

ON THE DIMENSION OF MODULES AND ALGEBRAS, IV

DIMENSION OF RESIDUE RINGS OF HEREDITARY RINGS

SAMUEL EILENBERG, HIROSI NAGAO and TADASI NAKAYAMA

A ring (with unit element) A is called semi-primary¹⁾ if it contains a nilpotent two-sided ideal N such that the residue ring $\Gamma = A/N$ is semi-simple (i.e. $\text{l. gl. dim } \Gamma = \text{r. gl. dim } \Gamma = 0$). N is then the (Jacobson) radical of A . Auslander [1] has shown that if A is semi-primary then

$$\text{l. gl. dim } A = \text{r. gl. dim } A = \text{l. dim}_A \Gamma = 1 + \text{l. dim}_A N.$$

The common value is denoted by $\text{gl. dim } A$. On the other hand, for any ring A the following conditions are equivalent: (a) $\text{l. gl. dim } A \leq 1$, (b) each left ideal in A is projective, (c) every submodule of a projective left A -module is projective. Rings satisfying conditions (a)-(c) are called *hereditary*. For integral domains the notions of "hereditary ring" and "Dedekind ring" coincide.

If α is a two-sided ideal in A , there is in general very little relation between $\text{l. gl. dim } A$ and $\text{l. gl. dim } (A/\alpha)$; it was however proved, substantially, in Eilenberg-Ikeda-Nakayama [4] that if A is semiprimary and α is contained in the radical N then

$$\text{gl. dim } A \leq \text{gl. dim } (A/\alpha) + \text{l. dim}_A (A/\alpha).$$

Now, we show in §1, of the present note, that if A is hereditary and the sequence α^i ($i = 1, 2, \dots$) becomes constant then $\text{gl. dim } (A/\alpha) < \infty$. Thus if A is hereditary and semi-primary then $\text{gl. dim } (A/\alpha) < \infty$, and we are able to give, in §2, rather precise estimates for $\text{gl. dim } (A/\alpha)$.

In §3 we show by examples that the above results are the best possible; thus for each pair (m, n) with $0 < m \leq \infty$, $0 < n \leq \infty$, $(m, n) \neq (1, \infty)$ we construct a semi-primary ring A and a two-sided ideal α of A such that $\text{gl. dim } A = m$, $\text{gl. dim } (A/\alpha) = n$.

Received October 15, 1955.

¹⁾ Our notion of "semi-primary" does not coincide with "halbprimär" of Deuring, *Algebren, Ergebn. Math.*

In § 4 we consider an algebra A over a field K with $(A : K) < \infty$. Here we consider the number $\dim A$ which is the highest integer n such that the cohomology group $H^n(A, A)$ is not zero for some two-sided A -module A . If no such integer exists then $\dim A = \infty$. We show that results analogous to those of §§ 2-3 hold.

§ 1. Residue rings of hereditary rings

PROPOSITION 1. *If A is a ring (with unit element), α a two-sided ideal in A and B a projective left A -module, then the left (A/α) -module $B/\alpha B$ is projective.*

Proof. The conclusion is trivial if $B = A$. Thus by a direct sum argument it is also valid if B is A -free. Again by a direct sum argument, the conclusion follows for B projective.

PROPOSITION 2. *Let A be a ring, α a nilpotent left ideal in A , A a left A -module and B a submodule of A . If $A = B + \alpha A$ then $A = B$.*

Proof. Assume $C = A/B \neq 0$. Then since α is nilpotent we have $\alpha C \neq C$. But $\alpha C = (B + \alpha A)/B$. Thus $B + \alpha A \neq A$, a contradiction.

PROPOSITION 3. *Let A be a hereditary ring, let α be a two-sided ideal in A and \mathfrak{b} a left ideal containing α . The following conditions are equivalent (for $i \geq 0$);*

- (i) $\text{l. dim}_{A/\alpha} (\mathfrak{b}/\alpha) \leq 2i$,
- (ii) $\alpha^i \mathfrak{b}/\alpha^{i+1}$ is (A/α) -projective,
- (iii) *there exists a left ideal \mathfrak{c} in A such that*

$$\mathfrak{c} + \alpha^{i+1} = \alpha^i \mathfrak{b}, \quad \mathfrak{c} \cap \alpha^{i+1} = \alpha^{i+1} \mathfrak{b}.$$

Similarly, the following conditions are equivalent (for $i > 0$):

- (i') $\text{l. dim}_{A/\alpha} (\mathfrak{b}/\alpha) \leq 2i - 1$,
- (ii') $\alpha^i/\alpha^i \mathfrak{b}$ is (A/α) -projective,
- (iii') *there exists a left ideal \mathfrak{c} in A such that*

$$\mathfrak{c} + \alpha^i \mathfrak{b} = \alpha^i, \quad \mathfrak{c} \cap \alpha^i \mathfrak{b} = \alpha^{i+1}.$$

Proof. Since in the ring A every left ideal \mathfrak{c} is A -projective, it follows from Prop. 1 that $\mathfrak{c}/\alpha \mathfrak{c}$ is (A/α) -projective. Consider the sequence of ideals

$$\dots \subset \alpha^{i+1} \mathfrak{b} \subset \alpha^{i+1} \subset \alpha^i \mathfrak{b} \subset \alpha^i \subset \dots \subset \alpha^2 \subset \alpha \mathfrak{b} \subset \alpha \subset \mathfrak{b}.$$

The inclusion maps induce an exact sequence

$$(1) \quad \dots \rightarrow a^i \mathfrak{b}/a^{i+1} \mathfrak{b} \rightarrow a^i/a^{i+1} \rightarrow a^{i-1} \mathfrak{b}/a^i \mathfrak{b} \rightarrow \dots \rightarrow a/a^2 \rightarrow \mathfrak{b}/a\mathfrak{b} \rightarrow \mathfrak{b}/a \rightarrow 0$$

which is a projective resolution of \mathfrak{b}/a as a (A/a) -module. Now replace the term $a^i \mathfrak{b}/a^{i+1} \mathfrak{b}$ (occurring in degree $2i$) by the image of $a^i \mathfrak{b}/a^{i+1} \mathfrak{b} \rightarrow a^i/a^{i+1}$ i.e. by $a^i \mathfrak{b}/a^{i+1}$, and replace the higher terms by zero. There results an exact sequence which implies the equivalence of (i) and (ii). Condition (iii) is simply an expression of the fact (equivalent with (ii)) that the exact sequence of (A/a) -modules

$$0 \rightarrow a^{i+1}/a^{i+1} \mathfrak{b} \rightarrow a^i \mathfrak{b}/a^{i+1} \mathfrak{b} \rightarrow a^i \mathfrak{b}/a^{i+1} \rightarrow 0$$

splits.

The second part is proved similarly by replacing the term $a^i \mathfrak{b}/a^{i+1}$ by $a^i/a^{i+1} \mathfrak{b}$.

PROPOSITION 4. *Let A be a hereditary ring and \mathfrak{b} a nilpotent two-sided ideal in A . Then for $k > 1$ we have:*

$$\begin{aligned} 1. \dim_{A/\mathfrak{b}^k} (\mathfrak{b}/\mathfrak{b}^k) &\leq 2i \iff \mathfrak{b}^{k(i+1)} = 0, && \text{for } i \geq 0; \\ 1. \dim_{A/\mathfrak{b}^k} (\mathfrak{b}/\mathfrak{b}^k) &\leq 2i - 1 \iff \mathfrak{b}^{ki+1} = 0, && \text{for } i > 0. \end{aligned}$$

Proof. We apply Prop. 3 with $a = \mathfrak{b}^k$. Condition (iii) then reads $c + \mathfrak{b}^{k(i+1)} = \mathfrak{b}^{ki+1}$, $c \cap \mathfrak{b}^{k(i+1)} = \mathfrak{b}^{k(i+1)+1}$. Since $k > 1$ we have $\mathfrak{b}^{k(i+1)} \subset \mathfrak{b}^{ki+1}$. Thus by Prop. 2 the first condition becomes $c = \mathfrak{b}^{ki+1}$. The second condition then becomes $\mathfrak{b}^{k(i+1)} = \mathfrak{b}^{k(i+1)+1}$. Since \mathfrak{b} is nilpotent, this is equivalent with $\mathfrak{b}^{k(i+1)} = 0$.

Condition (iii') reads: $c + \mathfrak{b}^{ki+1} = \mathfrak{b}^{ki}$, $c \cap \mathfrak{b}^{ki+1} = \mathfrak{b}^{k(i+1)}$. By Prop. 2, the first condition becomes $c = \mathfrak{b}^{ki}$. Thus the second condition becomes $\mathfrak{b}^{ki+1} = \mathfrak{b}^{k(i+1)}$. Since $k > 1$, this is equivalent with $\mathfrak{b}^{ki+1} = 0$.

THEOREM 5. *Let A be a hereditary ring and let a be a two-sided ideal in A such that $a^n = a^{n+1}$ for some integer $n > 0$. Then*

$$1. \text{gl. dim } (A/a) \leq 2n - 1.$$

Proof. Let \mathfrak{b} be any left ideal in A containing a . Since in the projective resolution (1) of \mathfrak{b}/a the term a^n/a^{n+1} occurring in degree $2n - 1$ is zero, it follows that

$$1. \dim_{A/a} (\mathfrak{b}/a) \leq 2n - 2.$$

Since \mathfrak{b}/a is the most general left ideal in A/a it follows from a theorem of Auslander [1] that $1. \text{gl. dim } (A/a) \leq 2n - 1$.

COROLLARY 6. *Let A be a hereditary ring satisfying minimum condition for two-sided ideals. Then for any two-side ideal a in A*

$$1. \text{gl. dim } (A/a) < \infty.$$

§ 2. Residue rings of hereditary semi-primary rings

PROPOSITION 7. *Let A be a semi-primary ring and a a left ideal in A . For n sufficiently large we have $a^n = a^{n+1}$. Denoting $\bar{a} = a^n$ for n sufficiently large we have $(\bar{a})^2 = \bar{a}$ and*

$$a^k = \bar{a} + (a \cap N)^k, \quad k = 1, 2, \dots$$

Proof. Since A is semi-primary there exists an idempotent e such that

$$\bar{a} = Ae + \mathfrak{b}, \quad \mathfrak{b} = a \cap N.$$

We assert that

$$(2) \quad a^k = Ae a + \mathfrak{b}^k, \quad k = 1, 2, \dots$$

For $k = 1$ we have

$$Ae a + \mathfrak{b} = Ae Ae + Ae \mathfrak{b} + \mathfrak{b} = Ae + \mathfrak{b} = a.$$

Arguing by induction we find

$$a^{k+1} = a a^k = Ae Ae a + Ae \mathfrak{b}^k + \mathfrak{b} Ae a + \mathfrak{b}^{k+1} = Ae a + \mathfrak{b}^{k+1}.$$

Relation (2) implies $a^n = a^{n+1}$ for n large, since \mathfrak{b} is nilpotent. It follows that $\bar{a} = Ae a$ and the proof is complete.

THEOREM 8. *Let A be a hereditary semi-primary ring. Then for any two-sided ideal a in A we have*

$$\text{gl. dim } (A/a) < \infty.$$

This is an immediate consequence of Theorem 5 and Prop. 7. More precise results concerning $\text{gl. dim } (A/a)$ will be given in this section.

Consider the representation $a = \bar{a} + (a \cap N)$, $\bar{a}^2 = \bar{a}$ given in Prop. 7. It follows from Theorem 5 that the ring A/\bar{a} is still hereditary. Thus if we replace A by A/\bar{a} and a by a/\bar{a} , the problem is reduced to the case $a \subset N$ where N is the readical of A . This reduction however does not seem to simplify the discussion materially.

PROPOSITION 9. *Let A be a hereditary semi-primary ring and let α be a two-sided ideal in A . The following conditions are equivalent (for $i \geq 0$):*

- (i) $\text{gl. dim } (A/\alpha) \leq 2i + 1,$
- (ii) $\alpha^i(\alpha + N)/\alpha^{i+1}$ is (A/α) -projective,
- (iii) *there exists a left ideal c in A such that*

$$c + \alpha^{i+1} = \alpha^i(\alpha + N), \quad c \cap \alpha^{i+1} = \alpha^{i+1}(\alpha + N).$$

Similarly the following conditions are equivalent (for $i \geq 0$):

- (i') $\text{gl. dim } (A/\alpha) \leq 2i,$
- (ii') $\alpha^i/\alpha^i(\alpha + N)$ is (A/α) -projective,
- (iii') *there exists a left ideal c in A such that*

$$c + \alpha^i(\alpha + N) = \alpha^i, \quad c \cap \alpha^i(\alpha + N) = \alpha^{i+1}.$$

Proof. Let $A' = A/\alpha$. The radical of A' is then $N' = (\alpha + N)/\alpha$. Thus

$$\text{gl. dim } A' = 1 + \text{l. dim}_{N'} N' = 1 + \text{l. dim}_{N/\alpha} (\alpha + N)/\alpha.$$

The result now follows from Prop. 3 applied to the ideal $b = \alpha + N$.

Remark. The last condition in (iii') implies $\alpha^{i+1} \subset c$. Thus the conditions in (iii') may be rewritten as follows

$$(3) \quad c + \alpha^i N = \alpha^i, \quad c \cap \alpha^i N \subset \alpha^{i+1}.$$

Iterating the first condition of (3) we find $c + cN + \alpha^i N^2 = \alpha^i$. Further iteration yields $c + cN + \alpha^i N^k = \alpha^i$ and thus $c + cN = \alpha^i$. In particular, $\alpha^i N = cN$ and (3) may be rewritten as

$$(4) \quad c + cN = \alpha^i, \quad c \cap cN \subset \alpha^{i+1}.$$

If further $\alpha N \subset N\alpha$ then $\alpha^i N \subset N\alpha^i$, and, by Prop. 2, the first condition of (3) becomes $c = \alpha^i$. The second condition becomes $\alpha^i N \subset \alpha^{i+1}$. Thus in the case $\alpha N \subset N\alpha$ we find that for $i \geq 0$ $\text{gl. dim } (A/\alpha) \leq 2i$ is equivalent with

$$(iii'') \quad \alpha^i N \subset \alpha^{i+1}.$$

PROPOSITION 10. *Let A be a hereditary semi-primary ring and let α be a two-sided ideal in A such that $\alpha \subset N^2$, $\alpha N \subset N\alpha$. Then for $i \geq 0$ we have*

$$\begin{aligned} \text{gl. dim } (A/\alpha) \leq 2i + 1 &\iff \alpha^{i+1} = 0, \\ \text{gl. dim } (A/\alpha) \leq 2i &\iff \alpha^i N = 0. \end{aligned}$$

Proof. The conditions stated in (iii) of Prop. 9 read

$$c + a^{i+1} = a^i N, \quad c \cap a^{i+1} = a^{i+1} N.$$

Since $a^{i+1} = a^i a \subset a^i N^2 \subset Na^i N$, it follows from Prop. 2 that the first condition becomes $c = a^i N$. Since $a^{i+1} \subset a^i N \subset c$, the second condition becomes $a^{i+1} = a^{i+1} N$. This is equivalent with $a^{i+1} = 0$.

For the second part we may use the simpler condition (iii''). The relation $a^i N \subset a^{i+1}$ implies $a^i N \subset a^i a \subset a^i N^2$ and this is equivalent with $a^i N = 0$.

Remark. In the case $i = 0$, the condition $aN \subset Na$ is not needed.

COROLLARY 11. *Let A be a hereditary semi-primary ring. Then, for $n \geq 0$,*

$$\text{gl. dim } (A/N^2) = n \iff N^{n+1} = 0, \quad N^n \neq 0.$$

§ 3. Triangular matrices

PROPOSITION 12. *Consider the ring $A = T_{n+1}(K)$ of all triangular matrices of degree $n + 1$*

$$a(\alpha_{ij}) = \begin{pmatrix} \alpha_{00}, 0, \dots, 0 \\ \alpha_{10}, \alpha_{11}, \dots, 0 \\ \dots \dots \dots \\ \alpha_{n0}, \alpha_{n,1}, \dots, \alpha_{nn} \end{pmatrix}$$

with coefficients in a semi-simple ring K . Then A is semi-primary, hereditary ($\text{gl. dim } A = 1$) and $\text{gl. dim } (A/N^2) = n$, where N is the radical of A .

Proof. Consider the elements $e, c \in A$ defined by

$$e = a(\alpha_{00} = 1, \alpha_{ij} = 0 \text{ for } (i, j) \neq (0, 0)), \\ c = a(\alpha_{i+1,i} = 1, \alpha_{ij} = 0 \text{ for } i \neq j + 1).$$

We obtain an exact sequence

$$0 \longrightarrow Ae \xrightarrow{\psi} A \xrightarrow{\varphi} N \longrightarrow 0$$

where ψ is the inclusion while $\varphi\lambda = \lambda c$ ($\lambda \in A$). Since e is an idempotent Ae is a direct summand. Thus the exact sequence splits and therefore N is A -projective. Since $N \neq 0$ we have $\text{l. dim}_A N = 0$ so that $\text{gl. dim } A = 1$. The assertion $\text{gl. dim } (A/N^2) = n$ now follows from Cor. 11 since $N^n \neq 0 = N^{n+1}$.

PROPOSITION 13. *Let A be any ring, A a left A -module and let*

$$\dots \longrightarrow 0 \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} X_0 \xrightarrow{d_0} A \longrightarrow 0$$

be a projective resolution of A ($n \leq \infty$) such that $d_i \neq 0$ and X_i is indecomposable for all $i \leq n$. Then $\text{l. dim}_A A = n$.

Proof. Assume $\text{l. dim}_A A = k < n$, and denote

$$Z_k = \text{Ker } d_k = \text{Im } d_{k+1}.$$

Then $0 \neq Z_k \neq X_k$. From the exact sequence

$$0 \longrightarrow X_k/Z_k \longrightarrow X_{k-1} \longrightarrow \dots \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

it follows that X_k/Z_k is A -projective. Therefore the exact sequence

$$0 \longrightarrow Z_k \longrightarrow X_k \longrightarrow X_k/Z_k \longrightarrow 0$$

splits. This contradicts the indecomposability of X_k , since $Z_k \neq 0 \neq X_k/Z_k$.

PROPOSITION 14. Consider the ring of all matrices of the form

$$a(\alpha, \beta, \gamma, \delta, \epsilon) = \begin{vmatrix} \alpha & 0 & 0 \\ \gamma & \beta & 0 \\ \epsilon & \delta & \alpha \end{vmatrix}$$

with coefficients in a semi-simple ring K . Then A is semi-primary, $\text{gl. dim } A = 2$ and $\text{gl. dim } (A/N^2) = \infty$.

Proof. Consider the elements

$$\begin{aligned} e_1 &= a(1, 0, 0, 0, 0), & e_2 &= a(0, 1, 0, 0, 0), \\ c &= a(0, 0, 1, 1, 0). \end{aligned}$$

Clearly e_1 and e_2 are orthogonal primitive idempotents and $e_1 + e_2 = 1$. We obtain an isomorphism

$$\theta : Ae_2 \approx Ne_1$$

by setting $\theta e_2 = ce_1$; this shows that Ne_1 is projective and indecomposable. Further we obtain an exact sequence

$$0 \longrightarrow Ne_1 \xrightarrow{\psi} Ae_1 \xrightarrow{\varphi} Ne_2 \longrightarrow 0$$

where ψ is the inclusion while $\varphi e_1 = ce_2$. Since both Ne_1 and Ae_1 are indecomposable and projective and $\psi \neq 0 \neq \varphi$ it follows from Prop. 13 that $\text{l. dim}_A Ne_2 = 1$. Since $N = Ne_1 + Ne_2$ is a direct sum, we obtain $\text{l. dim}_A N = 1$ so that $\text{gl. dim } A = 2$.

Now set $A' = A/N^2$ and let e'_1, e'_2, c' denote the images of e_1, e_2, c in A' .

Then e'_1 and e'_2 are still orthogonal primitive idempotents in A' with $e'_1 + e'_2 = 1$. We obtain exact sequences

$$0 \longrightarrow N'e'_2 \xrightarrow{\psi_1} A'e'_2 \xrightarrow{\varphi_1} N'e'_1 \longrightarrow 0, \quad 0 \longrightarrow N'e'_1 \xrightarrow{\psi_2} A'e'_1 \xrightarrow{\varphi_2} N'e'_2 \longrightarrow 0$$

where ψ_1 and ψ_2 are inclusion while $\varphi_1 e'_2 = ce'_1$, $\varphi_2 e'_1 = ce'_2$. By composition we obtain a projective resolution

$$\dots \longrightarrow A'e'_2 \longrightarrow A'e'_1 \longrightarrow A'e'_2 \longrightarrow \dots \longrightarrow A'e'_1 \longrightarrow A'e'_2 \longrightarrow N'e'_1 \longrightarrow 0$$

where none of the maps is zero. Again by Prop. 13 we have $\text{l. dim}_{A'}(N'e'_1) = \infty$. It follows that $\text{gl. dim } A' = \infty$.

PROPOSITION 15. *For each pair (m, n) where $0 < m \leq \infty$, $0 < n \leq \infty$, $(m, n) \neq (1, \infty)$, there exist a semi-primary ring A and a two-sided ideal \mathfrak{a} of A contained in the radical of A such that*

$$\text{gl. dim } A = m, \quad \text{gl. dim } (A/\mathfrak{a}) = n.$$

Proof. The case $m = 1$ is settled by Prop. 12. Assume $m > 1$. We define A as a direct product $A_1 + A_2$ where A_1 and A_2 are semi-primary rings such that

$$\begin{aligned} \text{gl. dim } A_1 &= m, & \text{gl. dim } (A_1/N_1) &= 0, \\ \text{gl. dim } A_2 &\leq 2, & \text{gl. dim } (A_2/N_2^2) &= n. \end{aligned}$$

The existence of such rings follows from Prop. 12 and 14. Using the general fact that $\text{gl. dim } A = \sup(\text{gl. dim } A_1, \text{gl. dim } A_2)$ (see [2], Ch. VI, Exer. 8) we find $\text{gl. dim } A = m$. Now consider the two-sided ideal $\mathfrak{a} = N_1 + N_2^2 \subset N$. we have

$$A/\mathfrak{a} = A_1/N_1 + A_2/N_2^2$$

so that applying the general rule we obtain $\text{gl. dim } (A/\mathfrak{a}) = n$.

§ 4. Algebras

Let A be an algebra over a (commutative) field K such that $(A : K) < \infty$. Then A is a semi-primary ring. Consider the following statements:

- (a) $\dim A < \infty$,
- (b) $\Gamma = A/N$ is separable (i.e. $\dim \Gamma = 0$),
- (c) $\dim A = 1 + \text{l. dim}_A N = \text{l. dim}_A \Gamma = \text{gl. dim } A$.

The implications

$$(a) \implies (b) \implies (c)$$

have been established in Ikeda-Nagao-Nakayama [5] and Eilenberg [3]. We further note that if (b) holds for A then it also holds for $A' = A/a$ where a is any two-sided ideal in A . Thus also (c) holds for A' . These remarks imply that if in §2 A is an algebra as above then all the propositions stated hold with $\text{gl.dim } A$ and $\text{gl.dim } (A/a)$ replaced by $\text{dim } A$ and $\text{dim } (A/a)$.

In Propositions 12 and 14 of §3 we may take K to be the ground field, and then the rings constructed satisfy (b). Thus the conclusions remain valid with gl.dim replaced by dim . This incidentally supplies examples of algebras with $(A : K) < \infty$ and $\text{dim } A = n$ for any $0 \leq n \leq \infty$; such examples have been given by Rose [6] and Shih [7].

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Columbia University

Osaka City University

Nagoya University