On the Oscillation Functions derived from a Discontinuous Function.

By L. R. Ford.

(*Read 11th June 1915. Received 8th August 1915.*)

1. In this paper are introduced what we shall term "successive oscillation functions." These functions are derived from functions of a real variable. The word "function" as here used has its widest meaning. We say y is a function of x in an interval of the the x-axis, if given any value of x, in the interval one or more values of y are thereby determined. The values of the function may be determined by any arbitrary law whatsoever. We shall deal with discontinuous functions; the theorems will be true for continuous functions, but will be trivial, except in the case of functions which are discontinuous and whose points of discontinuity are infinite in number. We shall assume in what follows that the values of the function lie between finite limits.

2. *The Oscillation Function.*—Let us consider the behaviour of a function $y=f(x)$ in the neighbourhood of a point $x=a$. Let an interval about a, extending from $a-h$ to $a+h$ be considered. Let M_h be the upper limit of the function in the interval, and let m_h be the lower limit (*i.e.* the greatest and least values of the function in the interval, if it has greatest and least values). Now, let h approach zero. M_h will not increase; and since $M_h \geqq f(a)$ always, we conclude that M_h approaches a limit. This limit is called ***the maximum of the function at the point a***, and is represented by $M(f, a)$. Similarly, m_h does not decrease and $m_h \leqq f(a)$; hence m_h approaches a limit. This limit is called ***the minimum of the function at the point a***, and is written $m(f, a)$.

https://doi.org/10.1017/S0013091500002443 Published online by Cambridge University Press

The oscillation of the function at the point a, represented by $\omega(f, a)$, is defined as follows :—

$$\omega(f, a) = M(f, a) - m(f, a).*$$

At each point a maximum, a minimum, and an oscillation are defined ; hence they yield three functions of x: $M(f, x)$, $m(f, x)$, and $\omega(f, x)$. Since at each point $M(f, x) \gtreqless m(f, x)$, we see that $\omega(f, x)$ is never negative. At a point a at which $f(x)$ is continuous $M(f, a) = f(a) = m(f, a)$, and hence $\omega(f, a) = 0$. Conversely, if $\omega(f, a) = 0$, the function is continuous at a.

It can be shown that the oscillation function has the following properties :—

(1) $M(\omega, x) \equiv \omega(f, x)$.

(2) There are points in every interval, however small, at which $\omega(f, x)$ is continuous.†

The second property is a consequence of the first.

3. *Successive oscillation functions.*—Representing by $\omega_1(f, x)$ the ordinary oscillation function derived from $f(x)$, we shall represent by $\omega_2(f, x)$ the oscillation of $\omega_1(f, x)$, by $\omega_3(f, x)$ the oscillation of $\omega_2(f, x)$, and so on. That is,

$$\omega_2(f, x) = \omega(\omega_1, x)$$
$$\omega_3(f, x) = \omega(\omega_2, x)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

We shall now establish the following general theorem concerning these functions :—

THEOREM 1.—*If $f(x)$ is any function whatever lying between finite limits, then $\omega_n(f, x) \equiv \omega_2(f, x)$, $n = 3, 4, \ldots$.*

Since $f(x)$ is bounded, all the oscillation functions are finite. The function $\omega_1(f, x)$ has points of continuity in every interval, by (2) ; at these points $\omega(\omega_1, x)$, or $\omega_2(f, x)$, is zero. Now consider $\omega_3(f, x)$. $\omega_3(f, x) = M(\omega_2, x) - m(\omega_2, x)$. From property (1)

* Baire, Lecons sur les functions discontinues, Sec. 45, p. 70. Hobson [Theory of Functions of a Real Variable] uses the term oscillation somewhat differently. In forming $M(f, a)$ and $m(f, a)$ the values of the function at the points of the interval *exclusive of the point a* are considered. He uses the term *saltus* for oscillation as here defined.

† Baire, *loc. cit.*, Sec. 46, p. 73 ; Sec. 48, p. 77.

$M(\omega_2, x) \equiv \omega_2(f, x)$; and since $\omega_2(f, x) = 0$ at points in every interval and is nowhere negative, it follows that $m(\omega_2, x) \equiv 0$. Hence $\omega_3(f, x) \equiv \omega_2(f, x)$. Similarly all succeeding oscillation functions are equal to $\omega_2(f, x)$ and the theorem is established.*

As an example of this theorem consider a function $f(x)$ defined as follows :—

$f(x) = 1/n$, when x is the rational fraction m/n (in its lowest terms).

$f(x) = 0$, when x is the root of a rational fraction, $\surd(m/n)$.

$f(x) = -1$, when x is any other number.

We see that the points of each of the three sets are dense along the x-axis. We find without difficulty the first oscillation function :—

$\omega_1(f, x) = 1 + 1/n$, when x is the rational fraction m/n.

$\omega_1(f, x) = 1$, when x is irrational.

And the oscillation of this function is :—

$\omega_2(f, x) = 1/n$, when x is the rational fraction m/n.

$\omega_2(f, x) = 0$, when x is irrational.

And the oscillation of this function, and hence also any succeeding oscillation, is the function $\omega_2(f, x)$.

4. *Pointwise discontinuous functions.*—A function which has points of continuity in every interval is called *pointwise discontinuous.*† The pointwise discontinuous functions are the simplest of the discontinuous functions. We see from property (2) above that the oscillation functions are pointwise discontinuous. It will be observed that the continuous functions are special cases of pointwise discontinuous functions.

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* Neither this theorem nor the following one is true for the oscillation as defined by Hobson.

† Here again there is a difference of definition. Harnack [*Math. Ann.* **19** (1882), p. 242, and **24** (1884), p. 218] adds the restriction that the points of discontinuity shall be of content 0. While Harnack's definition is useful in the theory of integration, it narrows the application of these functions in other fields. In the work of Baire, done since the publication of Harnack's papers, on the approach to discontinuous functions by continuous functions, the definition is as we have given it.

THEOREM 2.—*The necessary and sufficient condition that a function $f(x)$ be pointwise discontinuous is that $\omega_2(f, x) \equiv \omega_1(f, x)$.*

The condition is necessary. For, if $f(x)$ is pointwise discontinuous, $\omega_1(f, x) = 0$ at points in every interval. Then

$$\omega_2(f, x) \equiv M(\omega_1, x) - m(\omega_1, x) \equiv \omega_1(f, x) - 0 \equiv \omega_1(f, x).$$

Conversely, let $\omega_2(f, x) \equiv \omega_1(f, x)$. We found in proving the preceding theorem that $\omega_2(f, x) = 0$ at points in every interval, whatever the function $f(x)$. Owing to the equality assumed, $\omega_1(f, x) = 0$ at points in every interval. At these points $f(x)$ is continuous; hence $f(x)$ is a pointwise discontinuous function.

5. *The sum of a series.*—Let $f(x)$ be defined as the sum of a convergent series,

$$f(x) = u_1(x) + u_2(x) + \ldots$$

Certain facts concerning series can be neatly expressed by means of oscillation functions. It is known that if the terms of the series are continuous the sum is continuous, provided the convergence is uniform, otherwise the sum is pointwise discontinuous; also that the sum is pointwise discontinuous if the terms are pointwise discontinuous and the convergence is uniform. These facts are expressed in the following table:—

	Uniform convergence.	Mere convergence.
$\omega_1(u_i, x) \equiv 0$	$\omega_1(f, x) \equiv 0$	$\omega_2(f, x) \equiv \omega_1(f, x)$
$\omega_2(u_i, x) \equiv \omega_1(u_i, x)$ $i = 1, 2, 3\ldots$	$\omega_2(f, x) \equiv \omega_1(f, x)$	$\omega_3(f, x) \equiv \omega_2(f, x)$

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