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Unit Elements in the Double Dual of a Subalgebra of the Fourier Algebra $A(G)$

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Abstract. Let $\mathcal{A}$ be a Banach algebra with a bounded right approximate identity and let $\mathcal{B}$ be a closed ideal of $\mathcal{A}$. We study the relationship between the right identities of the double duals $\mathcal{B}^{**}$ and $\mathcal{A}^{**}$ under the Arens product. We show that every right identity of $\mathcal{B}^{**}$ can be extended to a right identity of $\mathcal{A}^{**}$ in some sense. As a consequence, we answer a question of Lau and Ülger, showing that for the Fourier algebra $A(G)$ of a locally compact group G, an element $\phi \in A(G)^{**}$ is in $A(G)$ if and only if $A(G)\phi \subseteq A(G)$ and $E\phi = \phi$ for all right identities E of $A(G)^{**}$. We also prove some results about the topological centers of $\mathcal{B}^{**}$ and $\mathcal{A}^{**}$.

Introduction

Let $\mathcal{A}$ be a Banach algebra with a bounded right approximate identity and let its double dual $\mathcal{A}^{**}$ be equipped with the first Arens multiplication (see Arens [1]). If $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$, then $\mathcal{B}^{**}$ is embedded into $\mathcal{A}^{**}$ by inclusion. We study the relationship between the right identities of $\mathcal{B}^{**}$ and $\mathcal{A}^{**}$. The motivation is that sometimes we need to reduce a problem to the case of a subalgebra with a sequential bounded approximate identity (see Corollary 2.4 and Theorem 3.2). Let $\mathcal{B}$ be a closed ideal of $\mathcal{A}$ such that there is a projection which is also a multiplier from $\mathcal{A}$ to $\mathcal{B}$, *i.e.,* a bounded linear operator $m\colon \mathcal{A} \to \mathcal{B}$ satisfying $m(ab) = am(b) = m(a)b$ for $a, b \in \mathcal{A}$ and $m(c) = c$ for $c \in \mathcal{B}$. Then $\mathcal{B}^*$ is embedded into $\mathcal{A}^*$ by m^*. If we identify a right unit E of $\mathcal{B}^{**}$ with an element $i^{**}(E)$ in $\mathcal{A}^{**}$, where $i\colon \mathcal{B} \to \mathcal{A}$ is the inclusion map, then we prove that E can be extended to a right identity $\tilde{E}$ of $\mathcal{A}^{**}$ in the sense that $\tilde{E}$ is a right identity of $\mathcal{A}^{**}$ and $\tilde{E} = E$ on $m^*(\mathcal{B}^*)$. Then we apply this result to show that for the Fourier algebra $A(G)$ of a locally compact group G, an element $\phi \in A(G)^{**}$ is in $A(G)$ if and only if $A(G)\phi \subseteq A(G)$ and $E\phi = \phi$ for all right identities E of $A(G)^{**}$. This answers problem h) in Lau and Ülger [13]. In the second part of this paper, we study the relationship between the topological centers of $\mathcal{B}^{**}$ and $\mathcal{A}^{**}$. We prove that the topological center of $\mathcal{B}^{**}$ (or $\mathcal{A}^{**}$) can be embedded into the topological center of $\mathcal{A}^{**}$ (or $\mathcal{B}^{**}$). Then we apply these results to the case of Herz–Figà–Talamanca algebra $A_p(G)$ of a locally compact group to generalize results in Hu [9] and Hu and Neufang [10].

This paper is organized as follows. In Section 1, we recall some necessary notations and some preliminary results. In Section 2, we prove an extension theorem of a right identity in $\mathcal{B}^{**}$ to a right identity in $\mathcal{A}^{**}$ under certain conditions. Let G be a locally

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compact group, G_0 an open and closed subgroup and N a compact normal subgroup of G. We apply this result to extend right identities of $A_p(G_0)^{**}$ and $L_1(G/N)^{**}$ to $A_p(G)^{**}$ and $L_1(G)^{**}$, respectively. In Section 3, we use the extension theorem to answer an open problem and to improve results in [13]. In Section 4, we deal with the topological center problems for $\mathcal{B}^{**}$ and $\mathcal{A}^{**}$.

1 Preliminaries and Some Notations

Let $\mathcal{A}$ be a Banach algebra with a bounded right approximate identity. The duality between Banach spaces is denoted by $\langle\,\cdot\,,\,\cdot\,\rangle$. Recall the definition of the first Arens product on the double dual $\mathcal{A}^{**}$: for $a, b \in \mathcal{A}$ and $f \in \mathcal{A}^*$, we define $fa \in \mathcal{A}^*$ by $\langle fa, b\rangle = \langle f, ab\rangle$. Then, for $\phi, \psi \in \mathcal{A}^{**}$, $\psi f \in \mathcal{A}^*$ is defined by $\langle \psi f, a\rangle = \langle \psi, fa\rangle$ and finally, $\phi\psi \in \mathcal{A}^{**}$ is defined by $\langle \phi\psi, f\rangle = \langle \phi, \psi f\rangle$. Throughout this paper, we regard the first Arens product as the Arens product. It is easy to see that the map $\nu \to \nu\mu$ is weak* continuous on $\mathcal{A}^{**}$ for any $\mu \in \mathcal{A}^{**}$. But $\nu \to \mu\nu$ may not be weak* continuous. The set

$$\Lambda(\mathcal{A}^{**}) = \{\mu \in \mathcal{A}^{**} \colon \nu \to \mu\nu \text{ is continuous in the weak}^* \text{ topology}\}$$

is called the topological center of $\mathcal{A}^{**}$. It is obvious that $\mathcal{A} \subseteq \Lambda(\mathcal{A}^{**})$.

Let $\mathcal{A}^*\mathcal{A}$ be the norm closure of the linear span of $\{fa : f \in \mathcal{A}^*, a \in \mathcal{A}\}$. Then the dual of the space $\mathcal{A}^*\mathcal{A}$ equipped with the multiplication induced by that of $\mathcal{A}^{**}$ is also a Banach algebra. Let $\tilde{M} = \{\mu \in (\mathcal{A}^*\mathcal{A})^* : \mathcal{A}\mu \subseteq \mathcal{A}\}$ and the topological center of $(\mathcal{A}^*\mathcal{A})^*$ is defined by

$$\tilde{Z}_{\mathcal{A}} = \{\mu \in (\mathcal{A}^*\mathcal{A})^* \colon \nu \to \mu\nu \text{ is continuous in the weak}^* \text{ topology}\}.$$

An element E in $\mathcal{A}^{**}$ is called a right identity or right unit if $\phi E = \phi$ for all $\phi \in \mathcal{A}^{**}$. Let $\mathcal{E}$ denote the set of all right identities of $\mathcal{A}^{**}$. It is easy to see that an element of $\mathcal{A}^{**}$ is a right unit if and only if it is a weak* cluster point of some bounded right approximate identity in $\mathcal{A}$, see [3, p. 146].

Let S be a set. The characteristic function of S is denoted by 1_S. A Banach algebra is said to be weakly sequentially complete if every weakly Cauchy sequence converges in the weak topology.

For any locally compact group G equipped with a fixed left Haar measure λ, let $L^p(G)$, $1 \le p \le \infty$, be the usual Lebesgue spaces on G with norm $\|\cdot\|_p$. Suppose that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The Herz–Figà–Talamanca algebra $A_p(G)$ is the space of continuous functions u which can be represented as

$$u = \sum_{n=1}^{\infty} f_i * \check{g}_i \quad \text{with} \quad f_i \in L^q(G), g_i \in L^p(G), \quad \text{and} \quad \sum_{n=1}^{\infty} \|f_i\|_q \|g_i\|_p < \infty,$$

where $\check{g} \in L^p(G)$ is defined by $\check{g}(x) = g(x^{-1})$, $x \in G$. The norm of u is defined by

$$\|u\|_{A_p(G)} = \inf \sum_{n=1}^{\infty} \|f_i\|_q \|g_i\|_p,$$

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where the infimum is taken over all the representations of u above. It is known that $A_p(G)$ is a regular tauberian Banach algebra under the pointwise multiplication and $A_p(G)$ has a bounded approximate identity if and only if the group G is amenable (see Herz [8], Theorem 6). We emphasize that our $A_p(G)$ coincides with $A_q(G)$, $\frac{1}{p}+\frac{1}{q}=1$, in [16]. It follows that the dual $A_p(G)^*$ is the space of convolution operators on $L^p(G)$, denoted by $PM_p(G)$ as in Herz [8]. Let $PF_p(G)$ be the norm closure of $L^1(G)$ in $A_p(G)^*$. Then $PF_p(G)^* = W_p(G)$ is a Banach algebra under pointwise multiplication. For $p = 2$, $A_p(G) = A(G)$ is the Fourier algebra of G, $PM_p(G) = VN(G)$ is the group Von Neumann algebra of G, $PF_2(G) = C^*_\rho(G)$ is the reduced group C^* algebra and $W_2(G) = B_\rho(G)$ (see Eymard [5]). For more properties of $PM_p(G)$ and $PF_p(G)$, see Pier [16]. Throughout this paper, $B(G)$ denotes the Fourier–Stieltjes algebra of G as defined in Eymard [5].

Let $UC_p(\hat{G})$ be the norm closure of the subset of $PM_p(G)$ consisting of all fu for $u \in A_p(G)$ and $f \in PM_p(G)$. For $f \in PM_p(G)$, the support of f is defined to be the closed subset $\operatorname{supp}(f)$ of G such that $x \notin \operatorname{supp}(f)$ if and only if there exists a neighborhood U_x of x in G such that $\langle f, u\rangle = 0$ for all $u \in A_p(G)$ such that $\operatorname{supp} u \subseteq U_x$ as a function on G (see Herz [8, page 101]). Then it is well known that $UC_p(\hat{G})$ is the norm closure of the set of all elements of $PM_p(G)$ with compact support.

Let G_0 be an open subgroup of a locally compact group G. It is proved by Herz [8] (Proposition 5) that $A_p(G_0)$ is identified with the subalgebra of $A_p(G)$ consisting of functions in $A_p(G)$ which vanish outside G_0 and the restriction map from $A_p(G)$ onto $A_p(G_0)$, denoted by m_{G_0}, is a contraction. It is obvious that $A_p(G_0)$ is a closed ideal of $A_p(G)$, and m_{G_0} is a projection and is also a multiplier .

Some conventions. Let $i\colon \mathcal{B} \to \mathcal{A}$ be the inclusion map and $m\colon \mathcal{A} \to \mathcal{B}$ be any map. Then $\mathcal{B}^{**}$ is embedded into $\mathcal{A}^{**}$ by i^{**}. In this paper, we write $i^{**}(\phi)$ as ϕ, and sometimes we consider m as a map from $\mathcal{A}$ to $\mathcal{A}$ without confusion.

2 Extensions of a Unit

Let $\mathcal{B}$ be a subalgebra of a Banach algebra $\mathcal{A}$. In this section, we will extend a unit of $\mathcal{B}^{**}$ to a unit of $\mathcal{A}^{**}$.

Definition Let $\mathcal{B}$ be a closed subalgebra of a Banach algebra $\mathcal{A}$. An element E of $\mathcal{A}^{**}$ is called a right unit or identity of $\mathcal{B}^{**}$ in $\mathcal{A}^{**}$ if $\phi E = \phi$ for $\phi \in \mathcal{B}^{**}$.

Clearly, if $E_{\mathcal{B}} \in \mathcal{B}^{**}$ is a right unit of $\mathcal{B}^{**}$ in the normal sense, then $E_{\mathcal{B}}$ is a right unit of $\mathcal{B}^{**}$ in $\mathcal{A}^{**}$.

Proposition 2.1 *Let $\mathcal{B}$ be a closed subalgebra of a Banach algebra $\mathcal{A}$. Then an element E of $\mathcal{A}^{**}$ is a right unit of $\mathcal{B}^{**}$ in $\mathcal{A}^{**}$ if and only if there is a net $\{a_\alpha\}$ in $\mathcal{A}$ such that $a_\alpha \to E$ in the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology and $\|ba_\alpha - b\| \to 0$ for $b \in \mathcal{B}$.*

Proof Let E be a right unit of $\mathcal{B}^{**}$ in $\mathcal{A}^{**}$. By Goldstine's theorem, there exists a net $\{x_\beta\}$ in $\mathcal{A}$ such that $\|x_\beta\| \le \|E\|$ and $x_\beta \to E$ in the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology. Then for every $b \in \mathcal{B}$, it is clear that $bx_\beta \to bE$ in $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology. Since $b \in \mathcal{B}$ and E is

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a right unit of $\mathcal{B}^{**}$ in $\mathcal{A}^{**}$, we have $bE = b$. Thus, $bx_\beta \to b$ in the weak topology in $\mathcal{A}$.

For any finite set Γ of $\mathcal{A}^*$ and any positive integer k, since $x_\beta \to E$ in the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology, there exists β_k such that $|\langle x_\beta, f\rangle - \langle E, f\rangle| < \frac{1}{k}$ for all $f \in \Gamma$ and $\beta \geq \beta_k$. For every finite set $\Lambda = \{b_1, b_2, \dots, b_n\}$ of $\mathcal{B}$, since $(b_1x_\beta - b_1, b_2x_\beta - b_2, \dots, b_nx_\beta - b_n) \to (0, 0, \dots, 0)$ in the weak topology of the direct sum $\mathcal{A} \oplus \mathcal{A} \oplus \cdots \oplus \mathcal{A}$ (n copies), 0 is in the norm closure of the convex hull of $\{(b_1x_\beta - b_1, b_2x_\beta - b_2, \dots, b_nx_\beta - b_n) : \beta \geq \beta_k\}$. Hence, there exists a convex combination of elements from $\{x_\beta : \beta \geq \beta_k\}$, denoted by a_α, such that $\|(b_1a_\alpha - b_1, b_2a_\alpha - b_2, \dots, b_na_\alpha - b_n)\| < \frac{1}{k}$, and so $\|ba_\alpha - b\| < \frac{1}{k}$ for all $b \in \Lambda$, where $\alpha = (\Gamma, \Lambda, k)$ is directed as usual. Therefore, a_α satisfies the requirements.

Conversely, if $\{a_\alpha\}$ is a net in $\mathcal{A}$ such that $a_\alpha \to E$ in the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology and $\|ba_\alpha - b\| \to 0$ for $b \in \mathcal{B}$, then the weak* cluster point E of $\{a_\alpha\}$ is a right unit of $\mathcal{B}^{**}$ in $\mathcal{A}^{**}$. In fact, for $b \in \mathcal{B}$, $\|ba_\alpha - b\| \to 0$ and $ba_\alpha \to bE$ in the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology imply $bE = b$. If $\phi \in \mathcal{B}^{**}$, it is routine to check that $\phi E = \phi$. ■

Remark. If $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$ and $m\colon \mathcal{A} \to \mathcal{B}$ is a projection, then $\mathcal{B}^*$ is embedded into $\mathcal{A}^*$ by the mapping $m^*\colon \mathcal{B}^* \to \mathcal{A}^*$. $\mathcal{B}^*$ is identified with $m^*(\mathcal{A}^*)$ as follows. For any $f \in \mathcal{B}^*$, let $\tilde{f}$ be an extension of f to an element of $\mathcal{A}^*$. It is easy to see that the map $f \mapsto m^*(\tilde{f})$ is well-defined and is an isomorphism from $\mathcal{B}^*$ onto $m^*(\mathcal{A}^*)$. Furthermore, the map $f \mapsto m^*(\tilde{f})$ is an isometry from $\mathcal{B}^*$ onto $m^*(\mathcal{A}^*)$ if $\|m\| = 1$. We will extend a right unit of $\mathcal{B}^{**}$ to right units of $\mathcal{A}^{**}$. Precisely, for a right unit E of $\mathcal{B}^{**}$, we like to find right units $\tilde{E}$ of $\mathcal{A}^{**}$ such that $\tilde{E} = E$ on $m^*(\mathcal{A}^*)$.

Definition Let $\mathcal{B}$ be a closed subalgebra of a Banach algebra $\mathcal{A}$ and let $m\colon \mathcal{A} \to \mathcal{B}$ be a bounded projection. For a right unit E of $\mathcal{B}^{**}$ in $\mathcal{A}^{**}$, we say that a right unit $\tilde{E}$ of $\mathcal{A}^{**}$ is an extension of E if $\langle \tilde{E}, m^*(f)\rangle = \langle E, m^*(f)\rangle$ for $f \in \mathcal{A}^*$.

A natural question is whether there exists an extension for a given right unit of $\mathcal{B}^{**}$ in $\mathcal{A}^{**}$. The following example shows that the answer to this question is negative in general.

Example. Let $B(G)$ be the Fourier–Stieltjes algebra of an amenable locally compact group G (see Eymard [5]). Then $B(G)$ is a unital commutative Banach algebra. So $B(G)^{**}$ has a unique unit I. The Fourier algebra $A(G)$ is a closed ideal of $B(G)$. Since $B(G)$ is a direct sum of $A(G)$ and a subspace of $B(G)$ (see Miao [14]), there is a projection from $B(G)$ to $A(G)$. From Corollary 2.4 below we know that there are many right units in $A(G)^{**}$ if G is not compact. Therefore, the right units of $A(G)^{**}$ in $B(G)^{**}$ cannot be extended to right units of $B(G)^{**}$.

We have to put conditions on $\mathcal{B}$ and the projection m. If $\mathcal{B}$ is an ideal of $\mathcal{A}$, we say that an operator $m\colon \mathcal{A} \to \mathcal{B}$ is a multiplier if $m(ab) = m(a)b = am(b)$ for $a, b \in \mathcal{A}$.

Lemma 2.2 *If $\mathcal{B}$ is a closed ideal of a Banach algebra $\mathcal{A}$ such that there is a bounded projection $m\colon \mathcal{A} \to \mathcal{B}$ which is also a multiplier, then for any $a \in \mathcal{A}$, $f \in \mathcal{A}^*$ and $\varphi \in \mathcal{A}^{**}$, we have*

(i) $m^*(fa) = fm(a) = m^*(f)a$;

(ii) $m^*(\varphi f) = m^{**}(\varphi)f = \varphi m^*(f)$.

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Proof These can be verified directly by an elementary calculation using the definition of the Arens product and the fact that m is a multiplier. ■

Theorem 2.3 *Let $\mathcal{A}$ be a Banach algebra with a bounded right approximate identity. If $\mathcal{B}$ is a closed ideal of $\mathcal{A}$ such that there is a bounded projection $m\colon \mathcal{A} \to \mathcal{B}$ which is also a multiplier, then every right unit $E_{\mathcal{B}}$ of $\mathcal{B}^{**}$ in $\mathcal{A}^{**}$ can be extended to a right unit $\tilde{E}$ of $\mathcal{A}^{**}$.*

Proof Since $\mathcal{A}$ has a bounded right approximate identity, we can choose a right unit E of $\mathcal{A}^{**}$. We denote $\tilde{E} = E - m^{**}(E) + m^{**}(E_{\mathcal{B}})$. We claim that $\tilde{E}$ is an extension of $E_{\mathcal{B}}$. We show that $\tilde{E}$ is a right unit of $\mathcal{A}^{**}$ first. It follows from Proposition 2.1 that there is a net $\{a_\alpha\}$ in $\mathcal{A}$ such that $a_\alpha \to E_{\mathcal{B}}$ in the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology and $\|ba_\alpha - b\| \to 0$ for $b \in \mathcal{B}$. For any $a \in \mathcal{A}$, since $m(a) \in \mathcal{B}$ and $m^*(fa) = fm(a)$ by Lemma 2.2(i), we have

$$\begin{aligned}\langle m^{**}(E_{\mathcal{B}})f, a\rangle &= \langle m^{**}(E_{\mathcal{B}}), fa\rangle = \langle E_{\mathcal{B}}, m^*(fa)\rangle = \langle E_{\mathcal{B}}, fm(a)\rangle \\ &= \lim_\alpha \langle a_\alpha, fm(a)\rangle = \lim_\alpha \langle m(a)a_\alpha, f\rangle = \langle m(a), f\rangle.\end{aligned}$$

Thus, $\langle m^{**}(E_{\mathcal{B}})f, a\rangle = \langle m(a), f\rangle$. Similarly, $\langle m^{**}(E)f, a\rangle = \langle m(a), f\rangle$. Therefore, $m^{**}(E)f = m^{**}(E_{\mathcal{B}})f$. For $\varphi \in \mathcal{A}^{**}$, since $\varphi E = \varphi$, we have

$$\langle \varphi\tilde{E}, f\rangle = \langle \varphi E, f\rangle - \langle \varphi, m^{**}(E)f\rangle + \langle \varphi, m^{**}(E_{\mathcal{B}})f\rangle = \langle \varphi E, f\rangle = \langle \varphi, f\rangle.$$

Therefore $\tilde{E}$ is a right unit of $\mathcal{A}^{**}$.

If $f \in \mathcal{A}^*$, since m is a projection onto $\mathcal{B}$, $m^*(m^*(f)) = m^*(f)$. Hence, $\langle m^{**}(E), m^*(f)\rangle = \langle E, m^*(f)\rangle$, and so we have

$$\begin{aligned}\langle \tilde{E}, m^*(f)\rangle &= \langle E - m^{**}(E) + m^{**}(E_{\mathcal{B}}), m^*(f)\rangle \\ &= \langle m^{**}(E_{\mathcal{B}}), m^*(f)\rangle = \langle E_{\mathcal{B}}, m^*(f)\rangle.\end{aligned}$$

So $\tilde{E}$ is an extension of $E_{\mathcal{B}}$. ■

Corollary 2.4 *Let G be an amenable locally compact group and let G_0 be an open and closed subgroup of G. Then every right unit of $A_p(G_0)^{**}$ in $A_p(G)^{**}$ can be extended to a right unit of $A(G)_p^{**}$. In particular, when $p = 2$, $A(G)^{**}$ has a unique right unit only when G is compact.*

Proof Since $m_{G_0}\colon A_p(G) \to A_p(G_0)$ is a projection as well as a multiplier, the proof of the first part of this result is finished by using Theorem 2.3.

If G is not compact, then there is an open, closed σ-compact, and noncompact subgroup G_0 of G. Since G_0 is σ-compact, $A(G_0)$ has a sequential bounded approximate identity $\{a_n\}$. Then $\{a_n\}$ has at least two distinct w^* cluster points E_1 and E_2 in $A(G_0)^{**}$. In fact, if there were only one w^* cluster point, then $\{a_n\}$ would be a weakly Cauchy sequence. Since $A(G_0)$ is weakly sequentially complete, $\{a_n\}$ must converge to its cluster point in the weak topology in $A(G_0)$. Hence the cluster point must be in $A(G_0)$, and so $A(G_0)$ is unital. This is impossible since G_0 is not compact. Hence E_1 and E_2 are distinct right units of $A(G_0)^{**}$ in $A(G)^{**}$. By Theorem 2.3, there are extensions of E_1 and E_2 to $A(G)^{**}$ that would create two distinct right identities for $A(G)^{**}$. This is a contradiction. ■

Corollary 2.5 *Let G be a locally compact group, and let N be a compact normal subgroup of G. Then every right unit of $L^1(G/N)^{**}$ in $L^1(G)^{**}$ can be extended to a right unit of $L^1(G)^{**}$.*

Proof Let $\pi_N\colon G \to G/N$ be the canonical map. For any $f \in L^1(G)$, $\dot{f}(\dot{x}) = \int_N f(x\xi)\,d\xi$ defines a function in $L^1(G/N)$, where $\dot{x} = \pi_N(x)$ for $x \in G$. Moreover, there is a Haar measure on G/N such that

$$\int_{G/N} \Big\{ \int_N f(x\xi)d\xi \Big\}\, d\dot{x} = \int_G f(x)dx$$

for all $f \in L^1(G)$ (see Reiter and Stegeman [17, p. 100]). If we regard $L^1(G/N)$ as a subspace of $L^1(G)$ consisting of periodic functions on G with respect to N, then it is routine to check that $L^1(G/N)$ is a closed ideal of $L^1(G)$. Define $m\colon L^1(G) \to L^1(G/N)$ by $m(f) = \dot{f}$ for $f \in L^1(G)$. Since $m(f * g) = m(f) * m(g) = m(f) * g = f * m(g)$ for $f, g \in L^1(G)$ (see Reiter and Stegeman [17, Theorem 3.5.4]), m is a multiplier. It follows from Theorem 2.3 that any right unit of $L^1(G/N)^{**}$ can be extended to a right unit of $L^1(G)^{**}$. ∎

3 Applications of the Extension Theorem to $A(G)$

In this section we present some applications of the results given in Section 2. Theorem 3.2 settles an open problem in Lau and Ülger [13, open problem h, p. 1211]. To prove Theorems 3.2 and 4.4, we need the next lemma.

Lemma 3.1 *Let G be a locally compact group and $\varphi \in A_p(G)^{**}$. If for every open σ-compact subgroup G_0 of G, $m^{**}_{G_0}(\varphi)$ is in $A_p(G_0)$, then the restriction of φ onto $UC_p(\hat{G})$ is in $A_p(G)$.*

Proof We will show that for each n, there is a compact subset K_n of G such that $|\langle \varphi, f\rangle| < \frac{1}{n}$ for $f \in UC_p(\hat{G})$ with $\|f\| \le 1$ and $\operatorname{supp}(f) \subseteq G \setminus K_n$. Otherwise, there exists a positive number $\epsilon > 0$ and a function $f_1 \in UC_p(\hat{G})$ which has compact support, and is such that $\|f_1\| \le 1$ and $|\langle \varphi, f_1\rangle| \ge \epsilon$. Since G is locally compact, let U_1 be a symmetric open subset of G with a compact closure $\overline{U_1}$ such that the group unit e is in U_1 and $\operatorname{supp}(f_1) \subseteq U_1$. There exists an element $f_2 \in UC_p(\hat{G})$ with compact support $\operatorname{supp}(f_2) \subseteq G \setminus \overline{U_1}$ and $\|f_2\| \le 1$ satisfying that $|\langle \varphi, f_2\rangle| \ge \epsilon$. Let U_2 be a symmetric open subset of G such that $\operatorname{supp}(f_2) \subseteq U_2$, $\overline{U_2}$ is compact and $U_1^2 \subseteq U_2$. By continuing the same process, we have a sequence $\{f_n\}$ in $UC_p(\hat{G})$ and a sequence of symmetric open subsets $\{U_n\}$ of G satisfying for each n,

(1) $\|f_n\| \le 1$ and $\overline{U_n}$ is compact;
(2) $\operatorname{supp}(f_{n+1}) \subseteq G \setminus \overline{U_n}$, $U_n^2 \subseteq U_{n+1}$ and $\operatorname{supp}(f_n) \subseteq U_n$;
(3) $|\langle \varphi, f_n\rangle| \ge \epsilon$. Let $G_0 = \bigcup_n U_n$.

Then by condition (2), G_0 is an open σ-compact subgroup of G. So it is also closed. Then $m^{**}_{G_0}(\varphi)$ is in $A_p(G_0)$ by hypothesis. It follows that there is a compact subset K of G_0 such that $|\langle m^{**}_{G_0}(\varphi), f\rangle| \le \frac{1}{2}\epsilon$ for any $f \in PM_p(G)$ with $\operatorname{supp}(f) \subseteq G \setminus K$ (see Miao [15]). Since K is compact and the sequence $\{U_n\}$ of open sets is increasing, there is n such that $K \subseteq U_n$. Hence $\operatorname{supp}(f_{n+1}) \subseteq G \setminus \overline{U_n} \subseteq G \setminus K$. It follows that $|\langle m^{**}_{G_0}(\varphi), f_{n+1}\rangle| \leq \frac{1}{2}\epsilon$. This contradicts the fact that

$$|\langle m^{**}_{G_0}(\varphi), f_{n+1}\rangle| = |\langle \varphi, m^{*}_{G_0}(f_{n+1})\rangle| = |\langle \varphi, f_{n+1}\rangle| \geq \epsilon$$

for each n since $\operatorname{supp}(f_{n+1}) \subseteq U_{n+1} \subseteq G_0$ and so it is clear that $m^{*}_{G_0}(f_{n+1}) = f_{n+1}$ (see Eymard [5, Proposition 4.8]).

Let G_0 be an open, closed and σ-compact subgroup of G containing all K_n. Then it is easy to see that for any $f \in UC_p(\hat{G})$ with support in $G \setminus G_0$, we have $\langle \varphi, f\rangle = 0$. For any $f \in UC_p(\hat{G})$, it is routine to check that $\operatorname{supp}(f - m^{*}_{G_0}(f)) \subseteq G \setminus G_0$. Hence we have

$$\langle \varphi, f\rangle = \langle \varphi, m^{*}_{G_0}(f)\rangle + \langle \varphi, f - m^{*}_{G_0}(f)\rangle = \langle m^{**}_{G_0}(\varphi), f\rangle.$$

Since $m^{**}_{G_0}(\varphi)$ is in $A_p(G_0) \subseteq A_p(G)$, the restriction of φ to $UC_p(\hat{G})$ is in $A_p(G)$. ■

Theorem 3.2 *Let G be an amenable locally compact group. Then for an element $\varphi \in A(G)^{**}$, $\varphi \in A(G)$ if and only if $A(G)\varphi \subseteq A(G)$ and for any E in $\mathcal{E}$, $E\varphi = \varphi$.*

Proof One direction of the result is trivial. Conversely, let $\varphi \in A(G)^{**}$ satisfy the two conditions. If G is compact, then $A(G)$ is unital. So the result is trivial. Let G be noncompact and let G_0 be a σ-compact, open and closed subgroup of G. Let $G_0 = \bigcup_{i=1}^{\infty} K_i$, where $\{K_i\}$ is a sequence of compact subsets of G_0 such that $K_1 \subseteq K_2 \subseteq K_3, \dots$. For each i, choose an $a_i \in A(G_0)$ such that $a_i(x) = 1$ for $x \in K_i$ and $\|a_i\| \leq 1 + \frac{1}{i}$ by amenability of G_0 (see Pier [16, Proof of Theorem 10.4]). Then $a_i \to 1_{G_0}$ in the w^*-topology of $B(G_0)$. Hence $\|a_i a - a\| \to 0$ for $a \in A(G_0)$ (see Granirer and Leinert [7]).

For each i, $a_i\varphi \in A(G)$ by assumption. We claim that $\{a_i\varphi\}$ is a weakly Cauchy sequence. If not, then there exist two subnets $\{a_{i_\alpha}\varphi\}$ and $\{a_{i_\beta}\varphi\}$ of $\{a_i\varphi\}$ converge to different points of $A(G)^{**}$ in the $\sigma(A(G)^{**}, A(G)^*)$-topology. Assume that $a_{i_\alpha} \to E_1$ and $a_{i_\beta} \to E_2$ in $\sigma(A(G)^{**}, A(G)^*)$-topology without loss of generality by taking subnets. Then E_1 and E_2 are right identities of $A(G_0)^{**}$ in $A(G)^{**}$ by Proposition 2.1, and it is obvious that $a_{i_\alpha}\varphi \to E_1\varphi$ and $a_{i_\beta}\varphi \to E_2\varphi$ in the $\sigma(A(G)^{**}, A(G)^*)$-topology. Thus, $E_1\varphi \neq E_2\varphi$. There exists $f \in A(G)^*$ such that $\langle E_1\varphi, f\rangle \neq \langle E_2\varphi, f\rangle$.

It follows from $a_{i_\alpha} \to E_1$ in the $\sigma(A(G)^{**}, A(G)^*)$-topology, $a_{i_\alpha} \in A(G_0)$, and Lemma 2.2(ii) that

$$\begin{aligned}\langle E_1\varphi, m^{*}_{G_0}(f)\rangle &= \langle E_1, \varphi m^{*}_{G_0}(f)\rangle = \langle E_1, m^{*}_{G_0}(\varphi f)\rangle \\ &= \lim_{\alpha} \langle a_{i_\alpha}, m^{*}_{G_0}(\varphi f)\rangle = \lim_{\alpha} \langle a_{i_\alpha}, \varphi f\rangle = \langle E_1, \varphi f\rangle = \langle E_1\varphi, f\rangle.\end{aligned}$$

Similarly, $\langle E_2\varphi, m^{*}_{G_0}(f)\rangle = \langle E_2\varphi, f\rangle$. Hence $\langle E_1\varphi, m^{*}_{G_0}(f)\rangle \neq \langle E_2\varphi, m^{*}_{G_0}(f)\rangle$.

We extend E_1 and E_2 to right units $\tilde{E}_1$ and $\tilde{E}_2$ of $A(G)^{**}$ by Corollary 2.4. It follows from Lemma 2.2(ii), since $\tilde{E}_1$ is an extension of E_1, that

$$\begin{aligned}\langle \tilde{E}_1\varphi, m^{*}_{G_0}(f)\rangle &= \langle \tilde{E}_1, \varphi m^{*}_{G_0}(f)\rangle = \langle \tilde{E}_1, m^{*}_{G_0}(\varphi f)\rangle = \langle E_1, m^{*}_{G_0}(\varphi f)\rangle \\ &= \langle E_1, \varphi m^{*}_{G_0}(f)\rangle = \langle E_1\varphi, m^{*}_{G_0}(f)\rangle.\end{aligned}$$

By a similar argument we can obtain $\langle \tilde{E}_2\varphi, m^*_{G_0}(f)\rangle = \langle E_2\varphi, m^*_{G_0}(f)\rangle$. Thus, we have $\langle \tilde{E}_1\varphi, m^*_{G_0}(f)\rangle \neq \langle \tilde{E}_2\varphi, m^*_{G_0}(f)\rangle$. This contradicts the assumption $\tilde{E}_1\varphi = \tilde{E}_2\varphi = \varphi$. Hence $\{a_i\varphi\}$ is a weakly Cauchy sequence.

It follows from the fact that $A(G)$ is weakly complete that $\{a_i\varphi\}$ converges weakly to a point in $A(G)$. So the weak limit point of $\{a_i\varphi\}$ is uniquely determined by the elements of $L^1(G) \subseteq VN(G)$ (see Eymard [5]). Let $f \in L^1(G)$ have a compact support. Since each $a_i \in A(G_0)$, we have $\langle a_i\varphi, f\rangle = \langle a_i\varphi, 1_{G_0}f\rangle$. It is obvious $\langle m^{**}_{G_0}(\varphi), (1 - 1_{G_0})f\rangle = 0$. Also, $m^*_{G_0}((1_{G_0}f)a_i) = (1_{G_0}f)m_{G_0}(a_i)$ by Lemma 2.2(i). Therefore,

$$\begin{aligned}\langle a_i\varphi, f\rangle &= \langle a_i\varphi, 1_{G_0}f\rangle = \langle \varphi, (1_{G_0}f)a_i\rangle \\ &= \langle \varphi, (1_{G_0}f)m_{G_0}(a_i)\rangle = \langle \varphi, m^*_{G_0}((1_{G_0}f)a_i)\rangle \\ &= \langle m^{**}_{G_0}(\varphi), (1_{G_0}f)a_i\rangle \to \langle m^{**}_{G_0}(\varphi), (1_{G_0}f)\rangle = \langle m^{**}_{G_0}(\varphi), f\rangle,\end{aligned}$$

by the property of a_i. Hence $m^{**}_{G_0}(\varphi) \in A(G)$. Since each $a_i\varphi \in A(G_0)$ and $A(G_0)$ is closed in $A(G)$, we have $m^{**}_{G_0}(\varphi) \in A(G_0)$.

Let the restriction $\varphi|UC_2(\hat{G}) = u$. Then $u \in A(G)$ by Lemma 3.1. Let u_α be an approximate identity of $A(G)$. Then it is obvious that $u_\alpha\varphi \to E\varphi$ in the $\sigma(A(G)^{**}, A(G)^*)$ topology for some $E \in \mathcal{E}$. By assumption, $E\varphi = \varphi$. Thus, $u_\alpha\varphi \to \varphi$. Since $u_\alpha\varphi = u_\alpha u$ and $u_\alpha u \to u$ in norm, $u = \varphi$. Therefore φ is in $A(G)$. ■

Remark. This is analogous to a result for $L^1(G)$ proved by Lau and Ülger [13] (Theorem 5.4). However, their result for $L^1(G)$ holds for all locally compact groups G because $L^1(G)$ always has a bounded approximate identity for any G, and a stronger condition on the algebra is needed for their result (see Lau and Ülger [13, Theorem 5.4, condition(iii)]). Our result requires G to be amenable since the existence of a bounded approximate identity in $A(G)$ is essential in Theorem 3.2.

Corollary 3.3 *Let G be an amenable discrete group and $\varphi \in A(G)^{**}$. Then $\varphi \in A(G)$ if and only if $E\varphi = \varphi$ for all $E \in \mathcal{E}$.*

Proof This result follows from that fact that $A(G)$ is an ideal in $A(G)^{**}$ if G is discrete (see Lau [11] and Forrest [6]) and Theorem 3.2. ■

Now we apply Theorem 3.2 to the topological center problem.

Lemma 3.4 *Let $\mathcal{A}$ be a commutative Banach algebra with a bounded approximate identity. If $\mu \in \Lambda(\mathcal{A}^{**})$, then $E\mu = \mu$ for $E \in \mathcal{E}$.*

Proof Since $\mathcal{A}$ is commutative, it is easy to check that $a\varphi = \varphi a$ for any $a \in \mathcal{A}$ and $\varphi \in \mathcal{A}^{**}$. Let $E \in \mathcal{E}$. There is a net $\{a_\alpha\}$ in $\mathcal{A}$ such that $\|a_\alpha\| \leq \|E\|$ and $a_\alpha \to E$ in the weak* topology by Goldstine's theorem. So if $\mu \in \Lambda(\mathcal{A}^{**})$, we have

$$E\mu = \lim_\alpha a_\alpha\mu = \lim_\alpha \mu a_\alpha = \mu E = \mu. \qquad ■$$

It is proved in Lau and Losert [12, Theorem 6.5] that for a big class of groups, including amenable discrete groups, $\Lambda(A(G)^{**}) = A(G)$. The following corollary is a result in this direction and also a version of Theorem 2.1(i) in Baker, Lau and Pym [2] without assuming the sequential bounded approximate identity in the case of $A(G)$ (see also [2, Theorem 2.2 and Corollary 2.3]).

Corollary 3.5 *Let G be an amenable locally compact group. If $\mu \in \Lambda(A(G)^{**})$ and $A(G)\mu \subseteq A(G)$, then $\mu \in A(G)$. In particular, if G is discrete, then $\Lambda(A(G)^{**}) = A(G)$.*

Proof This corollary is a direct consequence of Theorem 3.2 and Lemma 3.4. ■

It is an open question of Lau and Ülger [13, question (g), p. 1211] as to whether $\mathcal{E}$ distinguishes the points of $\tilde{M} \setminus \mathcal{A}$ from those of $\mathcal{A}$ (*i.e.,* if $\phi \in \tilde{M} \setminus \mathcal{A}$, let $\tilde{\phi}$ be an extension of ϕ to an element in $\mathcal{A}^{**}$. Then there exist E_1 and E_2 in $\mathcal{E}$ such that $E_1\tilde{\phi} \neq E_2\tilde{\phi}$) when $\mathcal{A}$ is sequentially complete and nonunital (see Lau and Ülger [13, p. 1208]). The following result answers this question in the case of $A(G)$ and removes the condition of σ-compactness in Lau and Ülger [13, Lemma 5.13].

Corollary 3.6 *Let G be an amenable locally compact group and $\mathcal{A} = A(G)$. Then $\mathcal{E}$ distinguishes the points of $\tilde{M} \setminus A$ from those of $A(G)$*

Proof Let $\phi \in \tilde{M}$. Extend ϕ to an element of $A(G)^{**}$, and denote it by $\tilde{\phi}$. It is obvious that $a\tilde{\phi} = a\phi$ for $a \in A(G)$. Hence $A(G)\tilde{\phi} \subseteq A(G)$. Let E be in $\mathcal{E}$. If $\mathcal{E}$ does not distinguish the point ϕ from those of $A(G)$, then $E_1\tilde{\phi} = E_2\tilde{\phi}$ for any E_1 and E_2 in $\mathcal{E}$ (see Lau and Ülger [13], p 1208). Fix a E_0 from $\mathcal{E}$ and let $\psi = E_0\tilde{\phi}$. For any right unit $E \in \mathcal{E}$, we have $E\psi = EE_0\tilde{\phi} = E\tilde{\phi} = E_0\tilde{\phi} = \psi$. For any $a \in A(G)$, $a\psi = aE_0\tilde{\phi} = a\tilde{\phi} \in A(G)$. Hence $A(G)\psi \subseteq A(G)$. By Theorem 3.2, ψ is in $A(G)$. Let $u_\alpha \to E_0$ in the $\sigma(A(G)^{**}, A(G)^*)$ topology for a bounded approximate identity $\{u_\alpha\}$ of $A(G)$. For each $f \in VN(G)$ and $a \in A(G)$,

$$\begin{aligned}\langle \psi, fa\rangle = \langle E_0\tilde{\phi}, fa\rangle &= \lim \langle u_\alpha\tilde{\phi}, fa\rangle = \lim \langle u_\alpha\phi, fa\rangle \\ &= \lim \langle u_\alpha, \phi(fa)\rangle = \lim \langle \phi, (fa)u_\alpha\rangle = \langle \phi, fa\rangle.\end{aligned}$$

Hence $\phi = \psi$ on $\mathcal{A}^*\mathcal{A}$. Therefore $\phi \in A(G)$. ■

It is shown in Lau and Losert [12] that for a large class of locally compact groups G, if $\mathcal{A} = A(G)$, then $\tilde{Z}_{\mathcal{A}} = B(G)$. The following result is due to Lau and Losert [12, Theorem 6.4]. It follows immediately from our Corollary 3.6 and Lau and Ülger [13, Theorem 5.12] (see also Lau and Ülger [13, Corollary 5.14]).

Corollary 3.7 *Let G be an amenable locally compact group and $\mathcal{A} = A(G)$. Then $\Lambda(A(G)^{**}) = A(G)$ whenever $\tilde{Z}_{\mathcal{A}} = B(G)$.*

4 Topological Center of a Subalgebra

Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A} \subseteq \Lambda(\mathcal{A}^{**})$ holds. It is well known that $\Lambda(L^1(G)^{**}) = L^1(G)$ for all locally compact groups G, and $\Lambda(A(G)^{**}) = A(G)$ for a large class of groups G (see in Lau and Losert [12]). It is natural to ask: for what kind of locally compact group G does $\Lambda(A_p(G)^{**}) = A_p(G)$? For a subalgebra $\mathcal{B}$ of $\mathcal{A}$, we study the relationship between the topological centers $\Lambda(\mathcal{B}^{**})$ and $\Lambda(\mathcal{A}^{**})$. As a consequence, we show that the problem of whether $\Lambda(A_p(G)^{**}) = A_p(G)$ can be reduced to that for an open σ-compact subgroup. We generalize some results in Hu [9] and Hu and Neufang [10].

Lemma 4.1 *If $\mathcal{B}$ is a closed ideal of a Banach algebra $\mathcal{A}$ such that there is a bounded projection $m\colon \mathcal{A} \to \mathcal{B}$ which is also a multiplier, then we have*

(i) $i^*(\psi f) = m^{**}(\psi)i^*(f)$ *for* $f \in \mathcal{A}^*$ *and* $\psi \in \mathcal{A}^{**}$*;*
(ii) $m^*(\varphi g) = i^{**}(\varphi)m^*(g)$ *for* $g \in \mathcal{B}^*$ *and* $\varphi \in \mathcal{B}^{**}$.

Proof These follow from a routine verification by using the Arens product and the properties of i and m. ■

The following result is an abstract version of Lemma 8.1 in Hu and Neufang [10].

Theorem 4.2 *Let $\mathcal{B}$ be a closed ideal of a Banach algebra $\mathcal{A}$. If there exists a bounded projection $m\colon \mathcal{A} \to \mathcal{B}$ which is also a multiplier, then*

$$i^{**}(\Lambda(\mathcal{B}^{**})) \subseteq \Lambda(\mathcal{A}^{**}) \quad \text{and} \quad m^{**}(\Lambda(\mathcal{A}^{**})) \subseteq \Lambda(\mathcal{B}^{**}).$$

Proof Let $\varphi \in \Lambda(\mathcal{B}^{**})$. If $\psi_\alpha \in \mathcal{A}^{**}$ and $\psi_\alpha \to 0$ in the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$ topology, it is easy to see that $m^{**}(\psi_\alpha) \to 0$ in the $\sigma(\mathcal{B}^{**}, \mathcal{B}^*)$ topology. Moreover, for any $f \in \mathcal{A}^*$, $i^*(\psi_\alpha f) = m^{**}(\psi_\alpha)i^*(f)$ by Lemma 4.1(i), we have

$$\begin{aligned}\langle i^{**}(\varphi)\psi_\alpha, f\rangle &= \langle \varphi, i^*(\psi_\alpha f)\rangle = \langle \varphi, m^{**}(\psi_\alpha)i^*(f)\rangle \\ &= \langle \varphi m^{**}(\psi_\alpha), i^*(f)\rangle \to 0\end{aligned}$$

by the hypothesis $\varphi \in \Lambda(\mathcal{B}^{**})$. Hence $i^{**}(\varphi) \in \Lambda(\mathcal{A}^{**})$.

Conversely, let $\psi \in \Lambda(\mathcal{A}^{**})$ and let $\varphi_\alpha \in \mathcal{B}^{**}$ and $\varphi_\alpha \to 0$ in the $\sigma(\mathcal{B}^{**}, \mathcal{B}^*)$ topology. For any $g \in \mathcal{B}^*$, since $m^*(\varphi_\alpha g) = i^{**}(\varphi_\alpha)m^*(g)$ by Lemma 4.1(ii), and $i^{**}(\varphi_\alpha) \to 0$ in the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$ topology,

$$\langle m^{**}(\psi)\varphi_\alpha, g\rangle = \langle \psi, m^*(\varphi_\alpha g)\rangle = \langle \psi, i^{**}(\varphi_\alpha)m^*(g)\rangle = \langle \psi i^{**}(\varphi_\alpha), m^*(g)\rangle \to 0.$$

Therefore, $m^{**}(\psi) \in \Lambda(\mathcal{B}^{**})$. ■

If H is an open subgroup of a locally compact group G, we denote the inclusion map from $A_p(H)$ to $A_p(G)$ by i_H. So we have the following result.

Corollary 4.3 *Let G be a locally compact group. Then for any open subgroup H of G, the following is true*

$$i_H^{**}(\Lambda(A_p(H)^{**})) \subseteq \Lambda(A_p(G)^{**}) \quad \text{and} \quad m_H^{**}(\Lambda(A_p(G)^{**})) \subseteq \Lambda(A_p(H)^{**}).$$

Theorem 4.4 *Let G be an amenable locally compact group. Then $\Lambda(A_p(G)^{**}) = A_p(G)$ if and only if $\Lambda(A_p(H)^{**}) = A_p(H)$ for any open σ-compact subgroup H of G.*

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Proof Assume $\Lambda(A_p(G)^{**}) = A_p(G)$. For any open σ-compact subgroup H of G, if $\varphi \in \Lambda(A_p(H)^{**})$, then $i_H^{**}(\varphi) \in \Lambda(A_p(G)^{**})$ by Corollary 4.3. So $i_H^{**}(\varphi) \in A_p(G)$. It is clear that $i_H^{**}(\varphi) = 0$ on $G \setminus H$. By identitifying an element of $A_p(H)$ with an element in $A_p(G)$ as usual, we have $i_H^{**}(\varphi) = \varphi$ is in $A_p(H)$.

Conversely, assume $\Lambda(A_p(H)^{**}) = A_p(H)$ for any open σ-compact subgroup H of G. For any $\psi \in \Lambda(A_p(G)^{**})$, it follows from Corollary 4.3 that $m_H^{**}(\psi) \in \Lambda(A_p(H)^{**})$ for any open σ-compact subgroup H of G. Hence $m_H^{**}(\psi) \in A_p(H)$. By Lemma 3.1, the restriction of ψ onto $UC_p(\hat{G})$ denoted by u_ψ is in $A_p(G)$. Since G is amenable, $A_p(G)$ has a bounded approximate identity $\{a_\alpha\}$. Assume $a_\alpha \to E$ in the $\sigma(A_p(G)^{**}, A_p(G)^*)$ topology without loss of generality. For any $f \in PM_p(G)$, since $fa_\alpha \in UC_p(\hat{G})$ and $fa_\alpha = a_\alpha f$, we have

$$\langle \psi, a_\alpha f \rangle = \langle u_\psi, a_\alpha f \rangle = \langle u_\psi a_\alpha, f \rangle \to \langle u_\psi, f \rangle.$$

On the other hand, since $E \in \mathcal{E}$ and $\psi \in \Lambda(A_p(G)^{**})$,

$$\langle \psi, a_\alpha f \rangle = \langle \psi a_\alpha, f \rangle \to \langle \psi E, f \rangle = \langle \psi, f \rangle.$$

Hence, $\psi = u_\psi$ is in $A_p(G)$. ∎

Let $\mathcal{B}$ be a closed ideal of a Banach algebra $\mathcal{A}$. Next, we study the relationship between $\tilde{Z}_{\mathcal{B}}$ and $\tilde{Z}_{\mathcal{A}}$. Assume that there is a bounded projection $m\colon \mathcal{A} \to \mathcal{B}$ which is also a multiplier. Then i^* maps $\mathcal{A}^*\mathcal{A}$ to $\mathcal{B}^*\mathcal{B}$. In fact, if $f \in \mathcal{A}^*$ and $a \in \mathcal{A}$, we have

$$\langle i^*(fa), b \rangle = \langle fa, b \rangle = \langle f, ab \rangle = \langle f, m(ab) \rangle = \langle f, m(a)b \rangle = \langle i^*(f)m(a), b \rangle$$

for any $b \in \mathcal{B}$. Hence $i^*(fa) = i^*(f)m(a)$ is in $\mathcal{B}^*\mathcal{B}$. Therefore, $i^*(\mathcal{A}^*\mathcal{A}) \subseteq \mathcal{B}^*\mathcal{B}$. Similarly, it is easy to see that for any $g \in \mathcal{B}^*$ and $b \in \mathcal{B}$, $m^*(gb) = m^*(g)b$. So $m^*\colon \mathcal{B}^* \to \mathcal{A}^*$ maps $\mathcal{B}^*\mathcal{B}$ to $\mathcal{A}^*\mathcal{A}$.

Theorem 4.5 *Let $\mathcal{B}$ be a closed ideal of a Banach algebra $\mathcal{A}$. If there exists a bounded projection $m\colon \mathcal{A} \to \mathcal{B}$ which is also a multiplier, then*

$$i^{**}(\tilde{Z}_{\mathcal{B}}) \subseteq \tilde{Z}_{\mathcal{A}} \quad \textit{and} \quad m^{**}(\tilde{Z}_{\mathcal{A}}) \subseteq \tilde{Z}_{\mathcal{B}}.$$

Proof Let $\varphi \in \tilde{Z}_{\mathcal{B}}$. If $\psi_\alpha \in (\mathcal{A}^*\mathcal{A})^*$ and $\psi_\alpha \to 0$ in the $\sigma((\mathcal{A}^*\mathcal{A})^*, \mathcal{A}^*\mathcal{A})$ topology, it is easy to see that $m^{**}(\psi_\alpha) \to 0$ in the $\sigma((\mathcal{B}^*\mathcal{B})^*, \mathcal{B}^*\mathcal{B})$ topology. Let $\tilde{\psi}_\alpha$ be an extension of ψ_α to an element of $\mathcal{A}^{**}$. For any $f \in \mathcal{A}^*\mathcal{A}$, it is clear that $\psi_\alpha f = \tilde{\psi}_\alpha f$ as elements in $\mathcal{A}^*$, and $m^{**}(\psi_\alpha)i^*(f) = m^{**}(\tilde{\psi}_\alpha)i^*(f)$ as elements of $\mathcal{B}^{**}$. Hence, by Lemma 4.1(i), $i^*(\tilde{\psi}_\alpha f) = m^{**}(\tilde{\psi}_\alpha)i^*(f)$. It follows that

$$\begin{aligned}\langle i^{**}(\varphi)\psi_\alpha, f \rangle &= \langle \varphi, i^*(\psi_\alpha f) \rangle = \langle \varphi, i^*(\tilde{\psi}_\alpha f) \rangle \\ &= \langle \varphi, m^{**}(\tilde{\psi}_\alpha)i^*(f) \rangle = \langle \varphi, m^{**}(\psi_\alpha)i^*(f) \rangle \\ &= \langle \varphi m^{**}(\psi_\alpha), i^*(f) \rangle \to 0\end{aligned}$$

by the hypothesis $\varphi \in \tilde{Z}_{\mathcal{B}}$. Hence $i^{**}(\varphi) \in \tilde{Z}_{\mathcal{A}}$.

Conversely, let $\psi \in \tilde{Z}_{\mathcal{A}}$ and let $\varphi_\alpha \in \mathcal{B}^*\mathcal{B}^*$ and $\varphi_\alpha \to 0$ in the $\sigma((\mathcal{B}^*\mathcal{B})^*, (\mathcal{B}^*\mathcal{B}))$ topology. Let $\tilde{\varphi}_\alpha$ be an extension of φ_α to an element in $\mathcal{B}^{**}$. Then for any $g \in \mathcal{B}^*\mathcal{B}$, it is clear that $\varphi_\alpha g = \tilde{\varphi}_\alpha g$ as elements in $\mathcal{B}^*$ and $i^{**}(\tilde{\varphi}_\alpha)m^*(g) = i^{**}(\varphi_\alpha)m^*(g)$ as elements of $\mathcal{A}^{**}$. By Lemma 4.1(ii), $m^*(\tilde{\varphi}_\alpha g) = i^{**}(\tilde{\varphi}_\alpha)m^*(g)$, and $i^{**}(\varphi_\alpha) \to 0$ in the $\sigma((\mathcal{A}^*\mathcal{A})^*, \mathcal{A}^*\mathcal{A})$ topology. Thus, we have

$$\begin{aligned}\langle m^{**}(\psi)\varphi_\alpha, g\rangle &= \langle \psi, m^*(\varphi_\alpha g)\rangle = \langle \psi, m^*(\tilde{\varphi}_\alpha g)\rangle = \langle \psi, i^{**}(\tilde{\varphi}_\alpha)m^*(g)\rangle \\ &= \langle \psi, i^{**}(\varphi_\alpha)m^*(g)\rangle = \langle \psi i^{**}(\varphi_\alpha), m^*(g)\rangle \to 0.\end{aligned}$$

Therefore, $m^{**}(\psi) \in \tilde{Z}_{\mathcal{B}}$. ■

It is proved by Derighetti, Filali, and Monfared [4] that $W_p(G)$ can be embedded into $UC_p(\hat{G})^*$ as follows. For $b \in W_p(G)$ and $fu \in UC_p(\hat{G})$, where $u \in A_p(G)$ and $f \in PM_p(G)$, $\langle b, fu\rangle = \langle f, bu\rangle$. It is proved in Lau and Losert [12] that for $p = 2$, $B_\rho(G)$ is contained in $\tilde{Z}_{UC_2(\hat{G})}$. Their proof works for the case of $p \neq 2$ as well. In fact, we only need to show that, for $b \in W_p(G)$ and $\varphi \in UC_p(\hat{G})^*$, we have $b\varphi = \varphi b$ under the first Arens multiplication as the proof of Proposition 4.5 in Lau and Losert [12]. Since it is routine to check that $\langle b\varphi, fu\rangle = \langle \varphi b, fu\rangle$ for any $u \in A_p(G)$ and $f \in PM_p(G)$, therefore, $b\varphi = \varphi b$ and so $W_p(G) \subseteq \tilde{Z}_{UC_p(\hat{G})}$. The question is whether $\tilde{Z}_{UC_p(\hat{G})} = W_p(G)$. Lau and Losert in [12] showed that if G is second countable and the commutator subgroup $\overline{[G,G]}$ is not open in G, then it is true that $\tilde{Z}_{UC_2(\hat{G})} = B_\rho(G)$. The following result is the p-version of Theorem 3.6 in Hu [9], showing that this problem can be reduced to that for σ-compact open subgroups .

Corollary 4.6 *Let G be a locally compact group. Then*

(i) $W_p(G) \subseteq \tilde{Z}_{UC_p(\hat{G})}$;

(ii) $\tilde{Z}_{UC_p(\hat{G})} = W_p(G)$ *if and only if* $\tilde{Z}_{UC_p(\hat{G}_0)} = W_p(G_0)$ *for all σ-compact open and closed subgroups G_0 of G.*

Proof (i) is proved above. To prove (ii), suppose $\tilde{Z}_{UC_p(\hat{G})} = W_p(G)$. Let G_0 be a σ-compact open and closed subgroup of G. If $\varphi \in \tilde{Z}_{UC_p(\hat{G}_0)}$, since $PF_p(G_0)$ is a closed subspace of $UC_p(G_0)$, we denote the restriction of φ onto $PF_p(G_0)$ by b_φ. Then $b_\varphi \in W_p(G_0)$. We claim that $\varphi = b_\varphi$. By (i), we have $b_\varphi \in \tilde{Z}_{UC_p(\hat{G}_0)}$. It follows from Theorem 4.5 that $i^{**}_{G_0}(\varphi - b_\varphi) \in \tilde{Z}_{UC_p(\hat{G})} = W_p(G)$. For any $f \in L^1(G)$, it is easy to see that $i^*(f) = f1_{G_0}$ is in $L^1(G_0)$. Hence

$$\langle i^{**}_{G_0}(\varphi - b_\varphi), f\rangle = \langle \varphi - b_\varphi, i^*_{G_0}(f)\rangle = \langle \varphi - b_\varphi, f1_{G_0}\rangle = 0.$$

Thus, $i^{**}_{G_0}(\varphi - b_\varphi) = 0$. For any $fu \in UC_p(\hat{G}_0)$, where $f \in PM_p(G_0)$ and $u \in A_p(G_0)$, we can extend f to an element $\tilde{f}$ of $PM_p(G)$. Then $\tilde{f}u \in UC_p(G)$. It follows from the fact $i^*(\tilde{f}u) = fu$ that

$$\langle \varphi - b_\varphi, fu\rangle = \langle \varphi - b_\varphi, i^*(\tilde{f}u)\rangle = \langle i^{**}(\varphi - b_\varphi), \tilde{f}u\rangle = 0.$$

Therefore, $\varphi = b_\varphi$ is in $W_p(G_0)$.

Conversely, let $\varphi \in \tilde{Z}_{UC_p(\hat{G})}$. Note that $PF_p(G)$ is a closed subspace of $UC_p(\hat{G})$. The restriction of φ onto $PF_p(G)$, denoted by b_φ, is in $W_p(G)$. Let $\tilde{\varphi} = \varphi - b_\varphi$. If $\tilde{\varphi} \neq 0$ as an element of $UC_p(\hat{G})^*$, then there is fu in $UC_p(\hat{G})$ for $f \in PM_p(G)$ and $u \in A_p(G)$ such that $\langle \tilde{\varphi}, fu \rangle \neq 0$. Assume the support of u, denoted by K, is a compact subset of G without loss of generality. There is a σ-compact open subgroup G_0 of G such that $K \subseteq G_0$. By the hypothesis, $\tilde{Z}_{UC_p(\hat{G}_0)} = W_p(G_0)$. It follows from Theorem 4.5 that $m_{G_0}^{**}(\tilde{\varphi}) \in W_p(G_0)$. Since $fu \in UC_p(\hat{G}_0)$ and $\langle m_{G_0}^{**}(\tilde{\varphi}), fu \rangle = \langle \tilde{\varphi}, m_{G_0}^{*}(fu) \rangle = \langle \tilde{\varphi}, fu \rangle \neq 0$, $m_{G_0}^{**}(\tilde{\varphi}) \neq 0$. It follows from $W_p(G_0) = PF_p(G_0)^*$ and $m_{G_0}^{**}(\tilde{\varphi}) \in W_p(G_0)$ that there is $g \in L^1(G_0)$ such that $\langle m_{G_0}^{**}(\tilde{\varphi}), g \rangle \neq 0$. On the other hand, since $L^1(G_0) \subseteq L^1(G)$, we have $\langle m_{G_0}^{**}(\tilde{\varphi}), g \rangle = \langle \tilde{\varphi}, g \rangle = \langle \varphi, g \rangle - \langle b_\varphi, g \rangle = 0$. This is a contradiction. Hence $\varphi = b_\varphi$ as elements in $UC_p(\hat{G})^*$ is in $W_p(G)$. ■

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