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M-IDEALS AND FUNCTION ALGEBRAS

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ABSTRACT. Let $C(X)$ be the space of all continuous complex-valued functions defined on the compact Hausdorff space X. We characterize the M-ideals in a uniform algebra A of $C(X)$ in terms of singular measures. For a Banach function algebra B of $C(X)$ we determine the connection between strong hulls for B and its peak sets. We also show that $M(X)$ the space of complex regular Borel measures on X has no M-ideal.

1. **Introduction.** Since its inception some twenty years ago, the topic of M-ideals has proven useful and interesting in various branches of analysis, thanks in large part to the approximation properties M-ideals enjoy.

In this article we plan to characterize the M-ideals of a function space A in $C(X)$. The concept of an M-ideal is defined by Alfsen and Effros [1] and a growing body of literature has been built up on the study of such ideals, see [2], [10] and [6].

We characterize an M-ideal J in a uniform algebra A of $C(X)$ in terms of singular measures and similarly for a Banach function algebra. Using the notion of band of measures we show that $M(X)$ the space of measures on X contains no M-ideal.

For Banach function algebras a good substitute for peak sets is the notion of a strong hull introduced in [4]. We prove that for a normal Banach function algebra if $\ker(E)$, E closed, is an M-ideal then E is a strong hull. Two examples are given; one to show that normality is essential; another to show that the converse is not true.

2. **Preliminaries.** Let $C(X)$ be the Banach space of continuous complex-valued functions on the compact Hausdorff space X equipped with the sup-norm. A subalgebra A of $C(X)$ is said to be a *uniform algebra* if it is uniformly closed, contains the constants and separates the points of X. A subalgebra B of $C(X)$ is called a *Banach function algebra* on X if B contains the constants, separates the points of X and has a norm $\|.\|$ which makes it into a Banach algebra. Clearly $|f(x)| \leq \|f\|$ for $f \in A$ and $x \in X$. Hence $\|f\|_\infty \leq \|f\|$ and the embedding of B into $C(X)$ is continuous.

Let Y be a Banach space. A closed subspace N_1 of Y is called an *L-summand* if there is a closed subspace N_2 of Y such that $Y = N_1 \oplus N_2$ and $\|n_1+n_2\| = \|n_1\| + \|n_2\|$ for $n_1 \in N_1$ and $n_2 \in N_2$. Similarly, a closed subspace J_1 of Y is an *M-summand* if there is a closed subspace J_2 of Y such that $Y = J_1 \oplus J_2$ and $\|x_1 + x_2\| = \max(\|x_1\|, \|x_2\|)$ for $x_1 \in J_1$ and

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$x_2 \in J_2$. If J is a closed subspace of Y such that the polar of J, $J^\circ = \{f \in Y^* : f|_J = 0\}$, is an L-summand in Y^* then J is called an M-ideal in Y.

Let B be a Banach function algebra on X. The state space of B is defined by $S_B = \{p \in B^* : p(1) = \|p\| = 1\}$. The map $L: X \longrightarrow S_B$ given by $L_x(f) = f(x), f \in B$ is a homeomorphic embedding of X into S_B. The Choquet boundary of X with respect to B is defined by $\partial_B X = \{x \in X : L_x \in \partial_e S_B\}$, where $\partial_e S_B$ is the set of extreme points of S_B. Finally we denote by $M(\partial_B X)$ those complex measures μ on X for which the direct image measure $L(|\mu|)$ on S_B is an element of $M(\partial_e S_B)$, see [8].

3. **Singular measures.** We start with the following simple lemma.

LEMMA 1. *Let μ and ν be two measures in $M(X)$. Then $\|\mu \pm \nu\| = \|\mu\| + \|\nu\|$ if and only if $\mu \perp \nu$.*

PROOF. We suppose that $\|\mu \pm \nu\| = \|\mu\| + \|\nu\|$. Set $\lambda = |\mu| + |\nu|$ and write $\mu = g\lambda$, $\nu = h\lambda$ for g, h in $L^1(\lambda)$. Then $\|g \pm h\|_1 = \|g\|_1 + \|h\|_1$, where the norm is that of $L^1(\lambda)$. From this we have $|g \pm h| = |g| + |h|$ a.e. λ on X and in particular a.e. λ on $E \cap F$ where $E = \{x : g(x) \neq 0\}$ and $F = \{x : h(x) \neq 0\}$. If C is a subset of $E \cap F$ with $\lambda(C) = 0$ on which $|g \pm h \ \neq |g| + |h|$ then replacing $E(F)$ by $E \setminus C$ $(F \setminus C)$ in our argument we may assume that $|g \pm h| = |g| + |h|$ on $E \cap F$. Since the equality $|a \pm b| = |a| + |b|$, a, b, in $\mathbf{C} - \{0\}$, never holds; we get $E \cap F = \emptyset$. Because $\mu(\nu)$ is carried by $E(F)$ we have $\mu \perp \nu$. ∎

In the sequel note that if A and B are subsets of $M(X)$ then by $A \perp B$ we mean $\mu \perp \nu\ \forall \mu$ in A and $\forall \nu$ in B.

PROPOSITION 1. *Let J be a closed ideal of $C(X)$. If J is an M-ideal and $M(X) = J^\circ \oplus K$ then $J^\circ \perp K$. Assume N is w^*-closed, that $M(X) = N \oplus K$, and that $N \perp K$. Then N is an L-summand and hence ${}^\circ N$ is an M-ideal.*

In what follows we need some notation. If J is an M-ideal in A then J° is an L-summand in $A^* = M(X)/A^\perp$. Recall that for every Banach space $\mathcal{X}$ the action of $x^* \in \mathcal{X}^*$ on $x \in \mathcal{X}$ is denoted by $\langle x, x^* \rangle$ $(= x^*(x))$. Now let $\mu \in M(X)$. Then $\langle f, \mu + A^\perp \rangle = 0\ \forall f$ in J if and only if $\langle f, \mu \rangle = 0\ \forall f$ in J. Hence $J^\circ = \{\mu + A^\perp : \mu \in J^\perp\} = J^\perp / A^\perp$, and we can write $M(X)/A^\perp = J^\perp/A^\perp \oplus F/A^\perp$. If $\mu \in M(X)$ then $\|\mu + A^\perp\| = \sup\{|\langle f, \mu + A^\perp \rangle| : f \in A, \|f\|_\infty \leq 1\} = \sup\{|\langle f, \mu \rangle| : f \in A, \|f\|_\infty \leq 1\} = \|\mu|_A\|$. Here we regard μ as a bounded linear functional on $C(X)$.

THEOREM 1. *Let A be a uniform algebra in $C(X)$ and let J be a closed ideal of A. If J is an M-ideal in A and $\|\mu_1 \pm \mu_2 + A^\perp\| = \|\mu_1 + A^\perp\| + \|\mu_2 + A^\perp\|$, $\mu_1 \in J^\perp$ and $\mu_2 \in F$ then there exist λ_1, λ_2 in $M(X)$ such that $\mu_i + A^\perp = \lambda_i + A^\perp$, $i = 1, 2$, $\|\lambda_i\| = \|\mu_i|_A\|$ and $\lambda_1 \perp \lambda_2$. Conversely, suppose $A^* = J^\circ \oplus K$ where $K = F/A^\perp$ such that $\forall \mu_1 \in J^\perp$ and $\mu_2 \in F$ the restrictions $\mu_i|_A$ $(i = 1, 2)$ have norm-preserving extensions λ_i to $C(X)$ with $\lambda_1 \perp \lambda_2$ then J is an M-ideal.*

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PROOF. Assume J is an M-ideal and the above relation holds. Then $\|\mu_i + A^\perp\| = \|\mu_i|A\|$, $i = 1, 2$. Applying the Hahn-Banach theorem we get a norm-preserving extension λ_i to $C(X)$ of $\mu_i|_A$. Because $\nu_i = \lambda_i - \mu_i$ is in $A^\perp$ we have $\mu_i + A^\perp = \lambda_i + A^\perp$. Since $\lambda_1 \pm \lambda_2$ is an extension of $(\mu_1 \pm \mu_2)|_A$ to $C(X)$ we have $\|\mu_1 \pm \mu_2 + A^\perp\| = \|(\mu_1 \pm \mu_2)|_A\| \leq \|\lambda_1 \pm \lambda_2\| \leq \|\lambda_1\| + \|\lambda_2\| = \|\mu_1|_A\| + \|\mu_2|_A\| = \|\mu_1 + A^\perp\| + \|\mu_2 + A^\perp\| = \|\mu_1 \pm \mu_2 + A^\perp\|$, so $\|\lambda_1 \pm \lambda_2\| = \|\lambda_1\| + \|\lambda_2\|$. Therefore $\lambda_1 \perp \lambda_2$.

Conversely, suppose $\mu_i|_A$, $i = 1, 2$, have norm preserving extensions λ_i to $C(X)$ such that $\lambda_1 \perp \lambda_2$. Suppose λ_i is carried by E_i $(i = 1, 2)$ and $X = E_1 \cup E_2$, a disjoint union. Then $\|(\mu_1 \pm \mu_2)|_A\| = \|\chi_{E_1}(\mu_1 \pm \mu_2)|_A + \chi_{E_2}(\mu_1 \pm \mu_2)|_A\| = \|\chi_{E_1}\mu_1|_A \pm \chi_{E_2}\mu_2|_A\| = \|\chi_{E_1}\mu_1|_A\| + \|\chi_{E_2}\mu_2|_A\| = \|\mu_1|_A\| + \|\mu_2|_A\|$. The proof is now complete. ■

THEOREM 2. *Suppose B is a Banach function algebra in $C(X)$ and N_1 is a weak* closed ideal in B^*. If N_1 is an L-summand, $B^* = N_1 \oplus N_2$ and $\|p_1 + p_2\| = \|p_1\| + \|p_2\|$, $p_i \in N_i$ $(i = 1, 2)$ then there exist representing measures μ_i in $M(\partial_B X)$ for p_i $(i = 1, 2)$ with $\|p_i\| = \|\mu_i\|$ satisfying $\mu_1 \perp \mu_2$. Conversely, suppose $B^* = N_1 \oplus N_2$ such that every $p_i \in N_i$ has a representing measure μ_i in $M(\partial_B X)$ with $\|p_i\| = \|\mu_i\|$ $(i = 1, 2)$ such that $\mu_1 \perp \mu_2$ then N_1 is an L-summand.*

PROOF. Assume N_1 is an L-summand and $B^* = N_1 \oplus N_2$. If $p_i \in N_i$ $(i = 1, 2)$ and $\|p_1 + p_2\| = \|p_1\| + \|p_2\|$ then by a result of Hustad and Hirsberg [8, p. 142] there exist representing measures $\mu_i \in M(\partial_B X)$ for p_i $(i = 1, 2)$ such that $\|p_i\| = \|\mu_i\|$. Because $\mu_1 + \mu_2$ is a representing measure for $p_1 + p_2$ we have $\|\mu_1\| + \|\mu_2\| = \|p_1\| + \|p_2\| = \|p_1 + p_2\| \leq \|\mu_1 + \mu_2\| \leq \|\mu_1\| + \|\mu_2\|$, so $\|\mu_1\| + \|\mu_2\| = \|\mu_1 + \mu_2\|$. Therefore $\mu_1 \perp \mu_2$.

Conversely, suppose $B^* = N_1 \oplus N_2$ such that every $p_i \in N_i$ has a representing measure μ_i in $M(\partial_B X)$ with $\|p_i\| = \|\mu_i\|$ $(i = 1, 2)$ such that $\mu_1 \perp \mu_2$. We show that N_1 is an L-summand. If μ_i is carried by E_i $(i = 1, 2)$ and $\partial_B X = E_1 \cup E_2$, a disjoint union, then we can write $\|p_1 + p_2\| = \|\chi_{E_1}(p_1 + p_2) + \chi_{E_2}(p_1 + p_2)\| = \|\chi_{E_1}p_1 + \chi_{E_2}p_2\| = \|\chi_{E_1}p_1\| + \|\chi_{E_2}p_2\| = \|p_1\| + \|p_2\|$. Here $\chi_E p(f) = \int_E f\,d\mu$ $\forall f \in B$. This completes the proof. ■

For the existence of M-ideals in $M(X)$ we need the notion of band of measures. A closed linear subspace $\mathcal{B}$ of $M(X)$ is called a *band of measures* if whenever $\mu \in \mathcal{B}$ and $\nu \in M(X)$ such that $\nu \ll \mu$, then $\nu \in \mathcal{B}$. We note that $M(X)$, $\{0\}$, the space of all completely non-atomic measures μ in $M(X)$ and $L^1(\mu) = \{\nu : \nu \ll \mu\}$ are examples of bands.

For any band $\mathcal{B}$ define $L^\infty(\mathcal{B})$ to be the collection of all $f = \{f_\mu\}$ in the Cartesian product $\Pi\{L^\infty(\mu) : \mu \in \mathcal{B}\}$ such that if μ and $\nu \in \mathcal{B}$ and $\mu \ll \nu$, then $f_\mu = f_\nu$ a.e. μ. For $f \in L^\infty(\mathcal{B})$ the norm is defined by $\|f\| = \sup\{\|f_\mu\|_\infty : \mu \in \mathcal{B}\}$. For f in $L^\infty(\mathcal{B})$, if $L_f : \mathcal{B} \longrightarrow \mathbf{C}$ is defined by $L_f(\mu) = \int f_\mu\,d\mu$, then the map $f \longrightarrow L_f$ is an isometric isomorphism of $L^\infty(\mathcal{B})$ onto $\mathcal{B}^*$ [3, p. 79]. If $\mathcal{B}$ is a band and $\mathcal{B}' = \{\mu \in M(X) : \mu \perp \nu\ \forall \nu \in \mathcal{B}\}$ then $\mathcal{B}'$ is also a band called the *complementary band to* $\mathcal{B}$ and every μ in $M(X)$ can be decomposed as $\mu = \nu + \eta$ with $\nu \in \mathcal{B}$ and $\eta \in \mathcal{B}'$. That is, $M(X) = \mathcal{B} \oplus \mathcal{B}'$. It is also easy to see that $L^\infty\big(M(X)\big) = L^\infty(\mathcal{B}) \oplus L^\infty(\mathcal{B}')$.

PROPOSITION 2. *If $\mathcal{B}$ is a band of measures on X then $\mathcal{B}^\circ$, the polar of $\mathcal{B}$ in $M(X)^*$, is given by $\mathcal{B}^\circ = L^\infty(\mathcal{B}')$. Moreover, $\mathcal{B}$ can not be an M-ideal in $M(X)$. In fact $M(X)$ has no nontrivial M-ideal.*

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PROOF. Note that $\mathcal{B}^\circ = \{f \in M(X)^* : f|_{\mathcal{B}} = 0\} = \{f \in L^\infty(M(X)) : L_f(\mu) = 0$ for all μ in $\mathcal{B}\} = \{f \in L^\infty(M(X)) : \int f_\mu\, d\mu = 0\ \forall \mu$ in $\mathcal{B}\}$. Clearly $L^\infty(\mathcal{B}') \subset \mathcal{B}^\circ$. Now let L be a weak* continuous linear functional on $L^\infty(M(X))$ annihilating $L^\infty(\mathcal{B}')$. We show that L annihilates $\mathcal{B}^\circ$ too. Because $L^\infty(M(X)) = M(X)^*$ there is a measure μ in $M(X)$ such that $L(f) = \int f_\mu\, d\mu\ \forall f$ in $L^\infty(M(X))$. Write $\mu = \nu + \eta$, $\nu \in \mathcal{B}$ and $\eta \in \mathcal{B}'$. Because $\nu \perp \eta$, $\nu(\eta)$ is carried by $A(B)$ where $A \cap B = \emptyset$. Then $\{\chi_B\}$ is in $L^\infty(\mathcal{B}')$ so $L(\chi_B) = \int \chi_B\, d\mu = \mu(B) = \nu(B) + \eta(B) = \eta(B) = 0$. Hence $\eta = 0$ *i.e.* $\mu = \nu \in \mathcal{B}$. Now if $f \in \mathcal{B}^\circ$ then $L(f) = \int f_\mu\, d\mu = \int f_\nu\, d\nu = 0$. If $\mathcal{B}$ is an M-ideal then $\mathcal{B}^\circ = L^\infty(\mathcal{B}')$ is an L-summand in $M(X)^* = L^\infty(M(X))$. But $L^\infty(\mathcal{B}')$ is an M-summand which is clearly a contradiction.

To show that $M(X)$ has no nontrivial M-ideal assume J is a closed subspace of $M(X)$ and let $\mathcal{B}$ be the band generated by J. We then prove that $\mathcal{B}^\circ = J^\circ$. All we need to show is that $J' = \mathcal{B}'$ since then $J^\circ = L^\infty(J') = L^\infty(\mathcal{B}') = \mathcal{B}^\circ$. Let $\mu \perp J$ and $\tau \in \mathcal{B}$. We can then find measures $\tau_1, \tau_2, \ldots$ in J and functions $h_1, h_2, \ldots$ such that $\tau = \sum_{i=1}^\infty h_i\tau_i$ (convergent in norm). Because $\mu \perp h_i\tau_i$ we see that $\mu \perp \tau$. Therefore $\mu \perp \mathcal{B}$ and the proof is complete. ∎

4. **Strong hulls.** Suppose B is a Banach function algebra in $C(X)$. If J is a closed ideal in B with $E = \mathrm{hull}(J) = \{x \in X : f(x) = 0\ \forall f$ in $J\}$, then E is called *strong hull* if there is a constant C (depending on E) such that for each compact set S disjoint from E and $\varepsilon > 0$ there is a function f in B such that $f(E) = 0$, $|1 - f(S)| < \varepsilon$ and $\|f\| \leq C$.

The notion of a strong hull in a Banach function algebra was introduced in [4] where the connection between this concept and peak sets for uniform algebras is shown. In particular they generalize a result of T. W. Gamelin [5] concerning uniform algebras and obtain a peaking criterion for strong hulls which we state without proof.

THEOREM 3. *Suppose B is a Banach function algebra in $C(X)$ and let E be a strong hull for B. Let p be a positive continuous function such that $p = 1$ on E and $\|p\|_\infty = 1$. Then there exists f in B, $\|f\|_\infty = 1$ satisfying $E \subset \{x \in X : f(x) = 1\} \subset \{x \in X : p(x) = 1\}$ and $|f(x)| \leq |p(x)|\ \forall x$ in X.*

Recall that a closed subset E of X is a p-set, or *generalized peak set*, if it is the intersection of peak sets. It is clear that each strong hull is a p-set (take p to be the identity function in the statement of the theorem). On the other hand each p-set is a strong hull if B is a uniform algebra. Therefore the two notions coincide for uniform algbras. But for Banach function algebras the situation is quite different.

THEOREM 4. *Suppose $B = C^1[0, 1]$ is the Banach algebra of continuously differentiable functions on the unit interval $[0, 1]$ equipped with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, $f \in B$. Then each closed set in $[0, 1]$ is a peak set and B has no non-trivial strong hull.*

PROOF. Let E be a closed set and write E' as the disjoint union $E' = \bigcup_{n=1}^\infty I_n$, where $I_n = (a_n, b_n)$. Let $g_n(t) = \exp\left[-\left(1/(t - a_n) + 1/(t - b_n)\right)^2\right]$ and define $f_n(x) = 2^{-n}\left(g_n(x)/\|g_n\|\right)\ \forall x$ in I_n and 0 elsewhere. Let $f = \sum_{n=1}^\infty f_n$. Then f is in B since each f_n

is. Clearly $f = 0$ on E and $|f(x)| < 1$ on E'. If $g = 1 - f$ then g peaks on E, so E is a peak set.

This follows from the previous theorem. First we suppose that $S = \{1/2\}$ is a strong hull for B and we let $p(x) = 2x$ on $[0, 1/2]$ and $= 2(1-x)$ on $[1/2, 1]$. Then the function f in the theorem can not be differentiable and so f is not in B. From which we conclude that S is not a strong hull for B. On the other hand if E contains more than one point and x_0 is not in E then we define $\alpha = \sup\{y \in E : y < x_0\}$ and $\beta = \inf\{y \in E : y > x_0\}$. Using these values we define a function p by $p(x) = 1, 1, (x-x_0)/(\alpha-x_0)$, and $(x-x_0)/(\beta-x_0)$ if $x \leq \alpha, x \geq \beta, \alpha < x < x_0$, and $x_0 < x < \beta$ respectively. Then f cannot be differentiable and E is not a strong hull for B. ∎

REMARK. The above result is mentioned in [4] but no proof is given.

In the case of a Banach function algebra we can prove the following

PROPOSITION 3. *Let B be a normal Banach function algebra on X and let E be a closed subset of X such that $J = \{f \in B : f|_E = 0\}$ is an M-ideal. Then E is a strong hull.*

PROOF. By [10] J has an approximate identity $\{e_\alpha\}$ with $\|e_\alpha\| \leq 1$. Let S be a compact subset of X disjoint from E and let $\varepsilon > 0$ be given. Since B is normal, there is g in B such that $g = 0$ on E and $g = 1$ on S. Therefore $g \in J$ and by definition of the approximate identity there is α_0 such that $\alpha \geq \alpha_0$ and $x \in S$ imply

$$|e_\alpha(x)g(x) - g(x)| \leq \|e_\alpha g - g\|_\infty \leq \|e_\alpha g - g\| \leq \varepsilon.$$

Because $g = 1$ on S we have $|e_\alpha(x) - 1| < \varepsilon$ for all x in S. If we let $f = e_{\alpha_0}$ in the definition of the strong hull then we are done. ∎

The following example shows that the normality of B can not be omitted.

EXAMPLE. Let $\mathbf{D} = \{z : |z| < 1\}$ denote the open unit disk and let A be the *disc algebra*; the space of all functions continuous on $\bar{\mathbf{D}}$ and analytic in $\mathbf{D}$. Let $X = \{z : |z| \leq 2\}$, $B = \{f \in C(X) : f|_{\mathbf{D}} \in A\}$ and $E = \{z : |z| \leq 1/2\}$. Then B is a uniform algebra on X and $M_B = X$, where M_B is the maximal ideal space of B.

To see this, note that the restriction map $B \longrightarrow A(f \longrightarrow f|_{\mathbf{D}})$ is a continuous surjection with kernel $J = \{f \in C(X) : f|_{\bar{\mathbf{D}}} = 0\}$. Hence $B/J \cong A$. So $\bar{\mathbf{D}} = h(J)$ [9, Theorem 3.1.17] where $h(J) = \{\phi \in M_B : \phi = 0 \text{ on } J\}$. Now $M_B \setminus h(J) = M_J$ by [9, Theorem 3.1.18]. Hence $M_B \setminus \bar{\mathbf{D}} = M_J$. Regard J as $C_0(\Delta)$ where $\Delta = \{z : 1 < |z| \leq 2\}$ and get $M_J = \Delta$. Therefore $M_B = X$.

If $J_E = \{f \in B : f|_E = 0\}$ then J_E is a closed ideal with $h(J_E) = \bar{\mathbf{D}}$ but E is not a p-set. To see this, note that since X is metrizable the two notions of peak set and p-set coincide. If E is a peak set then there is f in B such that $f = 1$ on E and $|f| < 1$ on $X \setminus E$. Since $f = 1$ on $\mathbf{D}$ we obtain a contradiction. ∎

The following lemma shows that the converse is not true.

LEMMA 2. *Suppose $\mathbf{T}$ is the unit circle and $M(\mathbf{T})$ is the Banach algebra of all regular Borel measures on $\mathbf{T}$ with convolution as multiplication. Let Δ be the maximal ideal* *space of $M(\mathbf{T})$ and $E = \Delta \setminus \mathbf{Z}$ be the complement of integers. Then E is a strong hull for $M(\mathbf{T})$ and $J = \{\mu \in M(\mathbf{T}) : \hat{\mu}(\psi) = 0\ \forall \psi \text{ in } E\}$ is a closed ideal in $M(\mathbf{T})$ which is not an M-ideal.* ∎

PROOF. Since $\mathbf{Z}$ is the dual group of $\mathbf{T}$ [9, A.3.2] it follows from [7, Theorem 38.4] that $J = \operatorname{rad} L^1(T)$. That is, J is the intersection of all maximal ideals of $M(\mathbf{T})$ which contain $L^1(\mathbf{T})$. Therefore $L^1(\mathbf{T}) \subset J$. Suppose $\varepsilon > 0$ is given and S is a compact set disjoint from E. Then $S \subset \mathbf{Z}$, so S is finite.

Now $L^1(\mathbf{T})$ contains an approximate identity $\{e_\alpha\}$ with $\|e_\alpha\| \leq 1$ [9, p. 321]. Since S is finite there is $g \in L^1(\mathbf{T})$ with $\hat{g} = 1$ on S. Let α_0 be such that $\|e_\alpha^* g - g\| < \varepsilon$ for $\alpha \geq \alpha_0$. If $j \in S$ and $\alpha \geq \alpha_0$ then $|\hat{e_\alpha}(j)\hat{g}(j) - \hat{g}(j)| \leq \|\hat{e_\alpha}\hat{g} - \hat{g}\|_\infty \leq \|e_\alpha^* g - g\| < \varepsilon$. But $\hat{g} = 1$ on S so $|\hat{e_\alpha} - 1| < \varepsilon\ \forall j$ in S. We conclude that E is a strong hull by setting $f = e_{\alpha_0}$.

The fact that J is not an M-ideal follows from the second Proposition of Section 3. Another way to prove this is to note that $M(\mathbf{T}) = M_d(\mathbf{T}) \oplus M_c(\mathbf{T})$ by [4, Theorem 19.20] where $M_d(\mathbf{T})$ $\big(M_c(\mathbf{T})\big)$ is the set of all purely discontinuous (continuous) measures in $M(\mathbf{T})$. In fact, $M_d(\mathbf{T})$ and $M_c(\mathbf{T})$ are complementary non-trivial L-summands of $M(\mathbf{T})$. It now follows from [2, Corollary 1.14, p. 28] that J is not an M-ideal. ∎

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