GEOGRAPHY AND BOTANY OF IRREDUCIBLE NON-SPIN SYMPLECTIC 4-MANIFOLDS WITH ABELIAN FUNDAMENTAL GROUP

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Abstract. The geography and botany problems of irreducible non-spin symplectic 4-manifolds with a choice of fundamental group from $\{\mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_p, \mathbb{Z} \oplus \mathbb{Z}\}$ are studied by building upon the recent progress obtained on the simply connected realm. Results on the botany of simply connected 4-manifolds not available in the literature are extended.

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1. Introduction. Topologists' understanding of smooth 4-manifolds has witnessed a drastic improvement during the last 20 years. Advances in symplectic geometry [**8**, **25**, **31**, **35**, **65**–**67**, **71**] and inventions of gluing formulas [**39**, **46**] for diffeomorphism invariants [**73**] have paired elegantly with topological constructions [**16**, **18**, **19**–**21**], offering a noteworthy insight into the smooth four-dimensional story. The most recent series of successes [**2**–**4**, **6**, **11**, **16**, **21**] has answered as many questions as it has raised new ones, and the 4-manifolds theory remains to be an intriguing and active area of research.

Two factors responsible for the recent progress on the simply connected realm are the increase in the repertoire of techniques that manufacture symplectic 4-manifolds with small topological invariants, and a new perspective on the usage of already existing mechanisms of construction. The idea of using symplectic sums [**25**] of non-simply connected building blocks along genus 2 surfaces to kill fundamental groups in an efficient way was introduced in [**1**]. Its immediate outcome was the construction of an exotic symplectic $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$ and, later on, the existence of an exotic symplectic $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$ [**4**] was put on display. Shortly after, Luttinger surgery [**8**, **35**] was introduced to the list of symplectic constructions in [**11**, **56**]. The combinations of these techniques yielded another construction of an exotic symplectic $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$ in [**11**, **16**]. Eventually these techniques were successful in unveiling exotic smooth structures on $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ [**6**, **21**], the 4-manifolds with the smallest Euler characteristic known to admit exotic smooth structures at the time this paper was written.

Having in mind as a motivation Freedman's topological classification of simply connected 4-manifolds [**23**], the knowledge that has been accumulated on smooth 4-manifolds can be encoded to address the questions of existence and uniqueness on a possible classification scheme as follows. The *geography problem* for symplectic

4-manifolds with a given fundamental group [**25**, **37**] asks which homeomorphism classes are realized by an irreducible symplectic 4-manifold. The *botany problem* [**26**] asks how many diffeomorphism classes exist within a given homeomorphism type. The geography problem for symplectic manifolds with trivial fundamental group was first studied systematically in [**43**], where regions are populated using the techniques of [**25**]. The novel tools recently introduced to unveil a myriad of smooth structures [**16**, **18**, **20**] have allowed a hand-in-hand study of both problems. As a sample of the noteworthy successes in the area, it is worth pointing out that the symplectic geography question has been settled for simply connected 4-manifolds of negative signature, and it is known that these manifolds admit infinitely many smooth structures [**2**, **6**, **16**, **25**, **47**, **48**, **52**].

The non-simply connected realm remains to be a somewhat uncharted territory. The topological classification of closed oriented 4-manifolds for several choices of good fundamental groups [**24**] has been established in [**28**–**30**, **33**]. These considerations motivate curiosity minding the symplectic geography and botany for manifolds with nontrivial fundamental group.

The contribution and purpose of this paper is to show how current efforts set on the simply connected realm extend naturally to smooth 4-manifolds with abelian fundamental groups of small rank amongst other choices of fundamental groups. This has been pointed out previously by other authors [**2**, **6**, **13**, **25**], and extends work of the author done in [**69**, **70**]. We build greatly upon the recent constructions to systematically study the geography and botany of irreducible symplectic 4-manifolds with abelian fundamental groups. The first examples of exotic 4-manifolds with cyclic fundamental group were constructed in [**4**, **13**, **17**, **25**, **27**, **28**, **34**, **36**, **40**, **72**], and examples of 4-manifolds with finite fundamental group can be found in [**25**, **28**]. Examples of minimal symplectic 4-manifolds with fundamental group $\mathbb{Z} \oplus \mathbb{Z}_p$ had been previously built in [**25**, **41**, **55**]. Efforts towards more general fundamental groups can be found in [**10**, **12**, **13**, **25**, **51**, **74**].

The organization of the paper is as follows. The main result and the notation employed are presented in Section 2. The results on homeomorphism criteria are recalled in Sections 3.1, 3.2 and 3.3. The fundamental building blocks are presented in Sections 3.4 and 3.8, and the basic construction tool is presented in Section 3.5. Our two main technical results are presented in Section 3.7. With the purpose of providing the reader with hands-on constructions, two examples are worked out thoroughly in Section 3.6. Finally, in Section 4 the efforts are put together to fill in regions of the symplectic geography for the choices of fundamental groups involved, and address the botany. The region of 4-manifolds with negative signature is populated in Section 4.2, and a region of non-negative signature is in Section 4.3.

2. Notation and main results. A finite cyclic group of order $p \geq 2$ is denoted by $\mathbb{Z}_p$. The homeomorphism types of the 4-manifolds with non-spin universal cover, and whose fundamental group is among the choices considered are given by

- $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_q : b_2^+ \mathbb{CP}^2 \# b_2^- \overline{\mathbb{CP}^2} \# L_{p,q}$,
- $\pi_1 = \mathbb{Z}_p : b_2^+ \mathbb{CP}^2 \# b_2^- \overline{\mathbb{CP}^2} \# L_p$, and
- $\pi_1 = \mathbb{Z} : b_2^+ \mathbb{CP}^2 \# b_2^- \overline{\mathbb{CP}^2} \# S^1 \times S^3$

according to [**24**, **29**, **30**, **33**] (see Section 3.1). The pieces $L_{p,q}$ and L_p are oriented closed smooth 4-manifolds that satisfy $\pi_1(L_{p,q}) = \mathbb{Z}_p \oplus \mathbb{Z}_q$, $\pi_1(L_p) = \mathbb{Z}_p$, $e(L_{p,q}) = 2 = e(L_p)$

and $\sigma(L_{p,q}) = 0 = \sigma(L_p)$. These manifolds are constructed as follows: Consider the product $L(p, 1) \times S^1$ of a Lens space and a circle, and the map

$$\phi : L(p, 1) \times S^1 \to L(p, 1) \times S^1$$
$$\{pt\} \times \alpha \mapsto \{pt\} \times \alpha^q.$$

Carve out the loop α^q and glue in a copy of $S^2 \times D^2$ in order to kill the generator that corresponds to the $\mathbb{Z}$-factor in $\pi_1(L(p, 1) \times S^1) \cong \mathbb{Z}_p \oplus \mathbb{Z}$ to obtain a manifold

$$L_{p,q} := (L(p, 1) \times S^1 - (S^1 \times D^3)) \cup_\phi (S^2 \times D^2).$$

Using the Seifert-van Kampen theorem, one concludes $\pi_1(L_{p,q}) = \mathbb{Z}_p \oplus \mathbb{Z}_q$. The case $q = 1$ yields the manifold $L_p := L_{p,1}$ with finite cyclic fundamental group of order p.

A smooth closed 4-manifold X is *irreducible* if for every smooth connected sum decomposition $X = X_1 \# X_2$, either X_1 or X_2 is homeomorphic to S^4 [**26**, Definition 10.1.17]. It is called *minimal* if it is not the blowup of another manifold [**26**, p. 46]. The following definition will be used for our purposes (cf. [**7**, Definition 1]).

DEFINITION 1. The homeomorphism class of a 4-manifold X has the ∞-*property* if and only if there exists an infinite set $\{X_n : n \in \mathbb{N}\}$ that consists of minimal 4-manifolds such that
- if $n \neq m$, then X_n is non-diffeomorphic to X_m;
- there exists an $m \in \mathbb{N}$ for which there is an element $X_m \in \{X_n\}$ that is symplectic, while X_m is non-symplectic for every $m \neq n$; we will assume $m = 1$;
- X_n is homeomorphic to X for all $n \in \mathbb{N}$.

The homeomorphism class of a 4-manifold X has the ∞^2-*property* if and only if there exists an infinite set $\{X_n : n \in \mathbb{N}\}$ that contains infinitely many irreducible symplectic and infinitely many minimal non-symplectic 4-manifolds such that
- if $n \neq m$, then X_n is non-diffeomorphic to X_m,
- X_n is homeomorphic to X for all $n \in \mathbb{N}$.

The main result of the paper is the following theorem.

THEOREM 2. *Let* $G \in \{\mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z} \oplus \mathbb{Z}_q, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}\}$. *Let* (c, χ) *be any pair of non-negative integers* (*except for* $(1, 1)$), *which satisfies*

$$0 \leq c \leq 8\chi - 1.$$

There exists an irreducible symplectic 4-manifold X^G *such that*

$$\pi_1(X^G) = G, \ (c_1^2(X^G), \chi_h(X^G)) = (c, \chi).$$

If $\chi = 1$, *then the homeomorphism types*
- $\mathbb{CP}^2 \#(10 - c - 1)\overline{\mathbb{CP}^2}\#L_p$

have the ∞-*property.*
If $\chi \geq 2$ *and* q *is an odd prime number, then the homeomorphism types*
- $(2\chi - 1)\mathbb{CP}^2 \#(10\chi - c - 1)\overline{\mathbb{CP}^2}$,
- $(2\chi - 1)\mathbb{CP}^2 \#(10\chi - c - 1)\overline{\mathbb{CP}^2}\#L_p$,
- $(2\chi - 1)\mathbb{CP}^2 \#(10\chi - c - 1)\overline{\mathbb{CP}^2}\#L_{q,q}$, *and*
- $2\chi\mathbb{CP}^2 \#(10\chi - c)\overline{\mathbb{CP}^2}\#S^1 \times S^3$

have the ∞^2-*property.*

3. Background results, construction tools and building blocks.

3.1. Homeomorphism type of manifolds with $\pi_1 = \mathbb{Z}_p$. The topological classification of closed oriented 4-manifolds with finite cyclic group was obtained by Hambleton and Kreck [29]. Due to the existence of 2-torsion, one must be careful when determining the parity of its intersection form. Let $\widetilde{X} \to X$ be the universal cover of the non-simply connected 4-manifold X,

- ω_2-type (I) if $\omega_2(\widetilde{X}) \neq 0$,
- ω_2-type (II) if $\omega_2(X) = 0$,
- ω_2-type (III) if $\omega_2(X) \neq 0$, but $\omega_2(\widetilde{X}) = 0$.

THEOREM 3. (Hambleton–Kreck [29, Theorem C]). *A smooth closed oriented 4-manifold X with finite cyclic fundamental group is classified up to homeomorphism by the fundamental group, the intersection form on $H_2(X; \mathbb{Z})/Tors$, and the ω_2-type. Moreover, any isometry of the intersection form can be realized by a homeomorphism.*

By using the known work of Donaldson and Minkowski-Hasse [26] on the classification of bilinear forms over the integers that are realized as intersection forms of smooth 4-manifolds, the previous result can be restated as follows.

THEOREM 4. *A smooth, closed oriented 4-manifold with finite cyclic fundamental and indefinite intersection form is classified up to homeomorphism by the fundamental group, the Betti numbers b_2^+ and b_2^-, the parity of the intersection form and the ω_2-type.*

3.2. Homeomorphism type of manifolds with $\pi_1 = \mathbb{Z}_q \oplus \mathbb{Z}_q$, q an odd prime number. The classification up to homeomorphism of 4-manifolds with fundamental group among the choices of finite non-cyclic abelian groups under consideration is given in the following theorem.

THEOREM 5. (Hambleton–Kreck [29, Theorem B]). *Let X be a smooth closed oriented 4-manifold, and let $\pi_1(X) = \pi$ be a finite group of odd order. When $\omega_2(\tilde{X}) = 0$ (resp. $\omega_2(\tilde{X}) \neq 0$), assume that*

$$b_2(X) - |\sigma(X)| > 2d(\pi),$$

(resp. $> 2d(\pi) + 2$). Then X is classified up to homeomorphism by the signature, Euler characteristic, type and fundamental class in $H_4(\pi, \mathbb{Z})/Out(\pi)$.

In the statement of Theorem 5, the outer automorphism group of π is denoted by $Out(\pi)$. The stability condition required on the lower bound for the Euler characteristic, $d(\pi) \in \mathbb{Z}$ (see [29] for details), depends on the fundamental group of the manifold. In [69], it was proven that $d(\mathbb{Z}_q \oplus \mathbb{Z}_q) = 1$.

3.3. Homeomorphism type of manifolds with $\pi_1 = \mathbb{Z}$. Provided a stability condition holds, the homeomorphism criterion for oriented smooth 4-manifolds with $\pi_1 = \mathbb{Z}$ is similar to Freedman's known result [23] (see also [24, 33, 62]).

THEOREM 6. (Hambleton–Teichner [30, Corollary 3]). *If X is a closed oriented smooth 4-manifold with infinite cyclic fundamental group and satisfies the inequality*

$$b_2(X) - |\sigma(X)| \geq 6,$$

then X is homeomorphic to the connected sum of $S^1 \times S^3$ with a unique closed simply-connected 4-manifold. In particular, X is determined up to homeomorphism by its second Betti number $b_2(X)$, its signature $\sigma(X)$ and its ω_2-type. Moreover, X is either spin or non-spin depending on the parity of its intersection form.

3.4. Basic symplectic building blocks. Consider the 4-torus $T^4 = T^2 \times T^2$ equipped with the product symplectic form. Let x, y, a, b denote both the generators of the group $\pi_1(T^2 \times T^2) = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}a \oplus \mathbb{Z}b$ as well as the corresponding loops. This convention will be maintained throughout the paper. The tori

$$T_1 := x \times a, \ T_2 := y \times a$$

and their respective geometrically dual tori $T_1^d := y \times b$, $T_2^d := x \times b$ are Lagrangian, and the torus $T_3 := a \times b$ and its geometrically dual torus $T_3^d := x \times y$ are symplectic. The characteristic numbers are $c_1^2(T^4) = 0 = \chi_h(T^4)$.

The calculation of the fundamental group of the complement of surfaces inside a manifold plays a fundamental role in recent constructions of 4-manifolds. Choices of base points are fundamental in the application of the Seifert-van Kampen theorem, which is to be used with care. For these matters we build greatly in the analysis done in [11–13]. In particular, we have the following result.

PROPOSITION 7. (Baldridge–Kirk [11, Theorem 2]). *The fundamental group*

$$\pi_1(T^4 - (T_1 \cup T_2))$$

is generated by the loops x, y, a, b, and the relations $[x, a] = [y, a] = 1$ hold. The meridians of the tori and the two Lagrangian push-offs of their generators are given by

$$T_1 : \mu_1 = [b^{-1}, y^{-1}], \ m_1 = x, \ l_1 = a,$$
$$T_2 : \mu_2 = [x^{-1}, b], \ m_2 = y, \ l_2 = bab^{-1}.$$

As our next building block, we take the product of a genus 2 surface and a torus, $\Sigma_2 \times T^2$, and endow it with the product symplectic form. Its characteristic numbers are $c_1^2(\Sigma_2 \times T^2) = 0 = \chi_h(\Sigma_2 \times T^2)$. Let a_1, b_1, a_2, b_2 be the loops and generators of $\pi_1(\Sigma_2)$, and x, y be the loops and generators of $\pi_1(T^2)$. Inside this manifold there are four pairs of homologically essential Lagrangian tori, and a symplectic surface of genus 2 and self-intersection zero. The tori are displayed in the statement of the following proposition; the genus 2 surface is a parallel copy of the surface $\Sigma_2 \times \{pt\}$, and we will denote it by Σ_2.

PROPOSITION 8. (Baldridge–Kirk [13, Proposition 7]). *The fundamental group*

$$\pi_1(\Sigma_2 \times T^2 - (\Sigma_2 \cup T_1 \cup \cdots \cup T_4))$$

is generated by the loops x, y, a_1, b_1, a_2, b_2. Moreover, with respect to certain paths to the boundary of the tubular neighbourhoods of T_i and Σ_2, the meridians and two Lagrangian push-offs of the surfaces are given by

- $T_1 : m_1 = x, \ l_1 = a_1, \ \mu_1 = [b^{-1}, y^{-1}],$
- $T_2 : m_2 = y, \ l_2 = b_1 a_1 b^{-1}, \ \mu_2 = [x^{-1}, b_1],$
- $T_3 : m_3 = x, \ l_3 = a_2, \ \mu_3 = [b_2^{-1}, y^{-1}],$
- $T_4 : m_4 = y, \ l_4 = b_2 a_2 b_2^{-1}, \ \mu_4 = [x^{-1}, b_2],$
- $\mu_{\Sigma_2} = [x, y].$

The loops a_1, b_1, a_2, b_2 lie on the genus 2 surface and form a standard set of generators; the relation $[a_1, b_1][a_2, b_2] = 1$ holds.

The final building block is obtained by applying Luttinger surgeries along Lagrangian tori to the product of two genus 2 surfaces $\Sigma_2 \times \Sigma_2$ equipped with the product symplectic form. Let x_1, y_1, x_2, y_2 be the generators of $\pi_1(\Sigma \times \{x\})$, and a_1, b_1, a_2, b_2 be the generators of $\pi_1(\{x\} \times \Sigma_2)$.

PROPOSITION 9. ([2, Lemma 16], [16, Section 4]). *There exists a minimal symplectic 4-manifold Z with $c_1^2(Z) = 8$ and $\chi_h(Z) = 1$ that contains eight homologically essential Lagrangian tori $\{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$ (each S_i has a geometrically dual torus S_i^d so that all other intersections are zero). The fundamental group $\pi_1(Z - (S_1 \cup \cdots \cup S_8))$ is generated by x_1, y_1, x_2, y_2 and a_1, b_1, a_2, b_2, and the meridians and Lagrangian push-offs are given by*

- $S_1 : \mu_1 = [b_1^{-1}, y_1^{-1}], m_1 = x_1, l_1 = a_1,$
- $S_2 : \mu_2 = [x_1^{-1}, b_1], m_2 = y_1, l_2 = b_1 a_1 b_1^{-1},$
- $S_3 : \mu_3 = [b_2^{-1}, y_1^{-1}], m_3 = x_1, l_3 = a_2,$
- $S_4 : \mu_4 = [x_1^{-1}, b_2], m_4 = y_1, l_4 = b_2 a_2 b_2^{-1},$
- $S_5 : \mu_5 = [b_1 a_1^{-1} b_1^{-1}, y_2^{-1}], m_5 = x_2, l_5 = b_1^{-1},$
- $S_6 : \mu_6 = [x_2^{-1}, b_1 a_1 b_1^{-1}], m_6 = y_2, l_6 = b_1 a_1 b_1^{-1} a_1^{-1} b_1^{-1},$
- $S_7 : \mu_7 = [b_2 a_2^{-1} b_2^{-1}, y_2^{-1}], m_7 = x_2, l_7 = b_2^{-1},$
- $S_8 : \mu_8 = [x_2^{-1}, b_2 a_2 b_2^{-1}], m_8 = y_2, l_8 = b_2 a_2 b_2^{-1} a_2^{-1} b_2^{-1}.$

3.5. Luttinger and torus surgeries. Let T be a Lagrangian torus in a symplectic 4-manifold X. By the Darboux–Weinstein Theorem [38], there exists a parametrization $T^2 \times D^2 \cong N_T \subset X$ of a tubular neighbourhood N_T of T, for which the image of $T^2 \times \{d\}$ ($d \in D^2$) is Lagrangian. Let $d \in D^2 - \{0\}$, the *Lagrangian push-off* or *Lagrangian framing* of T is the push-off

$$F_d : T \to T^2 \times \{d\} \subset X - T$$

determined by d. The smooth isotopy class of $F_d : T \to X - T$ depends only on the symplectic structure of X around a neighbourhood of T. Let γ be an embedded curve in T. The *Lagrangian push-off* of γ is the isotopy class of the image $F_d(\gamma)$. A *meridian* of the torus T is a curve in the isotopy class of $\{t\} \times \partial D^2 \subset \partial(N_T)$, and it will be denoted by μ_T.

Let α and β be the generators of $\pi_1(T)$. Let $m_T = F_d(\alpha)$ and $l_T = F_d(\beta)$ for $d \in \partial D^2$ be the push-offs of the loops in $\partial N_T = T^3$. These are loops homologous in N_T to α and β respectively. The loops $\{m, l, \mu_T\}$ generate $H_1(\partial(N_T); \mathbb{Z}) = \mathbb{Z}^3$. We choose a base point t laying on $\partial(N_T)$, and set $m_T, l_T, \mu_T \in \pi_1(\partial(N_T), t) = \mathbb{Z}^3$.

The manifold obtained from X by performing a (p, q, n)-torus surgery on T along γ is defined as

$$X_{T,\gamma}(p, q, n) := (X - N_T) \cup_\phi (T^2 \times D^2),$$

where the gluing map $\phi : T^2 \times \partial D^2 \to \partial(X - N_T)$ satisfies

$$\phi_*([\partial D^2]) = p[m_T] + q[l_T] + n[\mu_T] \text{ in } H_1(\partial(X - N_T)); \mathbb{Z}).$$

For $n = 1$, the procedure described above is known as *Luttinger surgery on T along* $\gamma \subset T$, and $X_{T,\gamma}(p, q, 1)$ admits a symplectic structure [8, 35].

If the base point of X is not on the boundary of the tubular neighbourhood N_T of T, then the based loops μ and γ are to be joined by the same path in $X - T$. The core torus of the surgery $S^1 \times S^1 \times \{0\} \subset X_{T,\gamma}(p, q, 1)$ will be denoted by $T_{(p,q,1)}$. Regarding the fundamental group of the manifolds $X_{T,\gamma}(p, q, n)$ constructed in this paper, the following considerations suffice. We will fix generating curves α, β on the torus T, and express the curve as $\gamma = \alpha^a \beta^b$ in $\pi_1(T)$ for $a, b \in \{0, 1\}$ such that $a \neq b$.

LEMMA 10. (cf. [11, Lemma 4]). *The manifold obtained by applying a (p, q, n)-torus surgery to X on the torus T along the curve $\gamma = a\alpha + b\beta$ has fundamental group given by*

$$\pi_1(X_{T,\gamma}(p, q, n)) \cong \pi_1(X - T)/N\left(\mu_T^n m_T^{ap} l_T^{bq}\right),$$

where $N(\mu_T^n m_T^p l_T^q)$ denotes the normal subgroup generated by $\mu_T^n m_T^p l_T^q$.

Every torus surgery that is employed in this paper will require for only one of the two integers p or q to be non-zero. From now on, the data needed to specify a torus surgery will be encoded in the following terminology. A $\pm p/n$-torus surgery on the torus T along the curve m_T implies that we are taking $q = 0$; analogously, a $\pm q/n$-torus surgery on the torus T along the curve l_T implies that we are setting $p = 0$.

3.6. Two examples. Before we start filling vast regions, the results and building blocks from the previous sections are used to build hands-on examples.

EXAMPLE 11. With the goal of constructing irreducible 4-manifolds with $c_1^2 = 3$, and $\chi_h = 1$, consider $T^2 \times T^2$ and $T^2 \times S^2$ equipped with the product symplectic forms. We begin by finding symplectic surfaces of genus 2 inside each of these manifolds. For $t_1, t_2 \in T^2$, take $T^2 \times \{t_2\} \cup \{t_1\} \times T^2 \subset T^2 \times T^2$. By symplectically resolving the double point, we obtain a symplectic surface of genus 2 and self-intersection $([T^2 \times \{t_2\}] + [\{t_1\} \times T^2])^2 = 2$. Blow up at the two intersection points, and obtain a symplectic surface of genus 2 and self-intersection zero $\Sigma_2 \subset T^2 \times T^2 \# 2\overline{\mathbb{CP}^2}$ [25, Building block 5.7].

Analogously, we obtain a symplectic surface of genus 2 that has self-intersection zero $\Sigma_2' \subset T^2 \times S^2 \# 3\overline{\mathbb{CP}^2}$ as follows (cf. [15], [6, Section 3]): Consider the embedded torus T' that represents the homology class $2[T^2 \times \{s_1\}]$, where $s_1 \in S^2$. Take the union of this torus with $T^2 \times \{s_1\} \cup \{t_1\} \times S^2$ for a given $t_1 \in T^2$. Blow up one of the double points, and symplectically resolve [25] the remaining pair of double points to get a genus 2 surface of self-intersection 2. Blow up at these intersection points to obtain the desired $\Sigma_2' \subset T^2 \times S^2 \# 3\overline{\mathbb{CP}^2}$.

Since both Σ_2 and Σ_2' have trivial self-intersection, there exist diffeomorphisms $\nu(\Sigma_2) \to \Sigma_2 \times D^2$ and $\nu(\Sigma_2') \to \Sigma_2' \times D^2$, where $\nu(\Sigma_2), \nu(\Sigma_2')$ are tubular neighbourhoods of the surfaces inside $T^4 \# 2\overline{\mathbb{CP}^2}$ and $T^2 \times S^2 \# 3\overline{\mathbb{CP}^2}$ respectively. Build the generalized fibre sum along the genus 2 surfaces

$$Z := (T^2 \times T^2 \# 2\overline{\mathbb{CP}^2}) \#_{\Sigma_2 = \Sigma_2'} (T^2 \times S^2 \# 3\overline{\mathbb{CP}^2}).$$

A well-known result of Gompf [25] (cf. [37]) implies that Z is a symplectic manifold, and its characteristic numbers are given by $c_1^2(Z) = 3$ and $\chi_h(Z) = 1$. A result of Usher [71, Theorem 1.1] implies that Z is minimal. In order to conclude its irreducibility

using the criteria of Hamilton–Kotschick [**31**, Corollary 1], we proceed to compute its fundamental group using the Seifert-van Kampen theorem.

The standard presentations of the groups involved in the cut-and-paste construction of Z are

$$\pi_1(T^4) = \langle x, y, a, b | [x, y] = [a, b] = [x, a] = [x, b] = [y, a] = [y, b] = 1\rangle,$$

$$\pi_1(T^2 \times S^2) = \langle \alpha, \beta | [\alpha, \beta] = 1\rangle,$$

$$\pi_1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 | [a_1, b_1][a_2, b_2] = 1\rangle \text{ and}$$

$$\pi_1(\Sigma_2') = \langle a_1', b_1', a_2', b_2' | [a_1', b_1'][a_2', b_2'] = 1\rangle.$$

Let the inclusion $\Sigma_2 \hookrightarrow T^2 \times T^2 \# 2\overline{\mathbb{CP}^2}$ induce the homomorphism that maps the generators as follows:

$$a_1 \mapsto x, b_1 \mapsto y,$$
$$a_2 \mapsto a, b_2 \mapsto b.$$

Similarly, let the homomorphism induced by the inclusion $\Sigma_2' \hookrightarrow T^2 \times S^2 \# 3\overline{\mathbb{CP}^2}$ map the generators of the fundamental groups by

$$a_1' \mapsto \alpha, b_1' \mapsto \beta^2,$$
$$a_2' \mapsto \alpha^{-1}, b_2' \mapsto \beta^{-2}$$

(cf. [**6**, Section 3]). Take the diffeomorphism $\phi : \Sigma_2 \times \partial D^2 \to \Sigma_2' \times \partial D^2$ that induces the homomorphism on fundamental groups that identifies the generators as follows:

$$a_1 \mapsto a_1', b_1 \mapsto b_1',$$
$$a_2 \mapsto a_2', b_2 \mapsto b_2'.$$

Furthermore, in each case the meridians of the surface are homotopically trivial in the complement. Indeed, an exceptional sphere introduced by a blowup intersects the surface transversally in one point. Thus, the meridians $\mu_{\Sigma_2}, \mu_{\Sigma_2'}$ are nullhomotopic in $T^4 \# 2\overline{\mathbb{CP}^2} - \nu(\Sigma_2)$ and $T^2 \times S^2 \# 3\overline{\mathbb{CP}^2} - \nu(\Sigma_2')$ respectively.

Using the Seifert-van Kampen theorem, we conclude that the group $\pi_1(Z)$ is generated by the elements $x, y, a, b, \alpha, \beta$, and that they satisfy the relations

$$[x, y] = [a, b] = [x, a] = [x, b] = [y, a] = [y, b] = [\alpha, \beta] = 1,$$
$$x = \alpha, y = \beta^2, a = \alpha^{-1}, b = \beta^{-2}.$$

Thus, $\pi_1(Z) = \langle \alpha, \beta | [\alpha, \beta^2] = 1\rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

Since $\mathbb{Z} \oplus \mathbb{Z}_2$ is a residually finite group, the symplectic 4-manifold Z is irreducible [**31**, Corollary 1]. In order to produce irreducible and minimal manifolds with $\pi \in \{\mathbb{Z}_2, \mathbb{Z}_{p_1} \oplus \mathbb{Z}_2\}$, we proceed as follows. The work of Taubes indicates that the Seiberg–Witten invariant of the symplectic manifold Z is nontrivial [**65**]. Moreover, it contains two pairs of Lagrangian tori $\{T_1, T_1^d\}, \{T_2, T_2^d\}$ such that T_i intersects T_i^d ($i = 1, 2$) transversally at one point, and any other intersection is empty. These tori are inherited from the $T^2 \times T^2$ building block in the construction of Z. By Proposition 7, the Lagrangian push-off and meridian of torus T_1 are

$$T_1 : m_1 = \alpha, \mu_{T_1} = 1.$$

Perform a (p_1/n)-torus surgery on T_1 along m_1 to produce an infinite set of pairwise non-diffeomorphic manifolds

$$\{Z_{p_1,2,n} : p_1, n \in \mathbb{N}\}.$$

By the Seifert-van Kampen theorem (cf. Lemma 10),

$$\pi_1(Z_{p_1,2,n}) \cong \langle \alpha, \beta | \alpha^{p_1} = \beta^2 = 1^n = [\alpha, \beta^2] \rangle \cong \mathbb{Z}_{p_1} \oplus \mathbb{Z}_2.$$

The characteristic numbers are invariant under torus surgeries. This yields $c_1^2(Z_{p_1,2,n}) = c_1^2(Z) = 3$ and $\chi_h(Z_{p_1,2,n}) = \chi_h(Z) = 1$ for all $p_1, n \in \mathbb{N}$. As it was mentioned in Section 3.5, the $(p_1/1)$-torus surgery is known as the Luttinger surgery, and the manifolds $\{Z_{p_1,2,1}\}$ are symplectic [8, 35]. As in [3, 16, 20, 21] the diffeomorphism classes of the manifolds $\{Z_{p_1,2,n}\}$ are distinguished by the Seiberg–Witten invariants [20, Theorem 5.3], which are calculated according to the surgery coefficient n of the torus surgery by the Morgan–Mrowka–Szabó formula [39], using the formulation given in [64, Theorem 3.4] (see proof of Lemma 13 for more details). We have produced then an infinite set $\{Z_{p_1,2,n} : p_1, n \in \mathbb{N}\}$ of pairwise non-diffeomorphic, non-symplectic minimal 4-manifolds.

To finish the example, we claim that the manifold

$$\mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2} \# L_2$$

has the ∞-property 1 for $p_1 = 1$.

Let $\{Z_{1,2,n} : n \in \mathbb{N}\}$ be the infinite set of pairwise non-diffeomorphic minimal manifolds with fundamental group of order two that was previously constructed. We proceed to pin down the homeomorphism type of these manifolds. Minding the fundamental group, we have computed $\pi_1(Z_{1,2,n}) \cong \mathbb{Z}_2$. The characteristic numbers are $e(Z_{1,2,n}) = e(Z) = 9$, $\sigma(Z_{1,2,n}) = \sigma(Z) = -5$, since both the Euler characteristic and signature are invariant under torus surgeries. Let $\widetilde{Z_{1,2,n}}$ be the universal cover of $Z_{1,2,n}$. Its signature is $\sigma(\widetilde{Z_{p_1,1,n}}) = -10$; by Rokhlin's Theorem [54], $\widetilde{Z_{p_1,1,n}}$ is non-spin. This implies that, for arbitrary $n \in \mathbb{N}$, the manifold $Z_{1,2,n}$ has ω_2-type (I). By Theorem 3, $Z_{1,2,n}$ is homeomorphic to $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2} \# L_2$. The claim follows from the existence of the infinite set $\{Z_{1,2,n} : n \in \mathbb{N}\}$ of pairwise non-diffeomorphic manifolds that was constructed before.

EXAMPLE 12. In this example, irreducible 4-manifolds with $c_1^2 = 1$ and $\chi_h = 1$ are constructed by modifying our choice of gluing map in the construction of symplectic sum. We begin by unveiling a surface of genus 2 and self-intersection zero symplectically embedded in $T^2 \times S^2 \# 4\overline{\mathbb{CP}^2}$ that intersects each exceptional sphere at a point. Consider the product of a 2-torus and 2-sphere $T^2 \times S^2$ equipped with the product symplectic form. Take the union $T^2 \times \{s_1\} \cup \{t_1\} \times S^2 \cup T^2 \times \{s_2\}$ for $t_1 \in T^2$ and $s_1, s_2 \in S^2$. This wedge of symplectic surfaces has two double points, which come from the intersection of $\{t_1\} \times S^2$ with two tori. Resolving symplectically [25] results in a surface of genus 2 and self-intersection 4. Blow up four times, and obtain a surface of genus 2 and self-intersection zero Σ_2 that is symplectically embedded in $T^2 \times S^2 \# 4\overline{\mathbb{CP}^2}$.

We build the generalized fibre sum

$$Z_\phi := T^2 \times S^2 \# 3\overline{\mathbb{CP}^2} \#_{\Sigma_2' = \Sigma_2} T^2 \times S^2 \# 4\overline{\mathbb{CP}^2},$$

where $\Sigma'_2 \subset T^2 \times S^2 \# 3\overline{\mathbb{CP}^2}$ are taken as in Example 11, and the glueing map is the diffeomorphism $\phi : \partial\nu(\Sigma'_2) \to \partial\nu(\Sigma_2)$ of the tubular neighbourhoods of the genus 2 surfaces. The same arguments used in Example 11 imply that Z_ϕ is a minimal symplectic 4-manifold with characteristic numbers $c_1^2(Z_\phi) = 1$ and $\chi_h(Z_\phi) = 1$.

Minding the fundamental group, we have the following computations. Let the groups involved have the presentations

$$\pi_1(T^2 \times S^2 \# 3\overline{\mathbb{CP}^2}) = \langle \alpha, \beta | [\alpha, \beta] = 1 \rangle,$$

$$\pi_1(T^2 \times S^2 \# 4\overline{\mathbb{CP}^2}) = \langle x, y | [x, y] = 1 \rangle,$$

$$\pi_1(\Sigma'_2) = \langle a'_1, b'_1, a'_2, b'_2 | [a'_1, b'_1][a'_2, b'_2] = 1 \rangle,$$

$$\pi_1(\Sigma_2) = \langle a_1, b_1, a_2, b_2 | [a_1, b_1][a_2, b_2] = 1 \rangle, \text{ and}$$

let the homomorphism induced by the inclusion, $\Sigma'_2 \hookrightarrow T^2 \times S^2 \# 3\overline{\mathbb{CP}^2}$, map the generators of the fundamental groups as

$$a'_1 \mapsto \alpha, \ b'_1 \mapsto \beta^2,$$
$$a'_2 \mapsto \alpha^{-1}, \ b'_2 \mapsto \beta^{-2}.$$

Similarly, let the homomorphism induced by the inclusion, $\Sigma_2 \hookrightarrow T^2 \times S^2 \# 4\overline{\mathbb{CP}^2}$, map the generators of the fundamental groups as

$$a_1 \mapsto x, \ b_1 \mapsto y,$$
$$a_2 \mapsto x^{-1}, \ b_2 \mapsto y^{-1}.$$

Different choices of gluing maps in the construction of Z_ϕ yield different symplectic manifolds with abelian fundamental groups of rank of at most two. For instance, choose as gluing map the diffeomorphism $\phi : \Sigma_2 \times \partial D^2 \to \Sigma'_2 \times \partial D^2$ that induces the isomorphism on fundamental groups that identifies the generators as follows:

$$a'_1 \mapsto a_1, \ b'_1 \mapsto b_1,$$
$$a'_2 \mapsto a_2, \ b'_2 \mapsto b_2.$$

We conclude that $\pi_1(Z_\phi) = \langle x, y | y^2 = 1, [x, y] = 1 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2$ using the Seifert-van Kampen theorem.

We proceed to construct an irreducible 4-manifold with fundamental group of order two. Inside the surface Σ_2, let A and B be parallel curves to α and β respectively. Let $\phi_1 : \Sigma_2 \times \partial D^2 \to \Sigma'_2 \times \partial D^2$ be the diffeomorphism obtained from composing three Dehn twists $D_A D_B D_A$ with ϕ (cf. [11, Section 4.2]). The composition induces the isomorphism on fundamental groups that assigns the generators as follows:

$$a'_1 \mapsto b_1^{-1}, \ b'_1 \mapsto b_1 a_1 b_1^{-1},$$
$$a'_2 \mapsto a_2, \ b'_2 \mapsto b_2.$$

By the Seifert-van Kampen theorem, we have that $\pi_1(Z_{\phi_1})$ is generated by x, y, α, β and the following relations hold

$$[\alpha, \beta] = [x, y] = 1, \ y^2 = 1, \ \alpha = y^{-1}, \ \beta = yxy^{-1}, \ \alpha^{-1} = x^{-1}, \ \beta^{-1} = y^{-1}.$$

This implies $\alpha = \beta^{-1}$, $x = y^{-1}$ and $x^2 = 1$. Thus, Z_{ϕ_1} is an irreducible symplectic 4-manifold with cyclic fundamental group of order two.

3.7. Technical tools. We now present two useful lemmas that allow us to extend results on simply connected manifolds to manifolds with abelian fundamental groups.

LEMMA 13. *Let $G \in \{\{1\}, \mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z} \oplus \mathbb{Z}_q, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}\}$. Let X be an irreducible symplectic simply connected 4-manifold, which contains a homologically essential Lagrangian torus T of self-intersection zero such that $\pi_1(X - T) = \{1\}$. Then, there exists an irreducible symplectic 4-manifold X^G with*

- *$\pi_1(X^G) = G$,*
- *$c_1^2(X^G) = c_1^2(X)$, and*
- *$\chi_h(X^G) = \chi_h(X)$.*

If $G \in \{\{1\}, \mathbb{Z}_p, \mathbb{Z}_q \oplus \mathbb{Z}_q, \mathbb{Z}\}$, where q is an odd prime number, then X^G has the ∞-property. Moreover, suppose X contains a second homologically essential Lagrangian torus T' of self-intersection zero, disjoint from T and $\pi_1(X - T \cup T') = \{1\}$. Then there exists an infinite family $\{X_n^G : n \in \mathbb{N}\}$ of pairwise non-diffeomorphic 4-manifolds with ω_2-type (I) that consists of infinitely many irreducible symplectic and infinitely many minimal non-symplectic members such that

- *$\pi_1(X_n^G) = G$,*
- *$c_1^2(X_n^G) = c_1^2(X)$, and*
- *$\chi_h(X_n^G) = \chi_h(X)$*

for all $n \in \mathbb{N}$. If $G \in \{\{1\}, \mathbb{Z}_p, \mathbb{Z}_q \oplus \mathbb{Z}_q, \mathbb{Z}\}$, then X^G has the ∞^2-property.

Proof. Consider the 4-torus $T^2 \times T^2$ equipped with the product symplectic form. Perturb the symplectic form on X so that T becomes symplectic [**25**, Lemma 1.6]. Build the generalized fibre sum

$$Z := X \#_{T=T'} T^2 \times T^2,$$

where $T' := \{x\} \times T^2 \subset T^4$. Using the notation in Proposition 8 by the Seifert-van Kampen theorem we have $\pi_1(Z) \cong \mathbb{Z}x \oplus \mathbb{Z}y$; the elements x and y each generate an infinite cyclic factor of $\mathbb{Z}^2$. As done in Example 11, an iterated usage of [**25**, **31**, **71**] allows us to conclude that Z is an irreducible symplectic 4-manifold with characteristic numbers $c_1^2(Z) = c_1^2(X)$ and $\chi_h(Z) = \chi_h(X)$.

The two Lagrangian tori T_1 and T_2 are contained inside the manifold Z and are available for surgery. Using the notation of Proposition 8, apply a $(-p/1)$- and a $(-q/n)$-torus surgery on T_1 and T_2 along the curves x and y, respectively, to produce a set of 4-manifolds

$$\{Z_{p,q,n} : p, q, n \in \mathbb{N}\}.$$

Lemma 10 yields $\pi_1(Z_{p,q,n}) \cong \langle x, y | x^p = 1, y^q = 1^n, [x, y] = 1 \rangle$.

The characteristic numbers are invariant under torus surgeries. Therefore, we have $c_1^2(Z_{p,q,n}) = c_1^2(Z) = c_1^2(X)$ and $\chi_h(Z_{p,q,n}) = \chi_h(Z) = \chi_h(X)$ for all $p, q, n \in \mathbb{N}$. The manifolds $\{Z_{p,q,1} : p, q \in \mathbb{N}\}$ are symplectic since the case $n = 1$ corresponds to Luttinger surgeries [**8**, **35**].

The infinite set of pairwise non-diffeomorphic minimal non-symplectic 4-manifolds $\{Z_{p,q,n} : p, q, n \in \mathbb{N}\}$ can be constructed using the techniques of [**3**, **16**, **20**, **21**]. The set is reverse-engineered from the symplectic 4-manifold Z in the sense of [**16**, Section 2] and [**21**, Section 2], given that the work of Taubes [**65**] indicates that the Seiberg–Witten invariant of Z is nontrivial. To verify that the needed hypothesis is satisfied, we use an argument in [**13**, Remark, p. 343]. For clarity, we discuss the case $G = \mathbb{Z}$; the other choices of fundamental group follow from a similar argument. The group $\pi_1(\partial \nu(T_2)) \cong \pi_1(T^3)$ is generated by the loops m_2, l_2 and μ_2. In the

group $\pi_1(Z - \nu(T_2))$, the identities $m_2 = y$, $l_2 = 1$ and $\mu_2 = 1$ hold by Proposition 7. Construct a manifold with infinite cyclic fundamental group

$$Z_{1,0} = (Z - \nu(T_2)) \cup_{\phi_{Z_{1,0}}} T^2 \times D^2$$

using gluing map $\phi_{Z_{1,0}}$ that is defined as follows. Let $T^2 \times D^2 = S^1 \times S^1 \times D^2$, and define the curves $\alpha := S^1 \times \{1\} \times \{1\}$, $\beta := \{1\} \times S^1 \times \{1\}$ and $\mu_{Z_{1,0}} := \{(1, 1)\} \times \partial D^2$ that are contained in $S^1 \times S^1 \times D^2$. Let $\phi_{Z_{1,0}}$ be the gluing map defined by the identification of curves $\alpha \mapsto l_2$, $\beta \mapsto \mu_2$ and $\mu_{Z_{1,0}} \mapsto m_2^{-1}$.

The core torus $\widetilde{T} := T^2 \times \{0\} \subset T^2 \times D^2$ of the surgery is nullhomologous. Note that we have $Z - \nu(T_2) = Z_{1,0} - \nu(\widetilde{T})$. Indeed, the meridian $\mu_{\widetilde{T}}$ is nontrivial in $Z_{1,0} - \nu(\widetilde{T})$ due to our identification. The torus surgery that was applied kills a generator of the first homology group as well as two generators of the second homology group. These generators correspond to the class of the torus to which we apply the surgery, and the class of its dual. Moreover, the curve β is nullhomotopic in $Z - \nu(T_2) = Z_{1,0} - \nu(\widetilde{T})$, since it was identified with μ_2. Thus, the curve β is nullhomologous in the complement. The manifold Z can be recovered from $Z_{1,0}$ by applying a 1/0-torus surgery on $\widetilde{T}$ along the curve β. Applying a $n/1$-torus surgery on $\widetilde{T}$ along β for $n \in \mathbb{N}$ produces a 4-manifold $Z_{1,0,n}$ with infinite cyclic fundamental group. Another description of this manifold is as the product of applying surgery to Z on T_2 along m_2 with the identification $\mu_2^n m_2^{-1} \mapsto y^{-1}$, i.e. a $1/n$-torus surgery.

The hypotheses of reverse-engineering in [16, 20, 21] are satisfied. The diffeomorphism classes are distinguished by the Seiberg–Witten invariants [20, Theorem 5.3], which are calculated with the Morgan–Mrowka–Szabó formula [39] using the formulation given in [64, Theorem 3.4]. We have then produced an infinite set $\{Z_{1,0,n} : n \in \mathbb{N}\}$ of pairwise non-diffeomorphic minimal non-symplectic 4-manifolds ([20, Theorem 5.3], [21, Theorem 1]) with infinite cyclic fundamental group. Note that we are abusing notation regarding the infinite set. Since we do not know if the manifold Z has only one Seiberg–Witten basic class up to sign, we cannot conclude that the manifolds $Z_{1,0,n}$ are pairwise non-diffeomorphic (cf. [16, Corollary 1]). However, we do know that Z has a basic class [65], and [16, Corollary 2] implies that among the manifolds $\{Z_{1,0,n} : n \in \mathbb{N}\}$ there are infinitely many pairwise non-diffeomorphic. We have kept the same notation for them.

The presence of a homologically essential torus $T' \subset Z$ with the homomorphism $\pi_1(T') \to \pi_1(Z_{p,q,1}) \cong G$ induced by the inclusion allows us to perform Fintushel and Stern's Knot surgery ([18], [14, Theorem 4], [19, Lecture 3]; see [46, Corollary 24] as well), and produce infinitely many pairwise non-diffeomorphic 4-manifolds $\{X_n^G : n \in \mathbb{N}\}$. Given that T' is homologically essential, by [25, Lemma 1.6] one can perturb the symplectic form so that T' becomes symplectic. A result by Thurston [68] implies that for fibred knots, the manifolds constructed in this way are symplectic [26, Remark 10.3.5]. The presence of a symplectic structure allows us to conclude on irreducibility using the results in [31].

In order to conclude on the ∞- and ∞^2-properties, we examine the homeomorphism types of the constructed manifolds. The hypothesis on the symplectic torus of self-intersection zero contained in the minimal symplectic manifold X implies that $b_2^+(X) \geq 3$. Thus, the stability conditions of Theorems 5 and 6 are satisfied. Let $b^+ := b_2^+(X)$ and $b^- := b_2^-(X)$. By Theorem 4, the manifolds in the set $\{Z_{p,1,n} : n \in \mathbb{N}\}$ are homeomorphic to $b^+ \mathbb{CP}^2 \# b^- \overline{\mathbb{CP}^2} \# L_p$. Assuming $p = q$ to be an odd prime number, Theorem 5 implies that the manifolds in the set $\{Z_{q,q,n} : n \in \mathbb{N}\}$ are homeomorphic to

$b^+\mathbb{CP}^2\#b^-\overline{\mathbb{CP}^2}\#L_{q,q}$. Theorem 6 implies that every manifold with infinite fundamental group constructed above is homeomorphic to $(b^+ + 1)\mathbb{CP}^2\#(b^- + 1)\overline{\mathbb{CP}^2}\#S^3 \times S^1$. $\square$

The previous result can be extended when the manifold X in the hypothesis of Lemma 13 contains a submanifold of higher genus. The genus 2 case is exemplified in the following lemma.

LEMMA 14. *Let $G \in \{\{1\}, \mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z} \oplus \mathbb{Z}_q, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}\}$. Let X be an irreducible symplectic simply connected 4-manifold, which contains a symplectic surface Σ of genus 2 and self-intersection zero such that $\pi_1(X - \Sigma) = \{1\}$. Then there exists an infinite family $\{X_n^G : n \in \mathbb{N}\}$ of pairwise non-diffeomorphic 4-manifolds with ω_2-type (I) that consists of infinitely many irreducible symplectic and infinitely many minimal non-symplectic members, such that*

- $\pi_1(X_n^G) = G$,
- $c_1^2(X_n^G) = c_1^2(X) + 8$, *and*
- $\chi_h(X_n^G) = \chi_h(X) + 1$

for all $n \in \mathbb{N}$. If $G \in \{\{1\}, \mathbb{Z}_p, \mathbb{Z}_q \oplus \mathbb{Z}_q, \mathbb{Z}\}$, where q is an odd prime number, then X^G has the ∞^2-property.

Proof. Consider the product $T^2 \times \Sigma_2$ equipped with the product symplectic form. Build the generalized fibre sum

$$Z := X\#_{\Sigma=\Sigma_2}T^2 \times \Sigma_2$$

along the corresponding genus 2 surfaces. By Proposition 8, the loops a_1, b_1, a_2, b_2 lie on the genus 2 surface. The Seifert-van Kampen theorem implies that the fundamental group is $\pi_1(Z) \cong \mathbb{Z}x \oplus \mathbb{Z}y$, since we assumed that $\pi_1(X - \Sigma) = \{1\}$. Arguing as before, we conclude that Z is an irreducible symplectic manifold with characteristic numbers $c_1^2(Z) = c_1^2(X) + 8$ and $\chi_h(Z) = \chi_h(X) + 1$.

The symplectic manifold Z contains four homologically Lagrangian tori available for surgery, which are contained in the $T^2 \times \Sigma_2$ block of the symplectic sum. Perform $-1/1$ Luttinger surgeries on T_1 and T_2 along x and y respectively. This results in an irreducible symplectic simply connected 4-manifold Y that contains two pairs of homologically essential Lagrangian tori T_3 and T_4 such that $Y - (T_3 \cup T_4)$ is simply connected by Proposition 8. The lemma now follows from Lemma 13. $\square$

3.8. Telescoping triples. The basic building blocks used to systematically populate geographical regions of irreducible symplectic 4-manifolds are presented in this section.

DEFINITION 15. ([**2**, Definition 2]). An ordered triple (X, T_1, T_2) consisting of a symplectic 4-manifold X and two disjointly embedded Lagrangian tori T_1 and T_2 is called a telescoping triple if

1. the tori T_1 and T_2 span a two-dimensional subspace of $H_2(X; \mathbb{R})$.
2. $\pi_1(X) \cong \mathbb{Z}^2$ and the inclusion induces an isomorphism
   $$\pi_1(X - (T_1 \cup T_2)) \to \pi_1(X).$$
   In particular, the meridians of the tori are trivial in $\pi_1(X - (T_1 \cup T_2))$.
3. The image of the homomorphism induced by the corresponding inclusion $\pi_1(T_1) \to \pi_1(X)$ is a summand $\mathbb{Z} \subset \pi_1(X)$.
4. The homomorphism induced by inclusion $\pi_1(T_2) \to \pi_1(X)$ is an isomorphism.

The telescoping triple is called minimal if X itself is minimal.

The order of the tori in the definition is relevant. The meridians μ_{T_1} and μ_{T_2} in $\pi_1(X - (T_1 \cup T_2))$ are trivial, and the fundamental groups involved are abelian. The push-off of an oriented loop $\gamma \subset T_i$ into $X - (T_1 \cup T_2)$ with respect to any (Lagrangian) framing of the normal bundle of T_i represents a well-defined element of $\pi_1(X - (T_1 \cup T_2))$, which is independent of the choices of framing and base point.

The first condition says that the Lagrangian tori T_1 and T_2 are linearly independent in $H_2(X; \mathbb{R})$. The symplectic form on X can be slightly perturbed so that one of the T_i remains Lagrangian while the other becomes symplectic [25, Lemma 1.6].

Out of two telescoping triples, one is able to produce another as follows.

PROPOSITION 16. [2, Proposition 3]. *Let (X, T_1, T_2) and (X', T_1', T_2') be two telescoping triples. Then for an appropriate gluing map, the triple*

$$(X \#_{T_2 = T_1'} X', T_1, T_2')$$

is again a telescoping triple. If X and X' are minimal symplectic 4-manifolds, then the resulting telescoping triple is minimal. The characteristic number of $X \#_{T_2 = T_1'} X'$ is

$$c_1^2(X) + c_1^2(X') \text{ and } \chi_h(X) + \chi_h(X').$$

Minimal telescoping triples are irreducible by [31, Corollary 1]. The existing telescoping triples are gathered in the following result, which was proven in [2, Section 5] and [69].

THEOREM 17. *Existence of telescoping triples.*
- *There exists a minimal telescoping triple (A, T_1, T_2) satisfying $c_1^2(A) = 7$, $\chi_h(A) = 1$.*
- *For each $g \geq 0$, there exists a minimal telescoping triple (B_g, T_1, T_2) satisfying $c_1^2(B_g) = 6 + 8g$, $\chi(B_g) = 1 + g$.*
- *There exists a minimal telescoping triple (C, T_1, T_2) satisfying $c_1^2(C) = 5$, $\chi_h(C) = 1$.*
- *There exists a minimal telescoping triple (D, T_1, T_2) satisfying $c_1^2(D) = 4$, $\chi_h(D) = 1$.*
- *There exists a minimal telescoping triple (F, T_1, T_2) satisfying $c_1^2(F) = 2$, $\chi_h(F) = 1$.*

REMARK 18. The telescoping triples are obtained in the construction process of minimal symplectic 4-manifolds homeomorphic to $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ [1, 6, 4, 11, 13, 16]. In [21], Fintushel and Stern introduced a procedure to unveil exotic smooth structures on these manifolds by performing surgeries on nullhomologous tori that are contained inside rational surfaces (cf. [20]). Minimal 4-manifolds with abelian fundamental group of small rank lying on the line $\chi_h = 1$ can be constructed using the techniques of [21]. This requires only a small variation on the choices of surgeries on the pinwheel structures on [21, Sections 7, 8].

An immediate consequence of Theorem 17 is the study of the geography and botany of irreducible symplectic 4-manifolds for various fundamental groups (cf. [2, 4, 6, 10, 12, 13, 25, 27, 28, 29, 34, 36, 40, 41, 55, 69, 72, 74]). In particular, the following result extends [69, Theorem 1].

COROLLARY 19. *Let $G \in \{\mathbb{Z}_p, \mathbb{Z}_q \oplus \mathbb{Z}_p, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_q, \mathbb{Z} \oplus \mathbb{Z}\}$, and let $n \geq 1$ and $m \geq 1$. For each of the following pairs of integers,*
(1) $(c, \chi) = (7n, n)$,
(2) $(c, \chi) = (5n, n)$,
(3) $(c, \chi) = (4n, n)$,
(4) $(c, \chi) = (2n, n)$,

(5) $(c, \chi) = ((6 + 8g)n, (1 + g)n)$ *(for $g \geq 0$)*,
(6) $(c, \chi) = (7n + (6 + 8g)m, n + (1 + g)m)$,
(7) $(c, \chi) = (7n + 5m, n + m)$,
(8) $(c, \chi) = (7n + 4m, n + m)$,
(9) $(c, \chi) = (7n + 2m, n + m)$,
(10) $(c, \chi) = ((6 + 8g)n + 5m, (1 + g)n + m)$ *(for $g \geq 0$)*,
(11) $(c, \chi) = ((6 + 8g)n + 4m, (1 + g)n + m)$ *(for $g \geq 0$)*,
(12) $(c, \chi) = ((6 + 8g)n + 2m, (1 + g)n + m)$ *(for $g \geq 0$)*,
(13) $(c, \chi) = (5n + 4m, n + m)$,
(14) $(c, \chi) = (5n + 2m, n + m)$, *and*
(15) $(c, \chi) = (4n + 2m, n + m)$,

there exists an irreducible symplectic 4-manifold X^G with ω_2-type (I) and

$$\pi_1(X^G) = G \text{ and } (c_1^2(X^G), \chi_h(X^G)) = (c, \chi).$$

Moreover, if $G = \mathbb{Z}_p$, then X^G has the ∞-property. Assume q to be an odd prime number, and let $\chi > 1$. If $G \in \{\mathbb{Z}_q \oplus \mathbb{Z}_q, \mathbb{Z}\}$, then X^G has the ∞-property.

REMARK 20. *Symplectic universal covers*. It is natural to ask whether the universal covers of the symplectic manifolds constructed are standard or exotic (we thank Paul Kirk for bringing it to our attention). If they were standard, then the action of the fundamental group would be exotic. Exotic smooth actions were studied in [22]. The universal covers in our constructions admit a symplectic structure; thus, they are exotic. This phenomena has previously been observed in [17]. Indeed, let $\pi : \widetilde{X} \to X$ be the cover of a symplectic manifold X of ω_2-type (I) with finite cyclic fundamental group, and denote its symplectic structure by ω. By Freedman's theorem [23], $\widetilde{X}$ is homeomorphic to

$$(p(b_2^+ + 1) - 1)\mathbb{CP}^2 \# (p(b_2^- + 1) - 1)\overline{\mathbb{CP}^2}.$$

The cover $\widetilde{X}$ admits a symplectic structure, since the pullback $\pi^*\omega$ is a symplectic form on $\widetilde{X}$. By Taubes' [65] result, the Seiberg–Witten invariant of $\widetilde{X}$ is nontrivial. Since $(p(b_2^+ + 1) - 1)\mathbb{CP}^2 \# (p(b_2^- + 1) - 1)\overline{\mathbb{CP}^2}$ has vanishing Seiberg–Witten invariants [26, Theorem 2.4.6], $\widetilde{X}$ is not diffeomorphic to it.

4. Filling in geographical regions.
We prove Theorem 2 in this section. We will first construct irreducible 4-manifolds with negative signature for every possible lattice point on the line $\chi_h = 2$, and use these manifolds to construct more manifolds on the lines $\chi_h > 2$ using Proposition 9. The following bound is a guide to which 4-manifolds are to be constructed.

LEMMA 21. *Let X be a minimal symplectic 4-manifold. Then*

$$b_2^-(X) \leq 10\chi_h(X) - 1.$$

Proof. By the work of Taubes [66, 67] (see also [32, Corollary 4.9]), a minimal symplectic 4-manifold X must satisfy $c_1^2(X) \geq 0$ ([42, Corollary 13.1.12]). A straightforward computation shows that the restriction implies

$$b_2^-(X) \leq 5b_2^+(X) + 4 = 10(\chi_h(X)) - 1.$$

$\square$

Table 1. Minimal symplectic 4-manifolds of Proposition 22

b_2^-	(c_1^2, χ_h, σ)	Symplectic sum/manifold.	# of Luttinger surgeries	Reference
4	$(15, 2, -1)$	$X_{1,2} \#_{\Sigma_2} T^2 \times \Sigma_2$	Two	[**6**, Section 9]
5	$(14, 2, -2)$	$\overline{X_{3,5}}$	None	[**13**, Theorem 18]
6	$(13, 2, -3)$	$X_{1,2} \#_{\Sigma_2} (T^4 \# 2\overline{\mathbb{CP}^2})$	None	[**6**, Section 9]
7	$(12, 2, -4)$	$X_{3,7}$	None	[**13**, Corollary 15]
8	$(11, 2, -5)$	$X_{1,5} \#_{\Sigma_2} (T^4 \# \overline{\mathbb{CP}^2})$	None	[**6**, Sections 3 and 4]
9	$(10, 2, -6)$	$X_{1,5} \#_{\Sigma_2} (T^4 \# 2\overline{\mathbb{CP}^2})$	None	[**13**, Corollary 16]
10	$(9, 2, -7)$	$X_{1,6} \#_{\Sigma_2} (T^4 \# 2\overline{\mathbb{CP}^2})$	None	[**6**, Lemma 15]
11	$(8, 2, -8)$	$E(1) \#_{T^2} S$	Six	[**2**, Lemma 16]
12	$(7, 2, -9)$	$E(1) \#_{T^2} A$	None	Theorem 17
13	$(6, 2, -10)$	$E(1) \#_{T^2} B$	None	Theorem 17
14	$(5, 2, -11)$	$\mathbb{CP}^2 \# 12\overline{\mathbb{CP}^2} \#_{\Sigma_2} T^2 \times \Sigma_2$	Two	[**25**, Building Block 5.6]
15	$(4, 2, -12)$	$\mathbb{CP}^2 \# 13\overline{\mathbb{CP}^2} \#_{\Sigma_2} T^2 \times \Sigma_2$	Two	[**25**, Building Block 5.6]
16	$(3, 2, -13)$	$\mathbb{CP}^2 \# 12\overline{\mathbb{CP}^2} \#_{\Sigma_2} (T^4 \# 2\overline{\mathbb{CP}^2})$	None	[**25**, Building Blocks 5.6 and 5.6]
17	$(2, 2, -14)$	$\mathbb{CP}^2 \# 13\overline{\mathbb{CP}^2} \#_{\Sigma_2} (T^4 \# 2\overline{\mathbb{CP}^2})$	None	[**25**, Building Blocks 5.6 and 5.7]
18	$(1, 2, -15)$	$S_{1,1}$	None	[**25**, Example 5.4]
19	$(0, 2, -16)$	$E(1) \#_{T^2} T^4 \#_{T^2} E(1)$	None	[**64**]

There are other restrictions for 4-manifolds with $b_1 \leq 1$ [**59**, Lemma 2.6]; Lemma 21 suffices for our purposes.

4.1. Irreducible manifolds on the line $\chi_h = 2$. With the purpose of proving Theorem 2, we begin to systematically populate every lattice point on the line $\chi_h = n$ for every integer $n \geq 2$ that corresponds to manifolds with negative signature, with an irreducible 4-manifold. For the line $\chi_h = 2$, we employ the following result.

PROPOSITION 22. *Let $b^- \in \{4, 5, \ldots, 17, 18, 19\}$. There exists a minimal non-spin symplectic simply connected 4-manifold X_{3,b^-} with second Betti number given by $b_2^+(X) = 3$ and $b_2^-(X) = b^-$, which contains two homologically essential Lagrangian tori T_1 and T_2 such that*

$$\pi_1(X_{3,b^-} - (T_1 \cup T_2)) = \{1\}.$$

The manifolds of Proposition 22 unveil exotic smooth structures on $3\mathbb{CP}^2 \# b_2^- \overline{\mathbb{CP}^2}$. This enterprise has been done previously [**1**–**4**, **6**, **10**, **12**, **13**, **25**, **44**, **45**, **47**, **49**, **50**, **53**, **57**, **60**, **61**, **63**]. What concerns us for our purposes is the existence of tori T_1 and T_2 with the required traits (cf. [**13**, Theorem 18]) in order to be able to apply Lemma 13.

Proof. Table 1 provides a guideline of where the manifolds of Proposition 22 are built or obtained from as well as the number of Luttinger surgeries required in their construction. We will prove the proposition for one case, given that the argument for the other manifolds is analogous. We proceed to construct an irreducible symplectic simply connected 4-manifold $X_{3,15}$ with

$$(c_1^2(X_{3,15}), \chi_h(X_{3,15}), \sigma(X_{3,15})) = (4, 2, -12)$$

that contains the two tori as in the statement of the proposition. Take a quartic curve inside $\mathbb{CP}^2$ with a single transverse double point. By blowing up at this singular point, a surface of genus 2 and self-intersection 12 is obtained in $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. Proceed to blow up

at each of these different points. The proper transform F of the 12 blowups is a complex curve of genus 2 and self-intersection zero, which is contained in $\mathbb{CP}^2 \# 13 \overline{\mathbb{CP}^2}$ (cf. [**25**, Building block 5.6]). From the construction it is clear that the surface F intersects every exceptional sphere from each of the 13 blowups. Moreover, $\pi_1(\mathbb{CP}^2 \# 13 \overline{\mathbb{CP}^2} - F) = \{1\}$, since the exceptional sphere introduced during the 13th blowup provides a nullhomotopy for the meridian of F once it is removed from the manifold.

Build the generalized fibre sum

$$Z_{4,2} := (\mathbb{CP}^2 \# 13 \overline{\mathbb{CP}^2}) \#_{F = \Sigma_2} (T^2 \times \Sigma_2).$$

Again, the manifold $Z_{4,2}$ is symplectic by [**25**], it is minimal by [**71**, Theorem 1.1]. Its characteristic numbers are readily computed to be $c_1^2(Z_{4,2}) = c_1^2(\mathbb{CP}^2 \# 13 \overline{\mathbb{CP}^2}) + c_1^2(T^2 \times \Sigma_2) + 8 = 4$ and $\chi_h(Z_{4,2}) = \chi_h(\mathbb{CP}^2 \# 13 \overline{\mathbb{CP}^2}) + \chi_h(T^2 \times \Sigma_2) + 1 = 2$.

One concludes $\pi_1(Z_{4,2}) \cong \mathbb{Z}x \oplus \mathbb{Z}y$ from Proposition 8 and the Seifert-van Kampen theorem. Since the fundamental group is residually finite, $Z_{4,2}$ is irreducible by [**31**, Corollary 1]. There are eight pairs of geometrically dual Lagrangian tori contained in $Z_{4,2}$ that come from the $T^2 \times \Sigma_2$ block in the construction of the symplectic sum. We apply torus surgeries to $Z_{4,2}$ following Proposition 8 and the notation there. Perform $-1/1$-torus surgeries on T_3 and T_4 along $m_3 = x$ and $m_4 = y$, respectively, to obtain a simply connected irreducible symplectic 4-manifold $X_{3,15}$ [**8**, **35**], Proposition 8. Both tori T_1 and T_2 have each a geometrically dual torus inside $X_{3,15}$. Thus, T_1 and T_2 are homologically essential Lagrangian tori. One concludes from Proposition 8 that their meridians are trivial in the complement, and one has $\pi_1(X_{3,15} - (T_1 \cup T_2)) = \{1\}$. $\quad\square$

4.2. Region of negative signature: proof of Theorem 2.

We will fill out the region of the geography of irreducible symplectic 4-manifolds whose fundamental group is among the choices $\{\mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z} \oplus \mathbb{Z}_q, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}\}$ by constructing irreducible manifolds with the given choices of fundamental groups realizing all pairs of integers

$$(c_1^2, \chi_h) \text{ when } 0 \leq c_1^2 \leq 8\chi_h - 1.$$

Note that under the chosen coordinates, a 4-manifold with $c_1^2 = 8\chi + k$ has signature k. We proceed to prove the main result Theorem 2.

Proof. The homeomorphism types are pinned down by the work of Hambleton–Kreck and Hambleton–Teichner presented in Sections 3.1 and 3.2. A worked out argument on these matters was given in Example 11. The line $\chi_h = 1$ was populated, and its botany was studied in Corollary 19 and Example 11 (also see Remark 18). The line $\chi_h = 2$ is populated as a consequence of Proposition 22 and Lemma 13. Assume $n \geq 3$. The region

$$\chi_h = n,$$

$$8n - 16 \leq c_1^2 \leq 8n - 1$$

in statement of Theorem 2 is populated by an iteration on the construction of generalized fibre sums of the manifolds of Proposition 22 with n copies of the manifold Z of Proposition 9 along tori. Note that by [**25**, Lemma 1.6], the symplectic form on these manifolds can be perturbed so that the Lagrangian tori become symplectic. The

Seifert-van Kampen theorem implies that the fundamental group is trivial. The claim now follows from Lemma 13. The region

$$\chi_h = n,$$
$$0 \leq c_1^2 < 8n - 16$$

is populated by forming the generalized fibre sum of the manifolds in Proposition 22 and the telescoping triples of Theorem 17 along tori. Lemma 21 serves as guidance as of which telescoping triple is to be used according to its signature. The torus involved in the gluing of each telescoping triple is the torus T_2; the Seifert-van Kampen theorem implies that the fundamental group of the symplectic sum is trivial. The reader will note that there are several choices on the building blocks used for the construction of the generalized fibre sums; various points can be populated alternatively by using Lemma 14. The claim now follows from an iterated application of [25, 31, 71] and Lemma 13 as before.   □

4.3. Regions of non-negative signature.

The tools of Section 3.7 can also be used to extend results on simply connected 4-manifolds with non-negative signature [2, 5, 9, 51, 58] to other fundamental groups. We give a sample result in this section, first recalling the following theorem.

THEOREM 23. (Akhmedov–Park [5]). *For each of the following pairs* $(c, \chi) \in \mathbb{Z} \oplus \mathbb{Z}$
- $(c, \chi) = (200, 25)$,
- $(c, \chi) = (201, 25)$,
- $(c, \chi) = (204, 24)$,
- $(c, \chi) = (219, 27)$, *and*
- $(c, \chi) = (212, 26)$,

there exists a minimal symplectic simply connected 4-manifold X with characteristic numbers $c_1^2(X) = c$ *and* $\chi_h(X) = \chi$. *Moreover, X contains two disjoint homologically essential Lagrangian tori,* T_1, T_2, *each of self-intersection zero and such that* $\pi_1(X - T_1 \cup T_2) = \pi_1(X - T_i) = \{1\}$ *for* $i \in \{1, 2\}$.

The previous theorem and Lemma 13 yield the following result.

COROLLARY 24. *Let* $G \in \{\{1\}, \mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_q, \mathbb{Z} \oplus \mathbb{Z}\}$, *and assume m, r, s, t to be odd positive integers that satisfy* $m \geq 49$, $r \geq 47$, $s \geq 53$ *and* $t \geq 51$. *For each of the following pairs of integers,*
  (1) $(c, \chi) = (4m + 4, \frac{1}{2}(m + 1))$,
  (2) $(c, \chi) = (4m + 5, \frac{1}{2}(m + 1))$,
  (3) $(c, \chi) = (4r + 6, \frac{1}{2}(r + 1))$,
  (4) $(c, \chi) = (4s + 7, \frac{1}{2}(s + 1))$, *and*
  (5) $(c, \chi) = (4t + 8, \frac{1}{2}(t + 1))$,
  there exists a 4-manifold X^G *with* ω_2-*type (I) and*

$$\pi_1(X^G) = G \text{ and } (c_1^2(X^G), \chi_h(X^G)) = (c, \chi).$$

If $G \in \{\{1\}, \mathbb{Z}, \mathbb{Z}_p, \mathbb{Z}_q \oplus \mathbb{Z}_q\}$ *for q an odd prime number, then the homeomorphism type of* X^G *has the* ∞^2-*property.*

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