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CURVATURE PINCHING BASED ON INTEGRAL NORMS OF THE CURVATURE

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ABSTRACT A compact Riemannian manifold (M,g) of dimension 3 or higher admits a metric of constant (positive or negative) sectional curvature if the following conditions hold the diameter is bounded from above, the part of the Ricci curvature which lies below some fixed negative number is bounded in L^p norm for $p > n/2$, and the metric is almost spherical or almost hyperbolic in the L^p sense The idea of the proof is to obtain stronger ($i\,e$ L^∞) pinching by deforming the initial metric using the Ricci flow, thus reducing the problem to the theorems of Gromov in the case $r^g < 0$ and of Grove, Karcher and Ruh in the case $r^g > 0$ The reduced curvature tensor changes along the flow according to the heat equation, which implies a weak nonlinear parabolic inequality for its norm The iteration method of De Giorgi, Nash and Moser is applied to obtain the estimate for the maximum norm of the reduced curvature tensor The crucial step in the iteration consists of controlling the Sobolev constant of the appropriate imbedding (which also changes along the flow, but behaves well) by the isoperimetric constant, which, in turn, can be bounded in terms independent of the particular manifold

1. **Introduction.** Pinching theorems represent one of the possible ways to investigate the relation between the geometry of the manifold and its topology. In general, they assume strong properties of the curvature, like bounds on the Ricci or sectional curvature, and give precise theorems on the structure of the manifold. Most of the assumptions, in one way or another, measure the closeness of the curvature tensor of the manifold to the best possible one, namely (in the case of a positive curvature) to the curvature tensor of the standard sphere S^n.

In this paper we consider a compact Riemannian manifold of dimension greater than or equal 3 and assume that there are bounds on integral norms of the reduced curvature tensor and of the negative part of the Ricci curvature. These assumptions (and one more, on the diameter) imply that the manifold admits a metric of constant sectional curvature. This result (see Theorem 2.1) is a generalization of Theorem 1 in [M-R 2]. We show that the assumption on the upper bound of the sectional curvature is not needed—it is only necessary to control the part of the Ricci curvature which is smaller than some negative constant. In our case, the evolution inequality for the norm of the reduced curvature tensor (Theorem 3.3) is slightly more complicated, since it contains not only the norm $\| \widetilde{Rm} \|$ but also its square. De Giorgi-Nash-Moser iteration can still be carried out, but the Sobolev inequality used in it represents the obstruction to improving the theorem in the sense of replacing the $L^{\frac{n}{2}+\nu}$ bound by the $L^{\frac{n}{2}}$ bound. According to [Go 1,2]

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similar result can be obtained with an $L^{\frac{n}{2}}$ bound on the full curvature and a bound on the maximum norm of the Ricci curvature (if one imposes an additional assumption on the injectivity radius or on the local volume of geodesic balls). Our theorem represents an improvement in the sense that we only need the bound on some integral norm of (the negative part of) the Ricci curvature.

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Before stating the result we are going to introduce the notation and conventions used throughout this paper. By (M, g) we denote a compact, Riemannian manifold of dimension $n \geq 3$; g_{ij} is the Riemannian metric on M and g^{ij} its inverse. The induced measure on M is $d\mu = \sqrt{\det g_{ij}}\, dx$, where dx represents the Lebesgue measure. The diameter of M will be denoted by $d(M)$ and $V(M) = \int_M d\mu$ is the volume of M. The symbol $\|.\|_g$ denotes the norm with respect to the metric g.

The Levi-Civita connection on (M, g) is described by the Christoffel symbols $\Gamma^k_{ij} = \frac{1}{2} g^{kq}(\frac{\partial}{\partial x_i} g_{jq} + \frac{\partial}{\partial x_j} g_{iq} - \frac{\partial}{\partial x_q} g_{ij})$ in a local coordinate system $(x_1, \ldots, x_n)$. The Riemann curvature tensor of type $(0, 4)$ is given by $Rm = R_{ijkl} = g_{lq} R^q_{ijk}$, where $R^h_{ijl} = \frac{\partial}{\partial x_i}\Gamma^h_{jl} - \frac{\partial}{\partial x_j}\Gamma^h_{il} + \Gamma^h_{iq}\Gamma^q_{jl} - \Gamma^h_{jq}\Gamma^q_{il}$. The Ricci tensor is defined by $Rc = R_{ik} = g^{jl} R_{ijlk}$ and the scalar curvature is the contraction $R = g^{ik} R_{ik}$. The average scalar curvature is denoted by $r = \frac{1}{V(M)} \int_M R\, d\mu$. The sectional curvature of the plane $\sigma = \sigma\{v, w\}$ spanned by the vectors v^i and w^i is defined by $K(\sigma) = \frac{R(v,w,w,v)}{\|v\|^2\|w\|^2 - \langle v,w\rangle^2}$, where $R(u, v, w, z) = R_{ijkl} u^i v^j w^k z^l$.

The average integral of a function f is denoted by $\frac{1}{V(M)} \int_M f\, d\mu = \fint_M f\, d\mu$. Sign convention for the Laplace operator is $\Delta^g = -g^{pq}\partial_p \partial_q$.

By $C(\cdots)$ we denote positive constants (possibly different) whose value depends only on the arguments listed.

2. **The result.** In order to measure the deviation of metric g from a constant sectional curvature metric, we use the reduced curvature tensor

$$\widetilde{Rm} = \tilde{R}_{ijkl} = R_{ijkl} - \frac{r(0)}{n(n-1)} g_{ijkl}.$$

The tensor $g_{ijkl} = g_{jk} g_{il} - g_{jl} g_{ik}$ is the curvature tensor of the sphere S^n equipped with the canonical metric and $r(0)$ is the average scalar curvature of g. The g-trace of $\tilde{R}_{ijkl}$ is the reduced Ricci tensor

$$\widetilde{Rc} = \tilde{R}_{ij} = R_{ij} - \frac{r(0)}{n} g_{ij}.$$

In our situation all curvature tensors will evolve with time but the average scalar curvature will always be measured using the initial metric. Define

$$\mathrm{Ric}_x(v) = \sum_{i=2}^{n} K(v, e_i) \|v\|^2_{g_x},$$

where $x \in M$, $v \in T_xM$ and $\{e_i\}$ is an orthonormal basis of T_xM such that $e_1 = v/\|v\|_{g_x}$. Let

$$\rho(x) = \inf\left\{ \frac{\mathrm{Ric}_x(v)}{(n-1)\|v\|^2_{g_x}} \middle| v \in T_xM,\ v \neq 0 \right\}$$

and $\rho_-(x) = \sup\{-\rho(x), 0\}$. The integrand in assumption (b), Theorem 2.1, where $f_+(x) = \sup\{f(x), 0\}$, describes the part of the Ricci curvature which lies below $-(n-1)\alpha^2|r|$. Our goal is to prove the following statement.

THEOREM 2.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. For $\alpha > 0$, $D > 0$, $0 < \nu < 1$ there exists an ε depending on n, ν, α and D only, such that if $r \neq 0$ and*

(a) $d(M)^2|r| \leq D$

(b) $\frac{1}{V(M)} \int_M (\frac{\rho}{|r|\alpha^2} - 1)_+^{\frac{n+\iota}{2}}\, d\mu \leq A(n, \nu, \alpha, D)$

(c) $\frac{1}{V(M)} \int_M \| \widetilde{Rm} \|^{\frac{n}{2}+\nu}\, d\mu < \varepsilon |r|^{\frac{n}{2}+\nu}$

then M admits a metric of constant sectional curvature equal to $\frac{r}{n(n-1)}$. ∎

The constant A in (b) is given by $A(n, \nu, \alpha, D) = \frac{1}{2}(e^{B(n,\nu)\alpha D} - 1)^{-1}$, where $B(n, \nu)$ is a constant (see [Ga 2]) which depends on n and ν only.

We now present main ideas and concepts used in the proof.

(a) DECOMPOSITION OF THE CURVATURE TENSOR. The space of curvature tensors of a Riemannian manifold M of dimension $n \geq 4$ splits orthogonally into 3 irreducible subspaces: for the tensor R_{ijkl} there is the decomposition

$$R_{ijkl} = \frac{R}{n(n-1)} g_{ijkl} + Z_{ijkl} + W_{ijkl}, \tag{1}$$

where $(n-2)Z_{ijkl} = z_{jk}g_{il} + z_{il}g_{jk} - z_{ik}g_{jl} - z_{jl}g_{ik}$ is the traceless Ricci tensor with $z_{ij} = R_{ij} - \frac{R}{n} g_{ij}$, and W_{ijkl} is the Weyl conformal curvature tensor. In dimension 3 the Weyl tensor vanishes, and the corresponding decomposition is $R_{ijkl} = \frac{R}{6} g_{ijkl} + Z_{ijkl}$.

(b) RICCI FLOW. The idea of "deforming the metric in the direction of its Ricci curvature" has been successfully used in a number of occasions. It gives a system (2) of quasi-linear second order partial differential equations which can be integrated for some (maximal) time on any compact manifold, with curvature increasing beyond any bounds if that time is finite (see Theorem 3.1).

(c) EQUIVALENCE OF THE SOBOLEV AND THE ISOPERIMETRIC CONSTANTS. One of the crucial ingredients in our proof is the manifold-independent version of the Sobolev inequality, which is obtained in Theorem 4.1. We use [Ga 2] to estimate the Sobolev constant from above in terms of the isoperimetric constant, which, in turn, depends only on the parameters appearing in assumptions (a) and (b) of the Theorem 2.1.

(d) DE GIORGI-NASH-MOSER ITERATION. This powerful technique is used to estimate the maximum norm of a subsolution v of a certain type of partial differential equations on a subdomain $D' \subseteq D$ in terms of its L^p norm on D for $p > 1$. The idea is to obtain integral inequalities (like the one of Lemma 5.2) from where one can estimate $\iint |dv|^2\, dx\, dt$ and $\max_t \int_M v^2\, dx$ from above in terms of $\iint v^2\, dx\, dt$ with constants depending on the geometry of the region. Then the Sobolev inequality is used to estimate $\iint v^{2k}\, dx\, dt$(with $k > 1$) in terms of $\iint v^2\, dx\, dt$, so that one obtains an integral estimate for a higher power of v. The estimate for the maximum norm is then obtained by repeating this process. The ideas of the method carry over to our setup without any problems, since the Sobolev inequality holds globally on compact manifolds.

The proof of the theorem is given in several stages. We deform the initial metric on the manifold using the Ricci flow, compute the heat equation for the curvature tensor $\widetilde{Rm}$ and then the evolution inequality for its norm (Section 3), which involves functions on M only, and is therefore easier to work with. We then apply the iteration technique of De Giorgi, Nash and Moser twice: first to get the L^p estimate (for some $p > n$) of the norm $\| \widetilde{Rm} \|$ in terms of its $L^{\frac{n}{2}+\nu}$ norm, and then to obtain the L^∞ bound of $\| \widetilde{Rm} \|$ using that L^p bound. Thus the proof is reduced to the results given in [Gr] and [G-K-R]. In Section 4 we show that in our framework the Sobolev constant can be bounded from above by the isoperimetric constant. The proof of the theorem is completed by showing that for some small initial time interval the appropriate L^p norm of the reduced curvature tensor and the Sobolev inequalities are well behaved.

3. **Evolution of the reduced curvature.** Let (M, g) be a compact n-dimensional Riemannian manifold. The initial metric $g(0) = g$ on M is deformed according to the equation

$$\frac{\partial}{\partial t} g(t) = -2\widetilde{Rc}(t) = -2Rc(t) + \frac{2r(0)}{n} g(t). \tag{2}$$

Existence of such flows of metrics is guaranteed by the following Theorem [Ha]:

THEOREM 3.1 (R. HAMILTON). *Let (M, g) be a compact Riemannian manifold. The evolution equation (2) has a unique solution on a maximal time interval $0 \le t \le T \le \infty$. If $T < \infty$ then $\lim_{t\to T}(\max_M \| \widetilde{Rm}(t) \|_{g(t)}) = \infty$.* ∎

The equation we use differs from Hamilton's: we determine the average scalar curvature using $g(0)$ and not $g(t)$. Consequently the volume of the manifold could change during evolution: it could shrink or blow up.

THEOREM 3.2. *The reduced Riemann curvature tensor satisfies the following evolution equation:*

$$\Big(\frac{\partial}{\partial t} + \Delta\Big)\tilde{R}_{ijkl} = \tilde{Q}_{ijkl} + \frac{2r(0)}{n}\Big(\tilde{R}_{ijkl} - \frac{n}{n-1} Z_{ijkl} - 2W_{ijkl}\Big), \tag{3}$$

where $\tilde{Q}_{ijkl}$ is quadratic in the components of $\tilde{R}_{ijkl}$, Z_{ijkl} is the traceless Ricci tensor and W_{ijkl} the Weyl conformal curvature tensor.

PROOF. The time derivative of $\tilde{R}_{ijkl}$ is computed to be

$$\frac{\partial}{\partial t}\tilde{R}_{ijkl} = \tilde{B}_{ijkl} + \frac{2r(0)}{n(n-1)}(\tilde{R}_{jk}g_{il} + \tilde{R}_{il}g_{jk} - \tilde{R}_{ik}g_{jl} - \tilde{R}_{jl}g_{ik})$$
$$- \frac{r(0)}{n(n-1)}g^{qp}(g_{ijql}\tilde{R}_{kp} - g_{ijqk}\tilde{R}_{pl}) - g^{qp}(\tilde{R}_{ijql}\tilde{R}_{kp} - \tilde{R}_{ijqk}\tilde{R}_{pl}),$$

where $\tilde{B}_{ijkl} = \partial_i\partial_l\tilde{R}_{jk} - \partial_j\partial_l\tilde{R}_{ik} - \partial_i\partial_k\tilde{R}_{jl} + \partial_j\partial_k\tilde{R}_{il}$. Furthermore,

$$\Delta\tilde{R}_{ijkl} = -\tilde{B}_{ijkl} + \tilde{Q}_{ijkl} - \frac{2r(0)}{n}\tilde{R}_{ijkl} + \frac{3r(0)}{n(n-1)}(\tilde{R}_{ik}g_{jl} - \tilde{R}_{il}g_{jk} - \tilde{R}_{jk}g_{il} + \tilde{R}_{jl}g_{ik}),$$

where

$$\tilde{Q}_{ijkl} = g^{ab}(\tilde{R}_{ib}\tilde{R}_{jakl} - \tilde{R}_{jb}\tilde{R}_{iakl}) + 2g^{ab}g^{mn}(\tilde{R}_{imjb}\tilde{R}_{knla} + \tilde{R}_{imkb}\tilde{R}_{jnla} - \tilde{R}_{imjb}\tilde{R}_{lnka} - \tilde{R}_{imlb}\tilde{R}_{jnka}).$$

Combining the above two formulas we get

$$\left(\frac{\partial}{\partial t} + \Delta\right)\tilde{R}_{ijkl} = \tilde{Q}_{ijkl} - \frac{2r(0)}{n}(\tilde{R}_{ijkl}) + \frac{r(0)}{n(n-1)}(2g_{jk}\tilde{R}_{il} + 2g_{il}\tilde{R}_{jk} - 2g_{ik}\tilde{R}_{jl} + 2g_{jl}\tilde{R}_{ik})$$
$$= \tilde{Q}_{ijkl} - \frac{2r(0)}{n}\tilde{R}_{ijkl} + \frac{2r(0)}{n(n-1)}\left((n-2)Z_{ijkl} + 2\frac{\tilde{R}}{n}g_{ijkl}\right),$$

since $g_{jk}\tilde{R}_{il} = g_{jk}(z_{il} + \frac{\tilde{R}}{n}g_{il})$. Using the decomposition (1) to replace g_{ijkl} and the formula $\tilde{R} = R - r(0)$ we obtain the statement of the theorem. ∎

We now derive a weak parabolic inequality for the norm of the reduced curvature tensor. With little extra work we could compute the actual equation (see [Ha] or [Hu] for the case of Rm). The norm of a (time dependent) tensor $T(t)$ will be denoted by $\|T(t)\|_{g(t)}$ or just $\|T\|$, keeping in mind that it is taken with respect to the metric $g(t)$. We are also going to drop the indices and write $\widetilde{Rm}$ instead of $\tilde{R}_{ijkl}$, $\tilde{Q}$ instead of $\tilde{Q}_{ijkl}$, and so on.

Computing the scalar product of (3) with $\widetilde{Rm}$, using the identity

$$\Delta\langle T, T\rangle = -2\|\nabla T\|^2 + 2\langle T, \Delta T\rangle$$

(which holds for any tensor field T, with Laplacian defined by $\Delta = -g^{pq}\partial_p\partial_q$), and orthogonality of the decomposition of Rm we obtain

$$\frac{1}{2}\left(\frac{\partial}{\partial t} + \Delta\right) \| \widetilde{Rm} \|^2 + \|\nabla\widetilde{Rm}\|^2 = \langle\tilde{Q}, \widetilde{Rm}\rangle + \langle\widetilde{Rc} * \widetilde{Rm}, \widetilde{Rm}\rangle$$
$$+ \frac{2r(0)}{n}\left(\| \widetilde{Rm} \|^2 - \frac{n}{n-1}\|Z\|^2 - 2\|W\|^2\right),$$

where the term $\widetilde{Rc} * \widetilde{Rm}$, quadratic in curvature, comes from the derivative of the inner product. Therefore for any $r(0) \neq 0$

$$\left(\frac{\partial}{\partial t} + \Delta\right) \| \widetilde{Rm} \|^2 + 2\|\nabla\widetilde{Rm}\|^2 \leq A_1|r(0)| \, \| \widetilde{Rm} \|^2 + A_2 \, \| \widetilde{Rm} \|^3, \tag{4}$$

where A_1 and A_2 are positive constants depending on n only.

THEOREM 3.3. *The norm of the reduced curvature tensor satisfies the following weak parabolic inequality:*

$$\left(\frac{\partial}{\partial t}+\Delta\right) \| \widetilde{Rm} \| \leq A_1|r(0)| \| \widetilde{Rm} \| +A_2 \| \widetilde{Rm} \|^2 . \tag{5}$$

PROOF. We start with formula (4), use the chain rule and the second Kato inequality (which is true in the sense of distributions, see [Bé]):

$$\langle \widetilde{Rm}, \Delta\widetilde{Rm}\rangle \geq \| \widetilde{Rm} \| \Delta \| \widetilde{Rm} \|$$

and then divide both sides of the inequality by $\| \widetilde{Rm} \|$. ∎

4. **Sobolev and isoperimetric constants.** We are going to show that under assumptions (a) and (b) of the main theorem (Theorem 2.1) the Sobolev constant of the imbedding $W_2^1(M) \hookrightarrow L^r(M)$ with $2 \leq r \leq \frac{2(n+\nu)}{n+\nu-2}$, $0 < \nu < 1$ can be made independent of the particular manifold (*i.e.* it will depend on n, ν, α and D only).

For a compact manifold M, we define [Ga 2, p. 201]

$$\lambda_{r,2}(M) = \inf_{f\in C^\infty(M)}\left(\frac{\|\nabla f\|_2}{\inf_{a\in\mathbf{R}}\|f-a\|_r V(M)^{\frac{1}{2}-\frac{1}{r}}}\right), \tag{6}$$

where $C^\infty(M)$ denotes the space of smooth functions on M.

THEOREM 4.1. *Let (M,g) be a compact n-dimensional Riemannian manifold, α and D positive constants and $0<\nu<1$. Assume that*

(a) $\operatorname{diam}(M) \leq D$

(b) $\frac{1}{V(M)}\int_M(\frac{\rho_-}{\alpha^2}-1)_+^{\frac{n+\nu}{2}}\,d\mu \leq A(n,\nu,\alpha,D)$.

Then for $2 \leq r \leq \frac{2(n+\nu)}{n+\nu-2}$

$$\left(\fint_M f^r\,d\mu\right)^{\frac{2}{r}} \leq C_r(n,\nu,\alpha,D)\left(\fint_M |df|^2\,d\mu+\fint_M f^2\,d\mu\right). \tag{7}$$

(see Section 2 for the value of the constant A).

PROOF. Let $f_1 = f - \frac{1}{V(M)}\int_M f\,d\mu$ and $s=\frac{2(n+\nu)}{n+\nu-2}$. Then

$$\|f\|_s \leq \|f_1\|_s + V(M)^{\frac{1}{s}-\frac{1}{2}}\|f\|_2,$$

by the the Hölder inequality, and therefore

$$V(M)^{\frac{1}{2}-\frac{1}{s}}\|f\|_s \leq \lambda_{s,2}^{-1}(M)\|df\|_2+\|f\|_2, \tag{8}$$

by definition (6). Applying the theorem of S.Gallot [Ga 2, Theorem 6(iv)] with $\frac{1}{n+\nu}=\frac{1}{2}-\frac{1}{s}$ yields

$$\lambda_{s,2}^{-1}(M) \leq \frac{C(n,\nu)}{Is(n,\nu,M)} \leq C(n,\nu,\alpha,D),$$

where $Is(n,\nu,M)$ denotes the isoperimetric constant

$$Is(n,\nu,M)=\inf_{\Omega}\left(\frac{V(\partial\Omega)}{\min[V(\Omega),V(M-\Omega)]^{1-\frac{1}{n+\nu}}V(M)^{\frac{1}{n+\nu}}}\right),$$

the infimum being taken over all domains $\Omega\subset M$ with regular boundary. The result now follows from (8) and from the inequality $\|f\|_r\le V(M)^{\frac{1}{r}-\frac{1}{s}}\|f\|_s$ which holds for $2\le r\le s$. ■

5. **De Giorgi-Nash-Moser iteration.** As usual, (M^n,g) denotes a compact n-dimensional Riemannian manifold. Let $t\mapsto g(t)$ be a family of smooth Riemannian metrics defined for $0\le t\le T\le 1$, where $T>0$ and $g(0)=g$. We study solutions of the weak inequality

$$\left(\frac{\partial}{\partial t}+\Delta^{g(t)}\right)u(x,t)\le A_1u(x,t)+A_2u(x,t)^2 \tag{9}$$

where $u\colon M\times[0,T]\longrightarrow\mathbf{R}$ is a positive smooth function with square integrable first partial derivatives and A_1 and A_2 are constants. Let $2\le r\le\frac{2(n+\nu)}{n+\nu-2}$ with $0<\nu<1$.

LEMMA 5.1. *For $q>1$ and a smooth positive function u satisfying (9)*

$$\begin{aligned}\frac{\partial}{\partial t}\left(\fint_M u^q\,d\mu(t)\right)+\frac{4(q-1)}{q}\fint_M|d(u^{\frac{q}{2}})|^2\,d\mu(t)\\ \le qA_1\fint_M u^q\,d\mu(t)+qA_2\fint_M u^{q+1}\,d\mu(t)+\fint_M u^q|\tilde R|\,d\mu(t)\\ +\left(\fint_M|\tilde R|\,d\mu(t)\right)\left(\fint_M u^q\,d\mu(t)\right),\end{aligned}\tag{10}$$

where $\tilde R=R-r(0)$ is the reduced scalar curvature. ■

LEMMA 5.2. *Let u be a positive smooth solution of (9) and assume that*

$$\left(\fint_M f^r\,d\mu(t)\right)^{\frac{2}{r}}\le C_r\left(\fint_M|df|^2\,d\mu(t)+\fint_M f^2\,d\mu(t)\right)\tag{11}$$

for $t\in[0,T]$; C_r is a positive constant and f is smooth function on M.

(A) Suppose that

$$\fint_M u^{n+\nu}\,d\mu(t)\le\varepsilon \text{ and } \fint_M\|\widetilde{Rm(t)}\|_{g(t)}^{n+\nu}\,d\mu(t)\le\varepsilon$$

hold for $t\in[0,T]$. Then for $q\ge\frac{3}{2}$, $t\in[0,T]$ and $C_5=C(n,\nu,\varepsilon,C_r,A_1,A_2)$

$$\frac{\partial}{\partial t}\left(\fint_M u^q\,d\mu(t)\right)+\frac{2(q-1)}{q}\fint_M|d(u^{\frac{q}{2}})|^2\,d\mu(t)\le C_5q^{n+1}\fint_M u^q\,d\mu(t).\tag{12}$$

(B) Suppose that

$$\fint_M u^{\frac{n}{2}+\nu}\,d\mu(t)\le\varepsilon \text{ and } \fint_M\|\widetilde{Rm(t)}\|_{g(t)}^{\frac{n}{2}+\nu}\,d\mu(t)\le\varepsilon$$

hold for any $t\in[0,T]$ with $\varepsilon^{\frac{2}{n+2\nu}}\le\left(QC_r(n+A_2)\right)^{-1}$, for some $Q>\frac{3}{2}$. Then for $\frac{3}{2}\le q\le Q$, $t\in[0,T]$ and $C_6=C(n,\nu,C_r,A_1,A_2,Q)$

$$\frac{\partial}{\partial t}\left(\fint_M u^q\,d\mu(t)\right)+\frac{2(q-1)}{q}\fint_M|d(u^{\frac{q}{2}})|^2\,d\mu(t)\le C_6q^{n+1}\fint_M u^q\,d\mu(t).\tag{13}$$

PROOF. We have to estimate the terms on the right side of (11). Using the Hölder inequality and the inequality $2ab \leq \lambda a^2 + \frac{1}{\lambda} b^2$ with $\lambda > 0$ we obtain

$$\fint_M u^{q+1}\, d\mu(t) \leq \Big(\fint_M u^{n+\nu}\, d\mu(t)\Big)^{\frac{1}{n+\nu}} \Big(\fint_M u^{q\frac{n+\nu}{n+\nu-1}}\, d\mu(t)\Big)^{\frac{n+\nu-1}{n+\nu}}$$
$$\leq \varepsilon^{\frac{1}{n+\nu}} \Big(\frac{\lambda}{2}\Big(\fint_M u^{q\frac{n+\nu}{n+\nu-2}}\, d\mu(t)\Big)^{\frac{n+\nu-2}{n+\nu}} + \frac{1}{2\lambda}\fint_M u^q\, d\mu(t)\Big).$$

The remaining terms are estimated similarly, and thus

$$\frac{\partial}{\partial t}\Big(\fint_M u^q\, d\mu(t)\Big) + \frac{4(q-1)}{q}\fint_M |d(u^{\frac{q}{2}})|^2\, d\mu(t) \leq C_1 \fint_M u^q\, d\mu(t) + C_2 \Big(\fint_M u^{q\frac{n+\nu}{n+\nu-2}}\, d\mu(t)\Big)^{\frac{n+\nu-2}{n+\nu}},$$

where $C_1 = qA_1 + n\varepsilon^{\frac{1}{n+\nu}} + \frac{1}{2\lambda}\varepsilon^{\frac{1}{n+\nu}}(n+qA_2)$ and $C_2 = \frac{\lambda}{2}\varepsilon^{\frac{1}{n+\nu}}(n+qA_2)$. Applying the Sobolev inequality (11) with $f = u^{\frac{q}{2}}$ gives

$$\frac{\partial}{\partial t}\Big(\fint_M u^q\, d\mu(t)\Big) + C_3 \fint_M |d(u^{\frac{q}{2}})|^2\, d\mu(t) \leq C_4 \fint_M u^q\, d\mu(t),$$

where $C_3 = \frac{4(q-1)}{q} - C_r C_2$ and $C_4 = C_1 + C_2 C_r$.

Choose $\lambda = 4C_r^{-1}\varepsilon^{-\frac{1}{n+\nu}}(n+qA_2)^{-1}\frac{q-1}{q}$. It follows that $C_3 = \frac{2(q-1)}{q}$ and $C_4 \leq C_5 q^{n+1}$, where $C_5 = C(n, \nu, \varepsilon, C_r, A_1, A_2)$. This proves (A). Statement (B) is proved similarly. ∎

COROLLARY 5.3. *Let u satisfy (12) or (13) and let $0 \leq t \leq t' \leq T$. Then*

$$\fint_M u^q\, d\mu(t') \leq e^{C_8(t'-t)} \fint_M u^q\, d\mu(t) \tag{14}$$

holds if $q \geq \frac{3}{2}$ (then $C_8 = C_5 q^{n+1}$) or if $\frac{3}{2} \leq q \leq Q$ (then $C_8 = C_6 q^{n+1}$). ∎

In Lemma 5.2 we showed that for the positive smooth solution $u\colon M \times [0, T] \longrightarrow \mathbf{R}$, $0 < T \leq 1$ of (9) there is the inequality

$$\frac{\partial}{\partial t}\Big(\fint_M u^q\, d\mu(t)\Big) + \frac{2(q-1)}{q}\fint_M |d(u^{\frac{q}{2}})|^2\, d\mu(t) \leq C_7 q^{n+1}\fint_M u^q\, d\mu(t), \tag{15}$$

where C_7 is a constant and $q \geq \frac{3}{2}$.

THEOREM 5.4. *Let u satisfy (15) and assume that for every $t \in [0, T]$*

$$\Big(\fint_M f^r\, d\mu(t)\Big)^{\frac{2}{r}} \leq C_r\Big(\fint_M |df|^2\, d\mu(t) + \fint_M f^2\, d\mu(t)\Big), \tag{16}$$

where C_r is a constant, f is a smooth function on M and $2 \leq r \leq \frac{2(n+\nu)}{n+\nu-2}$ with $0 < \nu < 1$.

(A) Let $Q \geq \frac{3}{2}$. Then for any δ, $0 < \delta \leq \frac{2}{n+\nu}$, and $C_{13} = C(n, \nu, C_r, C_7, Q)$

$$\max_{M\times[\frac{2T}{3}, T]} u(x,t) \leq C_{13} T^{-\frac{1}{Q\delta}}\Big(\int_{T/3}^{T}\fint_M u^Q\, d\mu(t)\Big)^{\frac{1}{Q}}. \tag{17}$$

(B) Let $Q \geq Q' \geq \frac{3}{2}$ and assume that inequality (15) holds only for q satisfying $Q \geq q \geq Q'$. Then at the time $t = \frac{2T}{3}$ and for $\kappa = 1 + \delta$ with $0 < \delta \leq \frac{2}{n+\nu}$

$$\Big(\fint_M u^Q\, d\mu(2T/3)\Big)^{\frac{1}{Q}} \leq C_{14} T^{-\frac{1}{\delta Q'}(1-\kappa^{-C_{15}})}\Big(\int_{T/3}^{T}\fint_M u^{Q'}\, d\mu(t)\Big)^{\frac{1}{Q'}}, \tag{18}$$

where $C_{14} = C(n, \nu, C_r, C_7, Q, Q')$ and $C_{15} = C(n, \nu, Q, Q')$.

PROOF. The iteration method of De Giorgi, Nash and Moser (see [Mo]) on the compact manifold $M \times [0, T]$ will be applied. We use the following notation:

$$H_q(t) = \fint_t^T \fint_M u^q \, d\mu(t)\, dt,$$
$$D_q(t) = \fint_t^T \fint_M |d(u^{\frac{q}{2}})|^2 \, d\mu(t)\, dt, \text{ and}$$
$$M_q(t) = \max_{[t,T]} \fint_M u^q \, d\mu(t).$$

STEP 1. For $\frac{T}{3} \leq t < t' \leq \frac{2T}{3}$ and $q \geq \frac{3}{2}$ there are estimates

$$M_q(t') \leq 2T\Big(C_7 q^{n+1} + \frac{2}{t'-t}\Big) H_q(t) \text{ and} \tag{19}$$

$$D_q(t') \leq \frac{9}{2}\Big(C_7 q^{n+1} + \frac{2}{t'-t}\Big) H_q(t). \tag{20}$$

These estimates are proved as in [M-R 2], [Go 2] and [Ya 1].

The next two steps represent the crucial moment of the iteration - getting the estimate for H at some time t' in terms of certain power of H (bigger than 1), computed at some earlier time t. Let $\kappa = 1 + \delta$ with $0 < \delta \leq \frac{2}{n+\nu}$.

STEP 2. For $t \in [0, T]$ and $q \geq \frac{3}{2}$ there is the inequality

$$H_{q\kappa}(t) \leq C_r M_q(t)^{\delta}\big(H_q(t) + D_q(t)\big). \tag{21}$$

PROOF. We combine the Hölder inequality (with $\frac{2}{r} + \delta = 1$) and (16).

STEP 3. Let $\frac{T}{3} \leq t < t' \leq \frac{2T}{3}$ and $q \geq \frac{3}{2}$. Then

$$H_{q\kappa}(t') \leq C_9 T^{\kappa-1}\Big(C_{10} q^{n+1} + \frac{2}{t'-t}\Big)^{\kappa} H_q(t)^{\kappa}. \tag{22}$$

PROOF. Combining the inequalities (19), (20) and (21) we get

$$H_{q\kappa}(t') \leq C_r M_q(t')^{\delta}\big(H_q(t') + D_q(t')\big) \leq C_9 T^{\delta}\Big(C_{10} q^{n+1} + \frac{2}{t'-t}\Big)^{\kappa} H_q(t)^{\kappa},$$

where $C_9 = C(C_r)$ and $C_{10} = C(C_7)$.

STEP 4. For $j \geq 0$ define $t_j = \frac{2T}{3} - \frac{T}{3}\frac{1}{\kappa^j}$ and $q_j = Q\kappa^j$ (so that $t_0 = \frac{T}{3}$, $t_\infty = \frac{2T}{3}$, $q_0 = Q$ and $q_\infty = \infty$). Inequality (22) with $q = q_j$, $t' = t_{j+1}$ and $t = t_j$ gives

$$H_{q_j\kappa}(t_{j+1}) \leq C_{12} T^{-1} q_j^{(n+1)\kappa} \kappa^{j\kappa} H_{q_j}(t_j)^{\kappa},$$

where $C_{12} = C(C_r, C_7, \kappa)$. Let $L(q_j, t_j) = \big(H_{q_j}(t_j)\big)^{\frac{1}{q_j}}$ for $j \geq 0$. Then

$$\begin{aligned} L(q_j, t_j) &= \big(H_{\kappa q_{j-1}}(t_j)\big)^{\frac{1}{q_j}} \leq \big(C_{12} T^{-1} q_{j-1}^{(n+1)\kappa} \kappa^{(j-1)\kappa} H^{\kappa}_{q_{j-1}}(t_{j-1})\big)^{\frac{1}{q_j}} \\ &= C_{12}^{\frac{1}{q_j}} T^{-\frac{1}{q_j}} q_{j-1}^{\frac{(n+1)\kappa}{q_j}} \kappa^{\frac{(j-1)\kappa}{q_j}} L(q_{j-1}, t_{j-1}) \\ &\leq C_{12}^{\sum_1^j \frac{1}{q_i}} T^{-\sum_1^j \frac{1}{q_i}} \Big(\prod_1^j q_{i-1}^{\frac{(n+1)\kappa}{q_i}}\Big) \kappa^{\sum_1^j \frac{(i-1)\kappa}{q_i}} L(q_0, t_0). \end{aligned} \tag{23}$$

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STEP 5. To prove (A), let $j \longrightarrow \infty$ in (23). All sums and the product converge; in particular, $\sum_1^\infty \frac{1}{q_i} = \frac{1}{Q\delta}$. Hence, with $C_{13} = C(n, \nu, C_r, C_7, Q)$,

$$\max_{M\times[\frac{2T}{3},T]} u(x,t) \leq C_{13} T^{-\frac{1}{Q\delta}} \Big(\fint_{T/3}^{T} \fint_M u^Q \, d\mu(t)\, dt \Big)^{\frac{1}{Q}}.$$

STEP 6. To prove (B), first choose J to be the smallest positive integer such that $Q'\kappa^J \geq Q$. For $0 \leq j \leq J$ define $t_j = \frac{2T}{3} - \frac{T}{3}\frac{1}{\kappa^j}$ and $q_j = Q'\kappa^j$ (so that $t_0 = \frac{T}{3}$, $t_J = \frac{2T}{3} - \frac{T}{3}\kappa^{-J} < \frac{2T}{3}$, $q_0 = Q'$ and $q_J \geq Q$), and continue using (23) and (19). ∎

6. **Proof of Theorem 2.1.** Consider the Ricci flow $t \longmapsto g(t)$ of metrics (2) on $\big(M, g = g(0)\big)$. The initial metric g can be rescaled (all assumptions of the theorem are scale invariant), so that $|r| = |r(0)| = 1$. The evolution equation for the curvature implies the following weak parabolic inequality (see Theorem 3.3):

$$\Big(\frac{\partial}{\partial t} + \Delta^{g(t)}\Big) \| \widetilde{Rm}(t) \|_{g(t)} \leq A_1 \| \widetilde{Rm}(t) \|_{g(t)} + A_2 \| \widetilde{Rm}(t) \|^2_{g(t)}, \tag{24}$$

where A_1, A_2 depend on n only.

Let $[0, \tau), \tau > 0$ be the maximal interval such that

(i) equation (2) has a solution

(ii) $(\fint_M f^r \, d\mu)^{\frac{2}{r}} \leq 10 C_r (\fint_M |df|^2 \, d\mu + \fint_M f^2 \, d\mu)$

(iii) $\fint_M \| \widetilde{Rm}(t) \|^{\frac{n}{2}+\nu}_{g(t)} < 10\varepsilon$

hold on $[0, \tau)$. In (ii) $2 \leq r \leq \frac{2(n+\nu)}{n+\nu-2}$ with $0 < \nu < 1$ and $C_r = C(n, \nu, \alpha, D)$, and in assumption (iii) $\varepsilon = \varepsilon(n, \nu, \alpha, D) \leq \big(2nC_r(n + A_2)\big)^{-\frac{2}{n+2\nu}}$.

It is possible to find such τ since by Theorem 3.1 the Ricci flow exists for at least a short (universal) period of time, assumptions (a) and (b) of Theorem 2.1 with the metric g rescaled so that $|r| = 1$ and Theorem 4.1 imply (ii), and assumption (c) of Theorem 2.1 implies (iii). We can assume that $\tau \leq 1$.

LEMMA 6.1. *Assume that properties (i)–(iii) hold on the (maximal) time interval* $[0, \tau)$. *Then*

(iv) $\max_M \| \widetilde{Rm}(t) \|_{g(t)} \leq C_{20} t^{-1+C_{21}} \varepsilon^{\frac{2}{n+2\nu}}$

holds for every $t \in (0, \tau)$, *where* $C_{21} = C(n, \nu, \alpha, D)$ *is a positive constant.*

PROOF. Take any $t \in (0, \tau)$. We are going to use the De Giorgi-Nash-Moser iteration twice: first to obtain the estimate for some L^p norm of the reduced curvature (for $p > n$) in terms of its $L^{\frac{n}{2}+\nu}$ norm, and then to estimate its maximum norm in terms of that L^p norm.

Let $u = \| \widetilde{Rm}(t) \|_{g(t)}$. For the first iteration, choose the interval $[0, T] = [0, t/2]$, and let $Q' = \frac{n+2\nu}{2}$, $Q = \frac{n+2\nu}{2}(1 + \frac{2}{n+2\nu})^J \geq n + \nu$ (where J denotes the smallest positive integer for which this is true). Properties (i)–(iii) and the choice of ε, together with Lemma 5.2, part (B) guarantee that the assumptions of Theorem 5.4 are satisfied. Hence by Theorem 5.4 part (B)

$$\Big(\fint_M u^Q \, d\mu(t/3) \Big)^{\frac{1}{Q}} \leq C_{16} t^{-1+\kappa^{-J}} \Big(\fint_{t/6}^{t/2} \fint_M u^{\frac{n}{2}+\nu} \, d\mu(t) \Big)^{\frac{2}{n+2\nu}},$$

with $C_{16} = C(n,\nu,\alpha,D)$ and $\kappa = 1+\frac{2}{n+2\nu}$. For any s, $\frac{t}{3} \le s \le t$ it follows by Corollary 5.3 that, with $C_{17}, C_{18} = C(n,\nu,\alpha,D)$,

$$\Big(\fint_M u^Q\,d\mu(s)\Big)^{\frac{1}{Q}} \le e^{\frac{1}{Q}C_{17}(s-\frac{t}{3})}\Big(\fint_M u^Q\,d\mu(t/3)\Big)^{\frac{1}{Q}}$$
$$\le C_{18}t^{-1+\kappa^{-J}}\varepsilon^{\frac{2}{n+2\nu}}.$$

For the second iteration we let $[0,T] = [t/3,t]$. Part (A) of Lemma 5.2 and properties (i)–(iii) imply that the assumptions of Theorem 5.4 are satisfied, with $C_7 = C(n,\nu,\alpha,D)$. Therefore, according to part (A), Theorem 5.4, by choosing $\delta = \frac{2}{n+\nu}$ we conclude that

$$\max_{M\times[\frac{7t}{9},t]} u(x,t) \le C_{19}t^{-\frac{n+\nu}{n+2\nu}\kappa^{-J}}\Big(\int_{5t/9}^{t}\fint_M u^Q\,d\mu(t)\Big)^{\frac{1}{Q}}$$
$$\le C_{20}t^{-1+\kappa^{-J}-\frac{n+\nu}{n+2\nu}\kappa^{-J}}\varepsilon^{\frac{2}{n+2\nu}},$$

and therefore

$$\max_{M\times[\frac{7t}{9},t]} u(x,t) \le C_{20}t^{-1+C_{21}}\varepsilon^{\frac{2}{n+2\nu}},$$

where $C_{21} = \frac{\nu}{n+2\nu}\kappa^{-J} > 0$. All constants depend on n, ν, α and D. ∎

REMARK. From Corollary 5.3 it follows that for $0 \le t \le \tau$

$$\fint_M \|\widetilde{Rm}(t)\|_{g(t)}^{\frac{n}{2}+\nu}\,d\mu(t) \le e^{C_{22}t}\fint_M \|\widetilde{Rm}(0)\|_{g(0)}^{\frac{n}{2}+\nu}\,d\mu(0).$$

Therefore

$$\fint_M \|\widetilde{Rm}(t)\|_{g(t)}^{\frac{n}{2}+\nu}\,d\mu(t) \le 5\varepsilon < 10\varepsilon$$

whenever $t \le \frac{\ln 5}{C_{22}} = C(n,\nu,\alpha,D)$. ∎

We have to show that the Sobolev constant does not change much as we follow the flow for some short time. This will be accomplished by combining the inequalities of the next two lemmas. The fact that the exponent of t in (iv) is bigger than -1 plays a crucial rôle.

LEMMA 6.2. *For a smooth function f on M and $t \in (0,\tau)$*

$$\Big(\fint_M |f|^r\,d\mu(t)\Big)^{\frac{2}{r}} \le \exp(C_{24}t^{C_{21}})\Big(\fint_M |f|^r\,d\mu(0)\Big)^{\frac{2}{r}},$$

where $2 \le r \le \frac{2(n+\nu)}{n+\nu-2}$, $0 < \nu < 1$ *and* $C_{21}, C_{24} = C(n,\nu,\alpha,D)$.

PROOF. From the chain rule and Lemma 6.1 it follows that

$$\frac{\partial}{\partial t}\Big(\fint_M |f|^r\,d\mu(t)\Big)^{\frac{2}{r}} \le \frac{2}{r}2n\max_M \|\widetilde{Rm}(t)\|_{g(t)}\Big(\fint_M |f|^r\,d\mu(t)\Big)^{\frac{2}{r}}$$
$$\le C_{23}t^{-1+C_{21}}\Big(\fint_M |f|^r\,d\mu(t)\Big)^{\frac{2}{r}},$$

where $C_{23} = C(n,\nu,\alpha,D)$. ∎

LEMMA 6.3. *For a smooth function f on M and $0 < \nu < 1$ the following estimate holds for any $t \in (0, \tau)$, with $C_{21}, C_{27} = C(n, \nu, \alpha, D)$:*

$$\fint_M |df|^2 \, d\mu(t) + \fint_M f^2 \, d\mu(t) \geq \exp(-C_{27} t^{C_{21}}) \Big(\fint_M |df|^2 \, d\mu(0) + \fint_M f^2 \, d\mu(0) \Big).$$

PROOF. Since $\frac{\partial}{\partial t} |df|^2 = -2\widetilde{Rc}(df, df)$ it follows that

$$\fint_M \frac{\partial}{\partial t} |df|^2 \, d\mu(t) \geq -2\sqrt{n} \max_M \| \widetilde{Rm(t)} \|_{g(t)} \fint_M |df|^2 \, d\mu(t)$$

and therefore

$$\begin{aligned} \frac{\partial}{\partial t} \Big(\fint_M |df|^2 \, d\mu(t) + \fint_M f^2 \, d\mu(t) \Big) & \\ &\geq -C_{25} \max_M \| \widetilde{Rm(t)} \|_{g(t)} \Big(\fint_M |df|^2 \, d\mu(t) + \fint_M f^2 \, d\mu(t) \Big) \\ &\geq -C_{26} t^{-1+C_{21}} \Big(\fint_M |df|^2 \, d\mu(t) + \fint_M f^2 \, d\mu(t) \Big), \end{aligned}$$

where $C_{25}, C_{26} = C(n, \nu, \alpha, D)$. ■

REMARK. From Lemmas 6.2 and 6.3 and the Sobolev inequality (7) with $C_r = C(n, \nu, \alpha, D)$ we obtain

$$\begin{aligned} \Big(\fint_M |f|^r \, d\mu(t) \Big)^{\frac{2}{r}} &\leq \exp(C_{24} t^{C_{21}}) \Big(\fint_M |f|^r \, d\mu(0) \Big)^{\frac{2}{r}} \\ &\leq C_r \exp(C_{24} t^{C_{21}}) \Big(\fint_M |df|^2 \, d\mu(0) + \fint_M f^2 \, d\mu(0) \Big) \\ &\leq C_r \exp\big((C_{24} + C_{27}) t^{C_{21}} \big) \Big(\fint_M |df|^2 \, d\mu(t) + \fint_M f^2 \, d\mu(t) \Big). \end{aligned}$$

Consequently,

$$\Big(\fint_M |f|^r \, d\mu(t) \Big)^{\frac{2}{r}} < 10 C_r \Big(\fint_M |df|^2 \, d\mu(t) + \fint_M f^2 \, d\mu(t) \Big)$$

whenever $t \leq \big(\frac{\ln 5}{C_{24} + C_{27}} \big)^{C_{21}^{-1}}$. ■

LEMMA 6.4. *Let $T_0 = T_0(n, \nu, \alpha, D) = \min\Big(\frac{\ln 5}{C_{22}}, \big(\frac{\ln 5}{C_{24} + C_{27}} \big)^{C_{21}^{-1}} \Big)$. Then $\tau \geq T_0$.*

PROOF. Suppose not, *i.e.* let $\tau < T_0$. Since the norm of the curvature tensor is bounded as $t \longrightarrow \tau$, it follows that the evolution equation (i) has a solution on some $[0, \tilde{\tau}]$ for $\tau < \tilde{\tau} \leq T_0$. But (ii) and (iii) still hold on $[0, \tilde{\tau}]$, which means (by Lemma 6.1) that (iv) also holds on $[0, \tilde{\tau}]$, thus contradicting the maximality of τ. Therefore, $\tau \geq T_0$. ■

END OF THE PROOF OF THEOREM 2.1. Since $\tau \geq T_0$ it follows that

$$\max_M \| \widetilde{Rm(T_0)} \|_{g(T_0)} \leq C_{28} T_0^{-1+C_{21}} \varepsilon^{\frac{2}{n+2r}}, \tag{25}$$

where $C_{21}, C_{28}, T_0 = C(n, \nu, \alpha, D)$. Hence, after following the flow for some time, we obtain stronger pinching assumption. The theorem now follows from [Gr] in the case $r < 0$ and from [G-K-R] in the case $r > 0$. ■

REFERENCES

[Ad] R A Adams, *Sobolev Spaces*, New York, Academic Press, 1975

[Bé] P H Berard, *The Bochner Technique Revisited*, Bull Amer Math Soc (N S) (2) **19**(1988), 371–406

[Bo] J P Bourguignon, *L'Equation de la Chaleur Associee a la Courbure de Ricci* In Seminaire Bourbaki, 38eme annee, (653), 1985/1986, Asterisque **145-146**(1987), 45–61

[Ga 1] S Gallot, *Inegalites Isoperimetriques Courbure de Ricci et Invariants Geometriques*, C R Acad Sci Paris (I) **296**(1983), 333–336

[Ga 2] ———, *Isoperimetric Inequalities Based on Integral Norms of Ricci Curvature* In Colloque Paul Levy sur les Processus Stochastiques, Palaiseau, 1987, Asterisque **157-158**(1987), 191–216

[Go 1] L Z Gao, *Convergence of Riemannian Manifolds, Ricci and $L^{n/2}$ Curvature Pinching*, J Differ Geom **32**, 349–381 (1990)

[Go 2] ———, *$L^{n/2}$-Curvature Pinching*, J Differ Geom **32**(1990), 713–774

[Gr] M Gromov, *Manifolds of Negative Curvature*, J Differ Geom **13**(1978), 223–230

[G-K-R] K Grove, H Karcher and E A Ruh, *Jacobi Fields and Finsler Metrics on Compact Lie Groups with an Application to Differentiable Pinching Problems*, Math Ann **211**(1974), 7–21

[Ha] R S Hamilton, *Three Manifolds With Positive Ricci Curvature*, J Differ Geom **17**(1982), 255–306

[Hu] G Huisken, *Ricci Deformation of the Metric on a Riemannian Manifold*, J Differ Geom **21**(1985), 47–62

[M-R 1] M Min-Oo, E A Ruh, *Curvature Deformations* In Curvature and Topology of Riemannian Manifolds, (ed Shiohama, K Sakai and T Sunada), Proceedings, Katata 1985, Lecture Notes in Math **1201**, Berlin-Heidelberg-New York, Springer 1986, 180–190

[M-R 2] ———, *L^2 – Curvature Pinching*, Comment Math Helv **65**(1990), 36–51

[Mo] J Moser, *A Harnack Inequality for Parabolic Differential Equations*, Comm Pure Appl Math **XVII**(1964), 101–134

[Ya 1] D Yang, *Convergence of Manifolds With Integral Bounds on Curvature, I*, preprint

[Ya 2] ———, *L^p Pinching and Compactness Theorems for Compact Riemannian Manifolds*, Seminaire de Theorie Spectrale et Geometrie (6), Annee 1987–1988, Univ Grenoble I, Saint-Martin-d'Heres, 1988, 81–89

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