite, then* AG *is semi-perfect if idempotents can be lifted modulo* $(\text{Rad}A)G$. *If* AG *is commutative, the converse is true.*

Proof. By a remark above $\text{Rad}A \subset \text{Rad}AG$, so

$$AG/\text{Rad}AG \simeq \frac{AG/(\text{Rad}A)G}{\text{Rad}AG/(\text{Rad}A)G} \simeq \frac{(A/\text{Rad}A)G}{\text{Rad}(A/\text{Rad}A)G}$$

Now $(A/\text{Rad}A)G$ is Artinian, since by [3, Theorem 1] a group ring is Artinain if and only if the underlying ring is Artinian and the group is finite; thus, idempotents may be lifted modulo $\text{Rad}((A/\text{Rad}A)G)$.

The converse is proved, for example, in [4, Corollary 1.3].

The following proposition yields a sufficient condition for the lifting of idempotents modulo (RadA)G; however, much better results are known (see [1] or [4]). The Proposition is included because it seems interesting and it may have other applications.

PROPOSITION 9. *Let* A *be any ring*, N *an ideal* $N \subset \operatorname{Rad}A$, G *Abelian and torsion, then idempotents can be lifted from* AG/NG *to* AG/N^2G.

Proof. Let $e = h_1g_1 + \ldots + h_ng_n$ be an idempotent of AG modulo NG. Since the subgroup of G generated by $\{g_1, \ldots, g_n\}$ is finite and the idempotent modulo N^2G which will be constructed has the same support as e, it is assumed below that G is finite with elements $\{g_1, \ldots, g_n\}$. We have, for each $k = 1, \ldots, n$, $\sum_{ij=k} h_ih_j = h_k + p_k$, where $p_k \in N$ (here, as in what follows, the group element is referred to by its subscript). To lift this idempotent to AG we would need to find $m_i \in N$, $i = 1, \ldots, n$ so that

(1) $$\sum_{ij=k} (h_i + m_i)(h_j + m_j) = h_k + m_k \quad \text{for } k \quad 1, \ldots, n \text{ , or}$$

(2) $$\sum_{ij=k} (h_ih_j + m_ih_j + h_jm_i + m_im_j) = h_k + m_k \text{ .}$$

Since e is an idempotent modulo NG

(3) $$\sum_{ij=k} (h_im_j + h_jm_i) - m_k = - p_k - \sum_{ij=k} m_im_j \text{ ,}$$

so a solution of

(4) $$\sum_{ij\ k} (h_im_j \quad h_jm_i) - m_k = - p_k$$

would yield an idempotent modulo N^2G since the term $\sum m_im_j \in N^2$. Relabelling (4) gives

https://doi.org/10.4153/CMB-1969-083-9 Published online by Cambridge University Press

$$\sum_i (h_i m_{i^{-1}k} + h_{i^{-1}k} m_i) - m_k = - p_k$$

or, using the commutativity of G for the first time,

$$\sum_i 2h_i m_{i^{-1}k} - m_k = - p_k . \tag{5}$$

To demonstrate the existence of a solution for (5), it suffices to show that the matrix B of coefficients is a unit in A_n. For

$$B \begin{pmatrix} m_1 \\ \cdot \\ \cdot \\ \cdot \\ m_n \end{pmatrix} = \begin{pmatrix} -p_1 \\ \cdot \\ \cdot \\ \cdot \\ -p_n \end{pmatrix}$$ implies that each $m_i \varepsilon N$.

The matrix B has the form

$$b_{kq} = \begin{cases} 2h_{kq^{-1}} & \text{if } q \neq k , \\ 2h_1 - 1 & \text{if } q = k . \end{cases}$$

Hence $B = 2C - I_n$, where $c_{kq} = h_{kq^{-1}}$ and I_n is the identity matrix. Thus C is just the regular representation of e in A_n and this is an idempotent modulo N_n. So $C^2 - C \varepsilon N_n$ and $(2C - I_n)^2 = I_n + J$ where $J \varepsilon N_n$. But $N \subset \text{RadA}$ so $I_n + J$ is a unit in A_n and it follows that B is also a unit.

It should be remarked that the above argument does not work if G is non-abelian, for then $B = C + C' - I_n$ where C is the right and C' the left regular representation of e.

A corollary of this result is that if A is complete in the RadA-adic topology then idempotents of AG modulo $(\text{RadA})G$ can be lifted. However, this is true even when (A,RadA) is a Hensel pair (see [1] and [4]) and G is any finite group. Certainly there are Hensel rings without the completeness property.

REFERENCES

1. G. Azumaya, On maximally central algebras. Nagoya Math. J. 2 (1950) 119-150.

2. H. Bass, Finitistic dimension and a homological generalization of semi-primary rings. Trans. Amer. Math. Soc. 95 (1960) 466-488.

3. I.G. Connell, On the group ring. Canad. J. Math. 15 (1963) 650-685.

4. S. Greco, Algebras over nonlocal Hensel rings. J. Algebra 8 (1968) 45-59.

5. I.N. Herstein, Noncommutative rings. (Mathematical Association of America, 1968).

6. S. Keye, Ring theoretic properties of matrix rings. Canad. Math. Bull. 10 (1967) 365-374.

7. J. Lambek, Lectures on rings and modules. (Blaisdell, Waltham, Mass., U.S.A., 1966).

8. W. Rudin and H. Schneider, Idempotents in group rings. Duke Math. J. 31 (1964) 585-602.

9. H. Schneider and J. Weissglass, Group rings, semigroup rings and their radicals. J. Algebra 5 (1967) 1-15.

University of Ottawa

https://doi.org/10.4153/CMB-1969-083-9 Published online by Cambridge University Press