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A note on simultaneous polynomial approximation of exponential functions

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Let $\alpha_1, \ldots, \alpha_m$ be distinct complex numbers and $\tau(1), \ldots, \tau(m)$ be non-negative integers. We obtain conditions under which the functions

$$z^{\tau(1)}\exp\left(\alpha_1 z\right), \ldots, z^{\tau(m)}\exp\left(\alpha_m z\right)$$

form a perfect system, that is, for every set $\rho(1), \ldots, \rho(m)$ of non-negative integers, there are polynomials $a_1(z), \ldots, a_m(z)$, with respective degrees exactly $\rho(1)-1, \ldots, \rho(m)-1$, such that the function

$$R(z) = \sum_{k=1}^{m} a_k(z)\exp\left(\alpha_k z\right)$$

has a zero of order at least $\rho(1) + \ldots + \rho(m)-1$ at the origin. Moreover, subject to the evaluation of certain determinants, we give explicit formulae for the approximating polynomials $a_1(z), \ldots, a_m(z)$.

1. Introduction

In [4], Mahler has introduced the idea of a perfect system of functions, defined as follows. Let $f_1(z), \ldots, f_m(z)$ be functions of one complex variable which are regular at the origin and do not all vanish there. Let $\rho = \left(\rho(1), \ldots, \rho(m)\right)$ be an m-tuple of non-negative integers

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and set $\sigma = \rho(1) + \ldots + \rho(m)$. Then there are polynomials $a_1(z), \ldots, a_m(z)$, with respective degrees at most $\rho(1)-1, \ldots, \rho(m)-1$ and not all identically zero, such that the function

$$(1) \qquad R(z) = \sum_{k=1}^{m} a_k(z) f_k(z)$$

has a zero of order at least $\sigma - 1$ at the origin. The functions $f_1(z), \ldots, f_m(z)$ form a perfect system if, for every choice of ρ , there are polynomials $a_1(z), \ldots, a_m(z)$ with respective degrees exactly $\rho(1)-1, \ldots, \rho(m)-1$ such that the function $R(z)$ defined in (1) has a zero of order at least $\sigma - 1$ at the origin. The polynomials $a_1(z), \ldots, a_m(z)$ are then uniquely determined up to a common constant multiple; (see [4], page 113).

In [5], the second author gave several examples of sets of functions whose perfectness can be established by explicitly constructing the approximating polynomials $a_1(z), \ldots, a_m(z)$. In this note, we consider in the same spirit the perfectness of the system of functions

$$(2) \qquad z^{\tau(1)}\exp(\alpha_1 z), \ldots, z^{\tau(m)}\exp(\alpha_m z) ,$$

where $\alpha_1, \ldots, \alpha_m$ are complex numbers and $\tau(1), \ldots, \tau(m)$ are non-negative integers. The main result and the corresponding construction are given in Section 2.

The particular case $\tau(1) = \ldots = \tau(m) = 0$ gives the approximating polynomials constructed by Mahler [2, 3] and used by him to obtain arithmetic properties of the exponential function. A lemma on the simultaneous polynomial approximation of the general system (2) was used recently by Baker [1] in obtaining a new diophantine inequality involving the exponential function. Unfortunately, our construction, at least in its present form, does not appear to have any applications of this kind.

2. Construction of the approximating polynomials

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an m-tuple of complex numbers and $\rho = (\rho(1), \ldots, \rho(m))$ and $\tau = (\tau(1), \ldots, \tau(m))$ be m-tuples of non-negative integers. Set $\sigma = \rho(1) + \ldots + \rho(m)$. We denote by $D(\alpha, \rho, \tau)$ the determinant of order σ with the element

$$\binom{i-1}{\tau(r)+s-1}\alpha_r^{i-\tau(r)-s}$$

in the ith row and jth column, where $j = \rho(1) + \ldots + \rho(r-1)+s$ $\left(1 \le r \le m\ ,\ \ 1 \le s \le \rho(r)\right)$.

THEOREM. *Let* $\alpha_1, \ldots, \alpha_m$ *be distinct complex numbers and* $\tau(1), \ldots, \tau(m)$ *be non-negative integers with*

$$0 = \tau(1) \le \tau(2) \le \ldots \le \tau(m) .$$

If, for each m*-tuple* $\rho = \left(\rho(1), \ldots, \rho(m)\right)$ *of non-negative integers, the determinant* $D(\alpha, \rho, \tau)$ *defined above is non-zero, then the functions*

$$z^{\tau(1)}\exp\left(\alpha_1 z\right), \ldots, z^{\tau(m)}\exp\left(\alpha_m z\right) \tag{3}$$

form a perfect system.

Proof. Let $\rho = \left(\rho(1), \ldots, \rho(m)\right)$ be an m-tuple of non-negative integers and set $\sigma = \rho(1) + \ldots + \rho(m)$. Let ω_{rs} $\left(1 \le r \le m\ ,\right.$ $\left.1 \le s \le \rho(r)\right)$ be σ distinct complex numbers and denote their difference product by $\Delta(\omega)$. Thus

$$\Delta(\omega) = \Delta\left(\omega_{11}, \ldots, \omega_{m,\rho(m)}\right) = \left|\omega_{rs}^{i-1}\right|$$

is the determinant of order σ with ω_{rs}^{i-1} in the ith row and jth column, where $j = \rho(1) + \ldots + \rho(r-1)+s$ $\left(1 \le r \le m\ ,\ \ 1 \le s \le \rho(r)\right)$.

Define a function $S(z)$ by

$$S(z) = \frac{\Delta(\omega)}{2\pi i}\int_C \prod_{r,s} \left(\zeta-\omega_{rs}\right)^{-1}e^{\zeta z}d\zeta , \tag{4}$$

where C is a simple closed contour in the ζ-plane containing all the ω_{rs} . On the one hand, evaluating the integral by obtaining the residue of the integrand at each of the poles ω_{rs} inside C , we obtain

$$S(z) = \sum_{r,s} \Delta_{rs}(\omega)\exp\left(\omega_{rs}z\right) , \tag{5}$$

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where

$$\Delta_{rs}(\omega) = (-1)^{\sigma(r,s)}\Delta(\omega_{11}, \ldots, \hat{\omega}_{rs}, \ldots, \omega_{m,\rho(m)})$$

is, except for sign, the difference product of the ω_{kl} with ω_{rs} omitted, and we have introduced $\sigma(r, s) = \rho(1) + \ldots + \rho(r-1)+s-1$. In particular, $\Delta_{rs}(\omega)$ is independent of ω_{rs}. On the other hand, evaluating the integral (4) by considering the behaviour of the integrand at its remaining singularity at $\zeta = \infty$, we see that $S(z)$ has a Taylor expansion about the origin which begins

$$S(z) = \Delta(\omega) \frac{z^{\sigma-1}}{(\sigma-1)!} + \ldots . \tag{6}$$

Define the differential operators

$$L = \prod_{k,l} \frac{1}{(\tau(k)+l-1)!} \left(\frac{\partial}{\partial\omega_{kl}}\right)^{\tau(k)+l-1}\Bigg|_{\omega_{kl}=\alpha_k} ,$$

and, for each pair (r, s) with $1 \le r \le m$, $1 \le s \le \rho(r)$,

$$L_{rs} = \prod_{(k,l)\neq(r,s)} \frac{1}{(\tau(k)+l-1)!} \left(\frac{\partial}{\partial\omega_{kl}}\right)^{\tau(k)+l-1}\Bigg|_{\omega_{kl}=\alpha_k} ,$$

where, after differentiation, we replace each ω_{kl} by α_k. On applying the operator L to (5), we obtain

$$R(z) = LS(z) = \sum_{k=1}^{m} a_k(z) z^{\tau(k)} \exp(\alpha_k z) , \tag{7}$$

where

$$a_k(z) = \sum_{l=1}^{\rho(k)} \frac{1}{(\tau(k)+l-1)!} L_{kl}\Delta_{kl}(\omega) z^{l-1} \quad (1 \le k \le m) \tag{8}$$

is a polynomial of degree at most $\rho(k) - 1$ in z. From (6), the function $R(z)$ has a zero of order at least $\sigma - 1$ at the origin and its Taylor expansion about the origin begins

$$R(z) = L\Delta(\omega) \frac{z^{\sigma-1}}{(\sigma-1)!} + \ldots = D(\alpha, \rho, \tau) \frac{z^{\sigma-1}}{(\sigma-1)!} + \ldots . \tag{9}$$

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Moreover, the leading coefficient of the polynomial $a_k(z)$ is

$$\pm \frac{1}{(\tau(k)+\rho(k)-1)!} D(\alpha, \rho'_k, \tau) , \tag{10}$$

where $\rho'_k = (\rho(1), \ldots, \rho(k)-1, \ldots, \rho(m))$, so by hypothesis, $a_k(z)$ has exact degree $\rho(k) - 1$. Thus the functions (3) form a perfect system.

3. Construction of linearly independent approximations

Following the general theory of [4], pages 104-107, we can use the preceding work to construct explicitly systems of linearly independent forms in the functions (3).

As before, let $\alpha_1, \ldots, \alpha_m$ be distinct complex numbers and $\tau(1), \ldots, \tau(m)$ be non-negative integers satisfying the hypotheses of the theorem of Section 2. We carry out the construction of Section 2 with the parameters ρ replaced in turn by the m-tuple

$$\rho_h = (\rho(1), \ldots, \rho(h)+1, \ldots, \rho(m)) \quad (1 \le h \le m) ,$$

denoting quantities obtained from ρ_h by a subscript h . Thus, from (7) and (9), we obtain the functions

$$R_h(z) = \sum_{k=1}^{m} a_{hk}(z) z^{\tau(k)} \exp(\alpha_k z) \tag{11}$$

$$= D(\alpha, \rho_h, \tau) \frac{z^\sigma}{\sigma!} + \ldots \quad (1 \le h \le m) ,$$

where, by (8) and (10), $a_{hk}(z)$ is a polynomial in z of degree $\rho(k) + \delta_{hk}$ and the leading coefficient of $a_{kk}(z)$ is

$$\pm \frac{D(\alpha,\rho,\tau)}{(\tau(k)+\rho(k))!} . \tag{12}$$

Let $A(z)$ be the $m \times m$ determinant

$$A(z) = |a_{hk}(z)|_{1 \le h,k \le m} .$$

From (11) and the hypothesis $\tau(1) = 0$, it follows that $A(z)$ has a zero of order at least σ at the origin. On the other hand, from the above remarks, $A(z)$ is a polynomial of degree at most σ and, in the expansion

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of $A(z)$, a term of degree σ can only arise from the main diagonal. Using (12) to compute this term, we find

$$A(z) = \pm\left\{\prod_{k=1}^{m} \frac{D(\alpha,\rho,\tau)}{(\tau(k)+\rho(k))!}\right\}z^{\sigma} .$$

In particular, from our hypothesis, $A(1) \neq 0$, so on writing $z = 1$ in (11), we obtain m linearly independent forms in $\exp(\alpha_1), \ldots, \exp(\alpha_m)$, say

$$R_h = \sum_{k=1}^{m} a_{hk}\exp(\alpha_k) \quad (1 \le h \le m) .$$

However, it does not seem at all easy to estimate the size of the numbers a_{hk} and R_h , which the applications such as those in [1] and [2, 3] require.

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