Compositio Mathematica **125:** 55–68, 2001.

Global Smoothing of Calabi–Yau Threefolds II

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(Received: 21 April 1999; accepted: 12 November 1999)

Abstract. The moduli spaces of Calabi–Yau threefolds are conjectured to be connected by the combination of birational contraction maps and flat deformations. In this context, it is important to calculate $\dim \mathrm{Def}(X)$ from $\dim \mathrm{Def}(\tilde{X})$ in terms of certain geometric information of f, when we are given a birational morphism $f: \tilde{X} \to X$ from a smooth Calabi–Yau threefold $\tilde{X}$ to a singular Calabi–Yau threefold X. A typical case of this problem is a conjecture of Morrison-Seiberg which originally came from physics. In this paper we give a mathematical proof to this conjecture. Moreover, by using output of this conjecture, we prove that certain Calabi–Yau threefolds with nonisolated singularities have flat deformations to smooth Calabi–Yau threefolds. We shall use invariants of singularities closely related to Du Bois's work to calculate $\dim \mathrm{Def}(X)$ from $\dim \mathrm{Def}(\tilde{X})$.

Mathematics Subject Classifications (2000): 14Jxx, 32Gxx.

Key words: Calabi–Yau, deformation, moduli space.

Introduction

The moduli space of Calabi–Yau threefolds is far from irreducible or connected in the usual sense. A lot of Calabi–Yau threefolds are, however, connected by the combination of birational contraction maps and flat deformations. In studying such phenomena, the following is a fundamental problem:

PROBLEM. Let $f: \tilde{X} \to X$ be a birational morphism from a smooth Calabi–Yau threefold $\tilde{X}$ to a singular Calabi–Yau threefold X. When does X have a flat deformation to a smooth Calabi–Yau threefold? Calculate $\dim \mathrm{Def}(X) - \dim \mathrm{Def}(\tilde{X})$ in terms of geometric informations of f, where $\mathrm{Def}(X)$ is the Kuranishi space of X.

We shall treat this problem in the case of X having nonisolated rational Gorenstein singularities and as an application, we shall give a mathematical proof to a conjecture posed by Morrison and Seiberg from a physical view point [M]. The case of X having isolated rational Gorenstein singularities or the case f being a primitive birational morphism has been studied in [Na-St,Na 1, Na 2] or in [Gr 1, 2], respectively.

Let us assume that X is a Calabi–Yau threefold with nonisolated rational Gorenstein singularities. By [Re], the singular locus Σ of X is a one-dimensional locally trivial flat deformation of a rational double point on a surface at a general point $p \in \Sigma$ except finite numbers of bad points; one calls these bad points *dissident*. Our main idea is to get information on deformations of X by studying dissident points. For a dissident point $p \in X$, we can define two invariants $\mu(X, p)$ and $\sigma(X, p)$ in a similar way as the case of an isolated singularity; the definitions of them are given between Lemmas 1.3 and 1.4.

Let Σ_0 be the set of dissident singular points of X and put $U := X \setminus \Sigma_0$. By Proposition 1.2 there is a natural inclusion $\iota: H^1(U, \Theta_U) \to \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$. Let $\varphi: \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to H^0(X, T^1_X)$ be the natural map and put $\phi := \varphi \circ \iota$. The main theorem of this paper is the following:

THEOREM 1.9. *Let X be a Calabi–Yau threefold with rational Gorenstein singularities. Then the map ϕ has the following properties:*

(1) *The image of ϕ is contained in $H^0_{\Sigma_0}(X, T^1_X)$. Moreover,*

$$\varphi^{-1}(H^0_{\Sigma_0}(X, T^1_X)) = H^1(U, \Theta_U).$$

(2) $\dim \mathrm{im}(\phi) \geqslant \Sigma_{p\in\Sigma_0}\{\mu(X, p) + \sigma(X, p)\} - \sigma(X)$.

(3) *If* $\mu(X, p) = 0$ *for all* $p \in \Sigma_0$, *then* $\dim \mathrm{im}(\phi) = \Sigma_{p\in\Sigma_0}\sigma(X, p) - \sigma(X)$.

The theorem plays an essential role in calculating $\dim \mathrm{Def}(X) - \dim \mathrm{Def}(\tilde{X})$ when X is obtained from a smooth Calabi–Yau threefold $\tilde{X}$ by a birational contraction morphism. In the remainder, we shall explain this by using the most basic example in the Morrison–Seiberg conjecture.

Let $f: \tilde{X} \to X$ be a birational morphism from a smooth Calabi–Yau threefold $\tilde{X}$ to a singular Calabi–Yau threefold X. Assume that $\mathrm{Exc}(f) =: D$ is a smooth surface having a conic bundle structure over $f(D) \cong \mathbf{P}^1$ and that $f|_D$ has no multiple fiber and has exactly n reducible fibers. X has exactly n dissident points corresponding to these reducible fibers. Let $k = \rho(\tilde{X}) - \rho(X)$. By a local calculation [Gr 2], there is an exact sequence

$$0 \to \mathcal{F} \to T^1_X \to \mathcal{O}_{\mathbf{P}^1}(4 - D^3) \to 0,$$

where $\mathcal{F}$ has support only on Σ_0; $H^0_{\Sigma_0}(X, T^1_X) = H^0(\mathcal{F})$. Again, by the argument in [Gr 2, Theorem(1.9)], we can prove that the composed map of φ and $H^0(X, T^1_X) \to H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(4 - D^3))$ is surjective. Now, by Theorem 1.9(1) we have $\dim \mathrm{im}(\varphi) = \dim\ \mathrm{im}(\phi) + h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(4 - D^3))$. Since it is easily checked that $\mu(X, p) = 0$ for all $p \in \Sigma_0$, one can calculate $\dim \mathrm{im}(\varphi)$ by Theorem 1.9(3). In our case, $\mathrm{Ker}(\varphi) = H^1(X, \Theta_X)$ is isomorphic to $H^1(\tilde{X}, \Theta_{\tilde{X}})$. The result is:

THEOREM 2.1 (Morrison–Seiberg). *Let $\tilde{X}$ and X be as above. Assume that $n \geqslant 3$. Then the Kuranishi space* $\mathrm{Def}(X)$ *of X is smooth, and* $\dim \mathrm{Def}(X) = \dim \mathrm{Def}(\tilde{X}) + 2n - 2 - k$.

By using the dimension count in Theorem 2.1, we can prove the following theorem.

THEOREM 2.3 *Let* $\tilde{X}$ *and* X *be as above. Assume that* $n \geqslant 4$. *Then* X *has a flat deformation to a smooth Calabi–Yau threefold.*

NOTATION. Let $D \subset W$ be a simple normal crossing divisor in a smooth variety. Let Ω_D^p be the sheaf of p-forms on D. There is a subsheaf of Ω_D^p consisting of the sections whose supports are contained in the singular locus D_{sing} of D. By $\hat{\Omega}_D^p$ we mean the quotient sheaf of Ω_D^p by this subsheaf. We denote by $D^{[0]}$ the normalization of D. Let D_i $(1 \leqslant i \leqslant n)$ be irreducible components of $D^{[0]}$. Then we define $D^{[l]} := \coprod_{i_0 < \ldots < i_l} D_{i_0} \cap D_{i_1} \cap \ldots \cap D_{i_l}$. Let $h^{[l]} : D^{[l]} \to W$ be the natural map. Then we simply write $\mathcal{O}_{D^{[l]}}$ for $h_*^{[l]}\mathcal{O}_{D^{[l]}}$.

1. Deformation of Calabi–Yau Threefolds with Nonisolated Singularities

PROPOSITION 1.1. *Let* (X, p) *be the germ of a (possibly nonisolated) normal Gorenstein singularity of* $\dim \geqslant 3$ *at a point* p. *Then one has isomorphisms*

$$H^0_{\{p\}}(X, T^1_X) \cong H^1(U, \Theta_U) \quad \textit{and} \quad H^0(X, T^1_X) \cong \mathrm{Ext}^1(\Omega^1_U, \mathcal{O}_U),$$

where

$$U := X \setminus \{p\}, T^1_X := \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$$

and Θ_U *is the tangent sheaf on* U.

Proof. X is embedded into a smooth variety V with the defining ideal sheaf I. The exact sequence

$$0 \to I/I^{[2]} \to \Omega^1_V|_X \to \Omega^1_X \to 0$$

yields a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{Hom}(\Omega^1_V|_X, \mathcal{O}_X) & \longrightarrow & \mathrm{Hom}(I/I^{[2]}, \mathcal{O}_X) & \longrightarrow & \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}(\Omega^1_V|_U, \mathcal{O}_U) & \longrightarrow & \mathrm{Hom}(I/I^{[2]}|_U, \mathcal{O}_U) & \longrightarrow & \mathrm{Ext}^1(\Omega^1_U, \mathcal{O}_U) & \longrightarrow & 0
\end{array} \tag{1}$$

Note here that the second sequence is exact because

$$\mathrm{Ext}^1(\Omega^1_V|_U, \mathcal{O}_U) = H^1(U, \Theta_V|_U) \cong H^2_{\{p\}}(X, \Theta_V|_X) = 0$$

by the depth argument.

The two vertical maps on the left-hand side are isomorphisms because X is a normal variety, hence we have $H^0(X, T^1_X) \cong \mathrm{Ext}^1(\Omega^1_U, \mathcal{O}_U)$ by the diagram.

Next we consider the commutative diagram

$$\begin{array}{ccc} H^0(X, T^1_X) & \longrightarrow & H^0(U, T^1_U) \\ \downarrow & & \downarrow \\ \operatorname{Ext}^1(\Omega^1_U, \mathcal{O}_U) & \longrightarrow & H^0(U, T^1_U) \end{array} \tag{2}$$

The vertical maps in this diagram are both isomorphisms by the arguement above. The kernel of the first horizontal map is isomorphic to $H^0_{\{p\}}(X, T^1_X)$ and the kernel of the second horizontal map is isomorphic to $H^1(U, \Theta_U)$. This implies that $H^0_{\{p\}}(X, T^1_X)$ is isomorphically mapped onto $H^1(U, \Theta_U)$ by the isomorphism $H^0(X, T^1_X) \cong \operatorname{Ext}^1(\Omega^1_U, \mathcal{O}_U)$. □

PROPOSITION 1.2. *Let X be a compact, normal, Gorenstein analytic space of* $\dim \geqslant 3$. *Put $\Sigma = \operatorname{Sing}(X)$ and choose finite number of points $p_1, \dots, p_m \in \Sigma$. Let $U := X \setminus \{p_1, \dots, p_m\}$. Then there is an injection $H^1(U, \Theta_U) \to \operatorname{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$.*

Proof. Let X_i be a Stein open neighborhood of $p_i \in X$ and set $U_i := X_i \setminus \{p_i\}$. Then one has the exact sequence $(*)$

$$0 \to \operatorname{Ext}^1(\Omega^1_X, \mathcal{O}_X) \xrightarrow{\phi} \oplus_{1 \leqslant i \leqslant m} \operatorname{Ext}^1(\Omega^1_{X_i}, \mathcal{O}_{X_i}) \oplus \\ \oplus \operatorname{Ext}^1(\Omega^1_U, \mathcal{O}_U) \to \oplus_{1 \leqslant i \leqslant m} \operatorname{Ext}^1(\Omega^1_{U_i}, \mathcal{O}_{U_i})$$

The map ϕ is injective because the map

$$\oplus_{1 \leqslant i \leqslant m} \operatorname{Hom}(\Omega^1_{X_i}, \mathcal{O}_{X_i}) \oplus \operatorname{Hom}(\Omega^1_U, \mathcal{O}_U) \to \oplus_{1 \leqslant i \leqslant m} \operatorname{Hom}(\Omega^1_{U_i}, \mathcal{O}_{U_i})$$

is surjective.

On the other hand, by Proposition 1.1, $H^1(U_i, \Theta_{U_i}) \cong H^0_{\{p_i\}}(X_i, T^1_{X_i})$. Therefore one has the exact sequence $(**)$:

$$0 \to H^1(U, \Theta_U) \to \oplus_{1 \leqslant i \leqslant m} H^0_{\{p_i\}}(X_i, T^1_{X_i}) \oplus \\ \oplus H^1(U, \Theta_U) \to \oplus_{1 \leqslant i \leqslant m} H^1(U_i, \Theta_{U_i}).$$

There are injections from the third and fourth terms of the sequence $(**)$ to the third and fourth terms of the sequence $(*)$ respectively, and the square made by these injections commutes. Hence there is an injection from the second term of $(**)$ to the second term of $(*)$. □

In the remainder, X will be a Calabi–Yau threefold with rational Gorenstein singularities (equivalently, a projective threefold with rational Gorenstein singularities with $K_X \sim 0$, $H^1(X, \mathcal{O}_X) = 0$). By [Re], the singular locus Σ of X is a one-dimensional locally trivial flat deformation of a rational double point on a surface at a general point except finite number of bad points; one call these bad points *dissident*. Let Σ_0 be the set of dissident points on X and set $U := X \setminus \Sigma_0$.

https://doi.org/10.1023/A:1002614926312 Published online by Cambridge University Press

Let $\pi\colon \tilde{X} \to X$ be a resolution of singularities such that $E := \pi^{-1}(\Sigma)$ is a divisor of $\tilde{X}$ with simple normal crossings. We take the resolution in such a way that $C := \pi^{-1}(\Sigma_0)$ is also a simple normal crossing divisor.

Our basic object is the following commutative diagram with exact rows induced by local cohomology sequences:

$$\begin{array}{ccccc} H^1(\pi^{-1}(U), \Omega^2_{\tilde{X}}(\log E)(-E)) & \longrightarrow & H^2_C(\tilde{X}, \Omega^2_{\tilde{X}}(\log E)(-E)) & \xrightarrow{\gamma} & H^2(\tilde{X}, \Omega^2_{\tilde{X}}(\log E)(-E)) \\ \uparrow & & {\scriptstyle\beta}\uparrow & & \\ H^1(U, \Theta_U) & \xrightarrow{\phi} & H^2_{\Sigma_0}(X, \Theta_X) & & \end{array} \tag{3}$$

Note that $\Theta_U \cong \pi_* \Omega^2_{\tilde{X}}(\log E)(-E))|_U$. In fact, take π_* of the exact sequence

$$0 \to \Omega^2_{\tilde{X}}(\log E)(-E)) \to \Omega^2_{\tilde{X}} \to \hat{\Omega}^2_E \to 0.$$

All singularities on U are locally trivial deformation of rational double points of surfaces, from which we deduce that $\pi_* \hat{\Omega}^2_E|_U = 0$ and that $\pi_* \Omega^2_{\tilde{X}}|_U \cong \Theta_U$.

Let $\Sigma = \{p_1, \ldots, p_m\}$ and take X_i and U_i for each p_i in the same way as the proof of Proposition 1.2. Then

$$H^2_{\Sigma_0}(X, \Theta_X) \cong \oplus_{1 \leqslant i \leqslant m} H^1(U_i, \Theta_{U_i}) \cong \oplus_{1 \leqslant i \leqslant m} H^1(U_i, \pi_* \Omega^2_{\tilde{X}}(\log E)(-E))|_{U_i}).$$

Hence, ϕ is identified with the coboundary map

$$H^1(U, \pi_* \Omega^2_{\tilde{X}}(\log E)(-E)|_U) \to H^2_{\Sigma_0}(X, \pi_* \Omega^2_{\tilde{X}}(\log E)(-E)),$$

and the vertical maps in the diagram are natural ones.

LEMMA 1.3. *The vertical maps in the diagram are both surjective.*

Proof. As we remarked above, the vertical maps fit into the following exact sequences respectively. (The second sequence is exact because $R^2 \pi_* \Omega^2_{\tilde{X}}(\log E)(-E) = 0$ by the vanishing theorem of Guillen, Navarro-Aznar and Puerta.)

$$H^1(U, \Theta_U) \to H^1(\pi^{-1}(U), \Omega^2_{\tilde{X}}(\log E)(-E)) \to H^0(U, R^1 \pi_* \Omega^2_{\tilde{X}}(\log E)(-E)|_U),$$
$$H^2_{\Sigma_0}(X, \Theta_X) \to H^2_C(\tilde{X}, \Omega^2_{\tilde{X}}(\log E)(-E)) \to H^1_{\Sigma_0}(X, R^1 \pi_* \Omega^2_{\tilde{X}}(\log E)(-E)).$$

All singularities on U are locally trivial 1-parameter deformation of rational double points. It can be checked that $R^1 \pi_* \Omega^2_{\tilde{X}}(\log E)(-E)$ vanishes on U by using this fact. Therefore, the third term of each sequence vanishes. □

Let (V, p) be the germ of a rational Gorenstein singularity of dim 3 at a point p. Set $\Sigma = \mathrm{Sing}(V)$. Assume that (V, p) is realized as an open subset of a complex algebraic variety.

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Let $(K^{\cdot}_V, F)$ be the filtered de Rham complex defined by Du Bois [DuB]. (Here we shall use the notation in [St].)

Define $\mu(V, p) := \dim \mathcal{H}^2(\mathrm{Gr}^1_F(K^{\cdot}_V))$.

Note that, when (V, p) is a rational Gorenstein singularity of dim 3, the sheaf $\mathcal{H}^2(\mathrm{Gr}^1_F(K^{\cdot}_V))$ has a support only at p. In fact, according to [St], one can construct the filtered de Rham complex of a pair $(K^{\cdot}_{V,\Sigma})$ in such a way that, for each i,

$$\mathrm{Gr}^i_F K^{\cdot}_{V,\Sigma} \to \mathrm{Gr}^i_F K^{\cdot}_V \to \mathrm{Gr}^i_F K^{\cdot}_\Sigma \to \mathrm{Gr}^i_F K^{\cdot}_{V,\Sigma}[1]$$

is an exact triangle in the derived category $D^+(V, \mathbf{C})$. In particular,

$$\mathcal{H}^2(\mathrm{Gr}^1_F K^{\cdot}_{V,\Sigma}) \to \mathcal{H}^2(\mathrm{Gr}^1_F K^{\cdot}_V) \to \mathcal{H}^2(\mathrm{Gr}^1_F K^{\cdot}_\Sigma)$$

is an exact sequence.

Let $\nu: \tilde{V} \to V$ be a resolution such that $\nu^{-1}(\Sigma) =: F$ is a divisor with normal crossings. By [St, (3.4)] $\mathcal{H}^2(\mathrm{Gr}^1_F K^{\cdot}_{V,\Sigma}) \cong R^1\nu_*\Omega^1_{\tilde{V}}(\log F)(-F)$.

Since Σ is a locally trivial deformation of a rational double point outside p, the sheaf $R^1\nu_*\Omega^1_{\tilde{V}}(\log F)(-F)$ has a support only at p, hence the sheaf $\mathcal{H}^2(\mathrm{Gr}^1_F K^{\cdot}_{V,\Sigma})$ also has a support only at p.

On the other hand, $\mathcal{H}^2(\mathrm{Gr}^1_F K^{\cdot}_\Sigma) = 0$ because Σ is a curve or an isolated point. Thus the sheaf $\mathcal{H}^2(\mathrm{Gr}^1_F(K^{\cdot}_V))$ has a support only at p.

Remark. Let $\alpha: \tilde{\Sigma} \to \Sigma$ be the normalization map. From [DuB], there is a distinguished triangle in the filtererd derived category

$$0 \to K^{\cdot}_V \to K^{\cdot}_\Sigma \oplus \mathbf{R}\nu_* K^{\cdot}_{\tilde{V}} \to \mathbf{R}\nu_* K^{\cdot}_F \to 0.$$

Since $\mathcal{H}^1(\mathrm{Gr}^1_F K^{\cdot}_\Sigma) \cong \alpha_*\Omega^1_{\tilde{\Sigma}}$ and since $\mathcal{H}^2(\mathrm{Gr}^1_F K^{\cdot}_\Sigma) = 0$, one has an exact sequence

$$\alpha_*\Omega^1_{\tilde{\Sigma}} \oplus \nu_*\Omega^1_{\tilde{V}} \to \nu_*\hat{\Omega}^1_F \to \mathcal{H}^2(\mathrm{Gr}^1_F(K^{\cdot}_V)) \to R^1\nu_*\Omega^1_{\tilde{V}} \to R^1\nu_*\hat{\Omega}^1_F$$

by the distinguished triangle.

The map $\alpha_*\Omega^1_{\tilde{\Sigma}} \to \nu_*\hat{\Omega}^1_F$ obtained as a composition of the natural inclusion $\alpha_*\Omega^1_{\tilde{\Sigma}} \to \alpha_*\Omega^1_{\tilde{\Sigma}} \oplus \nu_*\Omega^1_{\tilde{V}}$ and the first map in the sequence coincides with the map induced by the natural map from the normalization $F^{[0]}$ of F to $\tilde{\Sigma}$. We can check that this map is surjective. Hence, we have

$$\mathcal{H}^2(\mathrm{Gr}^1_F(K^{\cdot}_V)) \cong \mathrm{Ker}[R^1\nu_*\Omega^1_{\tilde{V}} \to R^1\nu_*\hat{\Omega}^1_F].$$

On the other hand, one has an exact sequence

$$R^1\nu_*\Omega^1_{\tilde{V}}(\log F)(-F) \to R^1\nu_*\Omega^1_{\tilde{V}} \to R^1\nu_*\hat{\Omega}^1_F.$$

Therefore we have a surjection

$$R^1\nu_*\Omega^1_{\tilde{V}}(\log F)(-F) \to \mathcal{H}^2(\mathrm{Gr}^1_F(K^{\cdot}_V)).$$

If Σ is analytically isomorphic to the coordinate axis of $\mathbf{C}^n$ or Σ is isolated, then

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one can show that this map is an isomorphism, equivalently that $\mu(V,p) = \dim(R^1 \nu_* \Omega^1_{\tilde{V}}(\log F)(-F))_p$. (We omit the proof of this fact.) For example, if $(V,p) \cong (RDP) \times (\mathbf{C}^1, 0)$, then $\mu(V,p) = 0$, where (RDP) means the germ of a surface rational double point.

Denote by Weil(V,p) (resp. Cart(V,p)) the Abelian group of Weil divisors (resp. Cartier divisors) on (V,p). Then define $\sigma(V,p)$ to be the rank of Weil(V,p)/Cart(V,p).

LEMMA 1.4. *Assume that $\mu(X,p) = 0$ for all $p \in \Sigma_0$. Then the map β is an isomorphism.*

Proof. By Lemma 1.3, the map β is surjective. We only have to prove the injectivity. The map β fits into the exact sequence

$$H^0_{\Sigma_0}(X, R^1\pi_*\Omega^2_{\tilde{X}}(\log E)(-E)) \xrightarrow{\alpha} H^2_{\Sigma_0}(X, \Theta_X) \xrightarrow{\beta} H^2_C(\tilde{X}, \Omega^2_{\tilde{X}}(\log E)(-E)).$$

By the same argument as [Na-St, Theorem 1], the map

$$d: R^1\pi_*\Omega^1_{\tilde{X}}(\log E)(-E) \to R^1\pi_*\Omega^2_{\tilde{X}}(\log E)(-E)$$

is a surjection. Since $\mu(X,p) = 0$, we have an exact sequence (cf. Remark)

$$\pi_*\Omega^1_{\tilde{X}} \to \pi_*\hat{\Omega}^1_E \to R^1\pi_*\Omega^1_{\tilde{X}}(\log E)(-E) \to 0.$$

Let us consider the commutative diagram

$$\begin{array}{ccc} \pi_*\hat{\Omega}^1_E & \longrightarrow & R^1\pi_*\Omega^1_{\tilde{X}}(\log E)(-E) \\ \uparrow & & \uparrow \\ \pi_*\mathcal{O}_E & \longrightarrow & R^1\pi_*\mathcal{O}_{\tilde{X}}(-E). \end{array} \tag{4}$$

We shall prove that the vertical map on the left-hand side is surjective. First note that we have taken π in such a way that $C := \pi^{-1}(\Sigma_0)$ is a simple normal crossing divisor. Since X has rational singularities, we see that $H^1(C, \mathcal{O}_C) = 0$. By the mixed Hodge structure on $H^1(C, \mathbf{Z})$, we know $H^0(C, \hat{\Omega}^1_C) = 0$. Therefore $H^1(C, \mathbf{C}) = 0$. This implies that $R^1\pi_*\mathbf{C}_E = 0$. Look at the spectral sequence $E_1^{p,q} := R^q\pi_*\hat{\Omega}^p_E$ which converges to $E^{p+q} := R^{p+q}\pi_*\mathbf{C}_E$. Then $E_1^{1,0} = \mathrm{coker}(\pi_*\mathcal{O}_E \to \pi_*\hat{\Omega}^1_E)$ because $\pi_*\hat{\Omega}^2_E = 0$. Since $E_1^{1,0} = E_\infty^{1,0} = 0$, we conclude that the map $\pi_*\mathcal{O}_E \to \pi_*\hat{\Omega}^1_E$ is surjective.

Now by the commutative diagram, the map

$$d: R^1\pi_*\mathcal{O}_{\tilde{X}}(-E) \to R^1\pi_*\Omega^1_{\tilde{X}}(\log E)(-E)$$

is surjective, hence $R^1\pi_*\Omega^2_{\tilde{X}}(\log E)(-E) = 0$. From this it follows that the first term of the exact sequence is zero. □

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We shall next investigate $\dim H^2_C(\tilde{X}, \Omega^2_{\tilde{X}}(\log E)(-E))$.

PROPOSITION 1.5.

$$\dim H^2_C(\tilde{X}, \Omega^2_{\tilde{X}}(\log E)(-E)) = \Sigma_{p\in\Sigma_0}\{\mu(V,p) + \sigma(V,p)\}$$

Proof. Note that $H^2_C(\tilde{X}, \Omega^2_{\tilde{X}}(\log E)(-E))$ is the dual space to $\oplus_{p\in\Sigma_0} R^1\pi_*\Omega^1_{\tilde{X}}(\log E)_p$. Then we have the result by the following lemma:

LEMMA 1.6. *Let (V,p) be the germ of a rational Gorenstein singularity of* $\dim 3$ *which can be realized as an open subset of a complex algebraic variety. Set* $\Sigma = \mathrm{Sing}(V)$. *Assume that Σ is a one-dimensional locally trivial flat deformation of a surface rational double point outside p. Let $\nu\colon \tilde{V} \to V$ be a resolution such that $\nu^{-1}(\Sigma) = F$ is a divisor with simple normal crossings. Then $R^1\nu_*\Omega^1_{\tilde{V}}(\log F)$ has its support only at p and its dimension is $\mu(V,p) + \sigma(V,p)$.*

Proof. Take $\mathbf{R}\nu_*$ of the exact sequences:

$$0 \to \Omega^1_{\tilde{V}} \to \Omega^1_{\tilde{V}}(\log F) \to \mathcal{O}_{F^{[0]}} \to 0,$$
$$0 \to \Omega^1_{\tilde{V}}(\log F)(-F) \to \Omega^1_{\tilde{V}} \to \hat{\Omega}^1_F \to 0,$$

where $F^{[0]}$ is the normalization of F. Note that $R^1\nu_*\mathcal{O}_{F^{[0]}} \to R^2\nu_*\Omega^1_{\tilde{V}}$ is injective. This is proved in the following manner. It is enough to prove that the composition $\varphi\colon R^1\nu_*\mathcal{O}_{F^{[0]}} \to R^2\nu_*\Omega^1_{\tilde{V}} \to R^2\nu_*\Omega^1_{F^{[0]}}$ is injective. Let F_i $(1 \leqslant i \leqslant k)$ be irreducible components of $F^{[0]}$ which are contracted to p by ν. Now define $\varphi_i := \wedge c_1(N_{F_i/\tilde{V}}) : H^1(F_i, \mathcal{O}_{F_i}) \to H^2(F_i, \Omega^1_{F_i})$. The map φ is then identified with $\oplus_i\varphi_i$. It is enough to prove that φ_i are all injective. We can choose an effective divisor $B \subset \tilde{V}$ so that $-B$ is ν-ample. Take an F_i. We may assume that $H^1(F_i, \mathcal{O}_{F_i}) \neq 0$ because, if $H^1(F_i, \mathcal{O}_{F_i}) = 0$, then φ_i is obviously injective. The F_i then has a fibration over an irregular curve C_i whose general fiber l_i is isomorphic to $\mathbf{P}^1$. Since $(B.l_i) < 0$, B contains F_i as an irreducible component and $(F_i.l_i) < 0$. This implies that φ_i is an injection.

By an argument used in the proof [Na-St, Theorem (1.1)], one has $R^2\nu_*\Omega^1_{\tilde{V}}(\log F)(-F) = 0$. Since

$$\mathcal{H}^2(\mathrm{Gr}^1_F(K^{\cdot}_V)) \cong \mathrm{Ker}[R^1\pi_*\Omega^1_{\tilde{V}} \to R^1\pi_*\hat{\Omega}^1_F]$$

(cf. Remark), the second exact sequence yields the exact sequence

$$0 \to \mathcal{H}^2(\mathrm{Gr}^1_F(K^{\cdot}_V)) \to R^1\nu_*\Omega^1_{\tilde{V}} \to R^1\nu_*\hat{\Omega}^1_F \to 0.$$

On the other hand, since $R^1\nu_*\mathcal{O}_{F^{[0]}} \to R^2\nu_*\Omega^1_{\tilde{V}}$ is an injection, the first exact sequence yields the excat sequence

$$\nu_*\mathcal{O}_{F^{[0]}} \to R^1\nu_*\Omega^1_{\tilde{V}} \to R^1\nu_*\Omega^1_{\tilde{V}}(\log F) \to 0.$$

Let us consider the composite $\nu_*\mathcal{O}_{F^{[0]}} \to R^1\nu_*\Omega^1_{\tilde{V}} \to R^1\nu_*\hat{\Omega}^1_F$. The cokernel of the composite has support at p and its dimension as a $\mathbf{C}$ vector space is $\sigma(V, p)$. Then the result follows from the two exact sequences just above. □

The following lemma will be useful in the next proposition.

LEMMA 1.7. *Let* $\tilde{X}$ *be the same as above. Then the natural map* $H^2(\tilde{X}, \Omega^2_{\tilde{X}}(\log E)(-E)) \to H^2(\tilde{X}, \Omega^2_{\tilde{X}})$ *is an injection.*

Proof. Taking the dual, we need to show the map

$$H^1(\tilde{X}, \Omega^1_{\tilde{X}}) \to H^1(\tilde{X}, \Omega^1_{\tilde{X}}(\log E))$$

is surjective. By the theory of mixed Hodge structure, it suffices to show that the map $H^2(\tilde{X}, \mathbf{C}) \to H^2(\tilde{X} \setminus E, \mathbf{C})$ is surjective. By the exact sequence

$$H^2(\tilde{X}, \mathbf{C}) \to H^2(\tilde{X} \setminus E, \mathbf{C}) \to H^3_E(\tilde{X}, \mathbf{C}) \xrightarrow{\iota} H^3(\tilde{X}, \mathbf{C}),$$

we only have to prove that ι is injective. Let $\iota^*: H^3(\tilde{X}, \mathbf{C}) \to H^3(E, \mathbf{C})$ be the dual map of ι. Note that ι^* is the map of mixed Hodge structure. Let F be the Hodge filtration of the mixed Hodge structure. Since $\mathrm{Gr}^0_F H^3(E, \mathbf{C}) = \mathrm{Gr}^3_F H^3(E, \mathbf{C}) = 0$, we have to show that the maps $\iota^*_i: \mathrm{Gr}^i_F H^3(\tilde{X}, \mathbf{C}) \to \mathrm{Gr}^i_F H^3(E, \mathbf{C})$ are surjective for $i = 1, 2$. In the exact sequence

$$H^2(\tilde{X}, \Omega^1_{\tilde{X}}) \xrightarrow{\iota^*_1} H^2(E, \hat{\Omega}^1_E) \to H^3(\tilde{X}, \Omega^1_{\tilde{X}}(\log E)(-E)),$$

the last term vanishes because it is dual to

$$H^0(\tilde{X}, \Omega^2_{\tilde{X}}(\log E)) \cong H^0(X, \Theta_X) = 0.$$

Hence ι^*_1 is surjective.

Let $E^{[0]}$ (resp. $E^{[1]}$) be the normalization of E (resp. the double locus of E). By taking H^2 of the exact sequence

$$0 \to \hat{\Omega}^1_E \to \Omega^1_{E^{[0]}} \to \Omega^1_{E^{[1]}} \to 0$$

we see that the map $H^2(E, \hat{\Omega}^1_E) \to H^2(E^{[0]}, \Omega^1_{E^{[0]}})$ is surjective. Compose this with ι^*_1 and take the conjugation. Then we see that $H^1(\tilde{X}, \Omega^2_{\tilde{X}}) \to H^1(E^{[0]}, \Omega^2_{E^{[0]}})$ is surjective. Since

$$\mathrm{Gr}^2_F H^3(E, \mathbf{C}) = H^1(E, \hat{\Omega}^2_E) \cong H^1(E^{[0]}, \Omega^2_{E^{[0]}}),$$

the ι^*_2 is surjective. □

PROPOSITION 1.8. *Let* X *and* $\tilde{X}$ *be the same as above and let* $\sigma(X) := \mathrm{rank}_{\mathbf{Z}} \mathrm{Weil}(X)/\mathrm{Cart}(X)$. *In the diagram (3),* $\dim \mathrm{im}(\gamma \circ \beta) = \sigma(X)$.

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Proof. By Lemma 1.7, we calculate the dimension of the image of the composed map

$$H^2_{\Sigma_0}(X, \Theta_X) \to H^2(\tilde{X}, \Omega^2_{\tilde{X}}(\log E)(-E)) \to H^2(\tilde{X}, \Omega^2_{\tilde{X}}).$$

This map is factorized as

$$H^2_{\Sigma_0}(X, \Theta_X) \to H^2_C(\tilde{X}, \Omega^2_{\tilde{X}}(\log E)(-E)) \to H^2_C(\tilde{X}, \Omega^2_{\tilde{X}}) \to H^2(\tilde{X}, \Omega^2_{\tilde{X}}).$$

Note here that the first map is surjective by Lemma (1.3). Taking the dual of these maps we have the sequence of maps

$$\begin{aligned} H^1(\tilde{X}, \Omega^1_{\tilde{X}}) &\to \oplus_{p\in\Sigma_0}(R^1\pi_*\Omega^1_{\tilde{X}})_p \otimes \hat{\mathcal{O}}_{X,p} \to \\ &\to \oplus_{p\in\Sigma_0}(R^1\pi_*\Omega^1_{\tilde{X}}(\log E)_p \otimes \hat{\mathcal{O}}_{X,p} \to H^2_{\Sigma_0}(X, \Theta_X)^* \end{aligned}$$

The last map here is an injection. By taking $\mathbf{R}\pi_*$ of the exact sequence

$$0 \to \Omega^1_{\tilde{X}} \to \Omega^1_{\tilde{X}}(\log E) \to \mathcal{O}_{E^{[0]}} \to 0,$$

we have an injection $R^1\pi_*\Omega^1_{\tilde{X}}/\pi_*\mathcal{O}_{E^{[0]}} \to R^1\pi_*\Omega^1_{\tilde{X}}(\log E)$. Therefore, we need to calculate the dimension of

$$\mathrm{im}[H^1(\tilde{X}, \Omega^1_{\tilde{X}}) \to \oplus_{p\in\Sigma_0}(R^1\pi_*\Omega^1_{\tilde{X}})_p/(\pi_*\mathcal{O}_{E^{[0]}})_p],$$

which is nothing but $\sigma(X)$. □

We are now in a position to state the Main Theorem:

THEOREM 1.9. *Let X be a Calabi–Yau threefold with rational Gorenstein singularities. Let Σ_0 be the set of dissident singular points of X and put $U := X \setminus \Sigma_0$. Then the coboundary map $\phi : H^1(U, \Theta_U) \to H^2_{\Sigma_0}(X, \Theta_X) \cong H^0_{\Sigma_0}(X, T^1_X)$ has the following properties:*

(1) *The map ϕ coincides with the natural map $\varphi\colon \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to H^0(X, T^1_X)$ if one identifies $H^1(U, \Theta_U)$ (resp. $H^0_{\Sigma_0}(X, T^1_X)$) with a subspace of $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ (resp. $H^0(X, T^1_X)$) by the natural inclusion defined in Proposition 1.2. Moreover, $\varphi^{-1}(H^0_{\Sigma_0}(X, T^1_X) = H^1(U, \Theta_U)$.*
(2) *$\dim \mathrm{im}(\phi) \geqslant \Sigma_{p\in\Sigma_0}\{\mu(X, p) + \sigma(X, p)\} - \sigma(X)$.*
(3) *If $\mu(X, p) = 0$ for all $p \in \Sigma_0$, then $\dim \mathrm{im}(\phi) = \Sigma_{p\in\Sigma_0}\sigma(X, p) - \sigma(X)$.*

Proof. (1) restates Propositions 1.1 and 1.2 except the final part. If $\varphi(\zeta) \in H^0_{\Sigma_0}(X, T^1_X)$ for $\zeta \in \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$, then ζ is sent to zero by the composed map $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to \mathrm{Ext}^1(\Omega^1_U, \mathcal{O}_U) \to H^0(U, T^1_U)$, hence ζ is contained in $H^1(U, \Theta_U)$.

(2), (3): Consider the diagram

$$\begin{array}{ccccc} H^1(\pi^{-1}(U), \Omega^2_{\tilde{X}}(\log E)(-E)) & \longrightarrow & H^2_C(\tilde{X}, \Omega^2_{\tilde{X}}(\log E)(-E)) & \stackrel{\gamma}{\longrightarrow} & H^2(X, \Omega^2_{\tilde{X}}(\log E)(-E)) \\ \uparrow & & \beta\uparrow & & \\ H^2(U, \Theta_U) & \stackrel{\phi}{\longrightarrow} & H^2_{\Sigma_0}(X, \Theta_X) & & \end{array} \qquad (5)$$

By Lemma 1.3 and Propositions 1.5 and 1.8,

$$\dim \mathrm{Ker}(\gamma) = \Sigma_{p \in \Sigma_0}\{\mu(X, p) + \sigma(X, p)\} - \sigma(X).$$

Then (2) follows from the diagram above. Note that if the assumption of (3) holds, then the map β is an isomorphism by Lemma 1.4. □

2. Applications

Let $f: \tilde{X} \to X$ be a birational morphism from a smooth Calabi–Yau threefold $\tilde{X}$ to a singular Calabi–Yau threefold X. Assume that $\mathrm{Exc}(f) =: D$ is a smooth surface having a conic bundle structure over $f(D) \cong \mathbf{P}^1$, and that $f|_D$ has no multiple fiber and has exactly n reducible fibers. Let $k = \rho(\tilde{X}) - \rho(X)$.

In this section, we shall give a mathematical proof of the following result due to Morrison–Seiberg.

THEOREM 2.1. *If $n \geqslant 3$, then the Kuranishi space* $\mathrm{Def}(X)$ *of X is smooth, and* $\dim \mathrm{Def}(X) = \dim \mathrm{Def}(\tilde{X}) + 2n - 2 - k$.

Proof. The smoothness of $\mathrm{Def}(X)$ is already proved in [Gr 2].

Note that X has dissident points on $f(D)$ corresponding to reducible fibers of $f|_D$. Let $p \in \Sigma_0$ be one of these points. Since the singular locus $\Sigma \subset X$ is smooth at p, $\mu(X, p) = \dim(R^1 f_* \Omega^1_{\tilde{X}}(\log D)(-D))_p$. In our case we have an exact sequence

$$0 \to R^1 f_* \Omega^1_{\tilde{X}}(\log D)(-D) \to R^1 f_* \Omega^1_{\tilde{X}} \to R^1 f_* \Omega^1_D \to 0.$$

By the formal function theorem, we can check directly that $R^1 f_* \Omega^1_{\tilde{X}} \cong R^1 f_* \Omega^1_D$, which implies that $\mu(X, p) = 0$.

Since $f^{-1}(p)$ is a line pair in a conic bundle $D \to \Sigma$, $\mathrm{Pic}(\tilde{X}, f^{-1}(p)) \cong \mathbf{Z}^{\oplus 2}$. On the other hand, D is irreducible around $f^{-1}(p)$. Therefore $\sigma(X, p) = 1$.

By [Gr 2], there is an exact sequence

$$0 \to \mathcal{F} \to T^1_X \to \mathcal{O}_{\mathbf{P}^1}(4 - D^3) \to 0,$$

where $\mathcal{F}$ has the support on Σ_0; $H^0_{\Sigma_0}(X, T^1_X) = H^0(X, \mathcal{F})$. By Theorem 1.9(1), (3), we see that $\dim(\mathrm{im}(\varphi) \cap H^0(X, \mathcal{F})) = n - \sigma(X)$. Note that $\sigma(X) = k - 1$.

On the other hand, we have the following result:

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LEMMA 2.2. *The composed map* $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(4 - D^3))$ *of* φ *and* $H^0(X, T^1_X) \to H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(4 - D^3))$ *is surjective.*

Proof. According to [Gr 2], by the exact sequence

$$f^*\Omega^1_X \longrightarrow \Omega^1_{\tilde{X}} \xrightarrow{\psi} \Omega^1_{\tilde{X}/X} \to 0,$$

we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} & & H^0(\mathcal{E}xt^1(\mathrm{Ker}(\psi), \mathcal{O}_{\tilde{X}}) & \to & H^2((\mathrm{Ker}(\psi)^*) & \xrightarrow{\delta'} & \mathrm{Ext}^2(\mathrm{Ker}(\psi), \mathcal{O}_{\tilde{X}}) \\ & & \lambda\downarrow & & \cong\downarrow & & \tau\downarrow \\ \mathbf{T}^1_X & \to & H^0(\mathcal{E}xt^1(f^*\Omega^1_X, \mathcal{O}_X) & \to & H^2(\mathcal{H}om(f^*\Omega^1_X, \mathcal{O}_{\tilde{X}})) & \xrightarrow{\delta} & \mathrm{Ext}^2(f^*\Omega^1_X, \mathcal{O}_{\tilde{X}}) \end{array} \qquad (6)$$

Note here that $\mathrm{Ext}^1(f^*\Omega^1_X, \mathcal{O}_{\tilde{X}}) \cong \mathbf{T}^1_X$ and that the map λ is injective.

By the argument in [Gr2, Theorem(1.9)], τ is injective, hence $\mathrm{Ker}(\delta) = \mathrm{Ker}(\delta')$. By the diagram above, $H^0(\mathcal{E}xt^1(f^*\Omega^1_X, \mathcal{O}_{\mathcal{X}}))$ is generated by $H^0(\mathcal{E}xt^1(\mathrm{Ker}(\psi), \mathcal{O}_{\tilde{X}})$ and the image of the map $\mathbf{T}^1_X \to H^0(\mathcal{E}xt^1(f^*\Omega^1_X, \mathcal{O}_{\mathcal{X}}))$.

On the other hand, the support of the sheaf $f_*\mathcal{E}xt^1(\mathrm{Ker}(\psi), \mathcal{O}_{\tilde{X}})$ is contained in Σ_0 and there is an exact sequence (cf. [Gr 2, Theorem (1.4)])

$$0 \to f_*\mathcal{E}xt^1(\mathrm{Ker}(\psi), \mathcal{O}_{\tilde{X}}) \to f_*\mathcal{E}xt^1(f^*\Omega^1_X, \mathcal{O}_{\tilde{X}}) \to \mathcal{O}_{\mathbf{P}^1}(4 - D^3) \to 0.$$

Hence $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(4 - D^3)) = \mathrm{Coker}(\lambda)$, which implies our claim. □

Proof Continued. Since $D^3 = 8 - n$, by Lemma 2.2, we have

$$\dim \mathrm{im}\,[\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to H^0(X, T^1_X)] = 2n - 2 - k.$$

One may assume that the map $\mathrm{Def}(\tilde{X}) \to \mathrm{Def}(X)$ is unramified by deforming $\tilde{X}$ slightly if necessary. In particular, one may assume that the map $H^1(\tilde{X}, \Theta_{\tilde{X}}) \to \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ is injective. Since $f_*\Theta_{\tilde{X}} \cong \Theta_X$, and since any deformation of $\tilde{X}$ induces a locally trivial deformation of the small neighborhood of $D \subset \tilde{X}$, we have $H^1(X, \Theta_X) \cong H^1(\tilde{X}, \Theta_{\tilde{X}})$. Now the theorem follows from the exact sequence

$$0 \to H^1(X, \Theta_X) \to \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to H^0(X, T^1_X). \qquad \square$$

THEOREM 2.3. *Let* $\tilde{X}$ *and* X *be as above. Assume* $n \geqslant 4$. *Then* X *has a flat deformation to a smooth Calabi–Yau threefold.*

Proof. The condition $n \geqslant 4$ implies that $\dim \mathrm{Def}(X) > \dim \mathrm{Def}(\tilde{X})$ by Lemma 2.2. By [Gr2, Lemma 1.6], if we take a general 1-parameter deformation $\mathcal{X} \to \Delta^1$ of X, then, after possibly making a finite base change over Δ^1, there is a small partial resolution $\mathcal{X}' \to \mathcal{X}$ with $\mathcal{X}' \to \Delta^1$ flat such that $\mathcal{X}'_t$ $(t \neq 0)$ has **Q**-factorial terminal singularities. By [Na-St, Theorem (1.3)], $\mathcal{X}'_t$ has a flat deformation to a smooth Calabi–Yau threefold. Since X has finitely many crepant partial resolutions, and

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since Def(X) is smooth, there is a crepant partial resolution X' such that Def(X') $\to$ Def(X) is surjective. This X' should satisfy

(a) $\dim \mathrm{Def}(X') \geqslant \dim \mathrm{Def}(X)$, and
(b) X' is smoothable by a flat deformation.

If $X' = X$, nothing to prove. we shall derive a contradiction when X' is not X. Assume that X' has nonisolated singularities. Then X' is obtained from $\tilde{X}$ by flopping some $(-1,-1)$-curves $C_1, \dots, C_l$ in reducible fibers of $f|_D$, and by contracting the proper transform D' of D to a curve along rulings. Let $\tilde{X}'$ be the smooth threefold obtained from $\tilde{X}$ by the flops, and let $f': \tilde{X}' \to X'$ be the birational contraction of D' to a curve, for which we will define the number n' and k' in the same way as above. By the construction, we have $n' = n - l$, $k' \geqslant k - l$. By Theorems 1.9 and Theorem 2.1, we calculate $\dim T^1_{\tilde{X}'}$, in terms of $\dim T^1_{\tilde{X}'} = \dim T^1_{\tilde{X}}$, n' and k' :

(1) If $n' \leqslant 2$, then $\dim \mathbf{T}^1_{X'} = \dim \mathbf{T}^1_{\tilde{X}'} + n' - k' + 1$.
(2) If $n' \geqslant 3$, then $\dim \mathbf{T}^1_{X'} = \dim \mathbf{T}^1_{\tilde{X}'} + 2n' - 2 - k'$.

Then we conclude that $\dim T^1_X > \dim T^1_{X'}$ when $n \geqslant 4$, a contradiction.

Assume that X' has only isolated singularities. Then X' is obtained from $\tilde{X}$ by flopping some $(-1,-1)$-curves $C_1, \dots, C_l$ in reducible fibers of $f|_D$ and by contracting $(-1,-1)$-curves $C_{l+1}, \dots, C_s$ in reducible fibers to points. Note that X' has only ordinary double points. In the calculation of $\dim \mathbf{T}^1_{X'}$, the flopping curves have no effects, hence we may assume that $l = 0$ (equivalently that $\tilde{X} = \tilde{X}'$). Let F_j $(1 \leqslant j \leqslant \rho)$ be a generator of $\mathrm{Pic}(\tilde{X}) \otimes \mathbf{Q}$. Let r be the rank of the matrix $(C_i, F_j)_{\{1 \leqslant i \leqslant s, 1 \leqslant j \leqslant \rho\}}$. Then, by the definition of k, we should have $s - r \leqslant n - k + 1$. By [Fr], $\dim \mathbf{T}^1_{X'} = \dim \mathbf{T}^1_{\tilde{X}} + s - r$. Thus we have $\dim \mathbf{T}^1_{X'} \leqslant \dim \mathbf{T}^1_{\tilde{X}} + n - k + 1$. On the other hand, by Theorem 1.9 and Lemma 2.2, $\dim \mathbf{T}^1_X = \dim \mathbf{T}^1_{\tilde{X}} + 2n - k - 2$. Since $\dim \mathbf{T}^1_X \leqslant \dim \mathbf{T}^1_{X'}$, we obtain $n \leqslant 3$, a contradiction.

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