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A NOTE ON THE DOOB-MEYER-DECOMPOSITION OF L^P-VALUED SUBMARTINGALES

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Let $p > 1$ real. We Doob-Meyer-decompose $L^p(\mathbb{P})$-valued positive submartingales such that the martingale and predictable parts are also in $L^p(\mathbb{P})$. We give two variants of such a decomposition. The first one handles also not necessarily right continuous submartingales, since its proof is as discrete in its nature as Doob's archaically decomposition. The second decomposition acts *in* $L^p(\mathbb{R} \times \Omega, \mathcal{B} \otimes \mathcal{F}, \mu \otimes \mathbb{P})$ for some finite measure μ on $\mathbb{R}$.

1. INTRODUCTION

In the paper [**1**] a Doob-Meyer-decomposition ([**6**]) for a certain class of Hilbert space valued processes is shown. Its proof uses the reflexivity of Hilbert spaces, or more precisely, the weak compactness of its unit ball. (Beside, Rao [**8**] uses the weak $L^1(\mathbb{P})$-compactness theorem of Dunford [**5**, 3.13] to get a decomposition in $L^1(\mathbb{P})$). In this paper we extend this method to real-valued, $L^p(\mathbb{P})$-valued, positive submartingales by using the reflexivity of the $L^p(\mathbb{P})$-space in the same fashion. Hence we must assume $p > 1$ and the proof fails for $p = 1$.

Beside its autonomous method of proof (which, however, is already demonstrated in [**1**]) the present paper takes its worth from demonstrating that $L^p(\mathbb{P})$-boundedness can survive the DM-decomposition.

Two variants of the DM-decomposition are proved.

(I) In the first decomposition we have given a totally ordered set $\mathcal{R}$, a real $p > 1$ and a positive, real-valued submartingale

$$X : \mathcal{R} \to L^p(\Omega, \mathcal{F}, \mathbb{P})$$

(we call such a process $L^p(\mathbb{P})$*-valued*) with respect to an arbitrary filtration $(\mathcal{F}_t)_{t \in \mathcal{R}}$. Then we find a decomposition $X = M + A$ into a $L^p(\mathbb{P})$-valued martingale M, and a $L^p(\mathbb{P})$-valued, "predictable" (A is adapted to the lefthanded filtration) and increasing process A.

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This Doob–Meyer-decomposition (Theorem 2.2) is completely outlined in Section 2, and its relatively elementary proof may also appreciated for its brevity. Astonishing is the fact that we need no right continuity, neither of the filtration nor of the process, a fact which is rather curious in the field of stochastic processes.

(II) In the second DM-decomposition (Theorem 3.4) we consider a totally ordered set $\mathcal{R}$ which is separable with respect to the order topology and which has at most countably many successor elements. Every subset of $\mathbb{R}$ satisfies these properties.

Moreover we assume that μ is a finite measure on $\mathcal{R}$ and assume that the filtration is right continuous. Then we consider the space

$$W^p := L^p(\mathcal{R} \times \Omega, \mathcal{B} \otimes \mathcal{F}, \mu \otimes \mathbb{P})$$

for some $p > 1$. If $X \in W^p$ is then a positive, real-valued submartingale which is right continuous with respect to the weak topology in $L^p(\mathbb{P})$, then we find a decomposition $X = M + A$ into a weakly right continuous martingale $M \in W^p$ and a predictable, increasing, weakly right continuous process $A \in W^p$.

Its more extensive proof is not completely outlined here. We only outline the differences to the paper [**1**], where necessary, and omit everything what can almost copied from [**1**].

2. The discrete decomposition

We consider a totally ordered set $\mathcal{R}$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t \in \mathcal{R}}$ in the σ-algebra $\mathcal{F}$. We often write briefly $L^p(\mathbb{P})$ for $L^p(\Omega, \mathcal{F}, \mathbb{P})$ $(p \geqslant 1)$ and subsequently use the notation $P_t(f) = \mathbb{E}(f \mid \mathcal{F}_t)$ $(t \in \mathcal{R})$ for the conditional expectation operator.

We shall recall some facts about the weak topology and the conditional expectation operator P_t. By Jensen's inequality we have

$$\|P_t X\|_{L^p(\mathbb{P})} \leqslant \|X\|_{L^p(\mathbb{P})}$$

and thus it is evident that P_t is an operator in $B\big(L^p(\mathbb{P})\big)$ for every $p \geqslant 1$ on its own.

Moreover the operator P_t is *selfadjoint* in the following sense: $\forall X \in L^p(\mathbb{P}), Y \in L^p(\mathbb{P})^* = L^q(\mathbb{P})$ $(1 = p^{-1} + q^{-1})$ we have

$$\mathbb{E}\big((P_t X) Y\big) = \mathbb{E}\big(X (P_t Y)\big).$$

If $p > 1$ then we endow $L^p(\mathbb{P})$ with the weak topology which is just the w*-topology of $L^p(\mathbb{P})^{**} = L^p(\mathbb{P})$. Hence the closed unit ball of $L^p(\mathbb{P})$ is compact with respect to the weak topology due to Alaoglu's theorem. Also recall that norm closed convex sets are weakly closed in Banach spaces, thus in $L^p(\mathbb{P})$; a fact we shall use several times.

Any continuous linear operator $T \in B(L^p(\mathbb{P}))$ is continuous with respect to the weak topology, that is,

$$T(\text{wlim}_i x_i) = \text{wlim}_i T(x_i)$$

for weakly convergent nets $(x_i)_i \to x$ in $L^p(\mathbb{P})$. This is clear since for arbitrary $y^* \in L^p(\mathbb{P})^*$ we have

$$\lim_i \langle T(x_i - x), y^* \rangle = \lim_i \langle x_i - x, T^* y^* \rangle = 0.$$

We shall now come to the proof of the discrete DM-decomposition. Central to the discrete decomposition (and also to the integrable decomposition in Section 3) is the Lemma 2.1 below. It says, that applying Doob's decomposition on a positive submartingale X, the resulting martingale and predictable parts M and A do not explode in $L^p(\mathbb{P})$-norm. This is quite immediate for $p = 1$, but really not obvious for $p > 1$. (We could say, *the Doob-decomposition is bounded*, and we consider this fact as the deeper reason why Doob's decomposition can be extended to continuous time scales, by taking the limit of Doob's decomposition say, at all.) In fact, a more general version of Lemma 2.1 is due to Garsia [4] and Neveu [7] (or see for example, [**5**, Proposition 22.21]). However, we shall give an elementary proof of our simple version Lemma 2.1, which fits here exactly into our framework.

Lemma 2.1. *Let $p \geqslant 1$ be real, $n \geqslant 0$ and $X : \mathbb{N} \to L^p(\Omega, \mathcal{F}, \mathbb{P})$ be a positive submartingale. Then for the predictable part A of the Doob-decomposition ($A_n := \sum_{k=1}^{n} P_{k-1}(X_k - X_{k-1})$) we have*

$$\mathbb{E}(A_n^p) \leqslant \mathbb{E}(X_n A_n^{p-1} p) \quad \text{and} \quad \|A_n\|_{L^p(\mathbb{P})} \leqslant p \|X_n\|_{L^p(\mathbb{P})}. \tag{1}$$

Proof: Note that $\mathbb{E}((P_n X_{n+1}) A_{n+1}^{p-1} p) = \mathbb{E}(X_{n+1} A_{n+1}^{p-1} p)$ since A_{n+1} is $\mathcal{F}_n$-measurable. Thus we have (abbreviate $\Delta X_n := X_{n+1} - X_n$)

$$0 = \mathbb{E}\big(X_{n+1} A_{n+1}^{p-1} p - (P_n \Delta X_n) A_{n+1}^{p-1} p - X_n A_{n+1}^{p-1} p\big).$$

Hence, by simply adding some same expressions on both sides, we get

$$\begin{aligned}\mathbb{E}(A_{n+1}^p) = \mathbb{E}\big(&A_n^p - X_n A_n^{p-1} p + X_{n+1} A_{n+1}^{p-1} p \\ &+ A_{n+1}^p - A_n^p - (P_n \Delta X_n) A_{n+1}^{p-1} p - X_n A_{n+1}^{p-1} p + X_n A_n^{p-1} p\big).\end{aligned}$$

Now inequality (1) is trivially true for $n = 0$ since $A_0 = 0$. Assume that inequality (1) holds for n by induction hypothesis. Using it and the obvious inequality $-X_n p(A_{n+1}^{p-1} - A_n^{p-1}) \leqslant 0$ we obtain from the previous equation

$$\begin{aligned}\mathbb{E}(A_{n+1}^p) &\leqslant \mathbb{E}\big(X_{n+1} A_{n+1}^{p-1} p + A_{n+1}^p - A_n^p - (P_n \Delta X_n) A_{n+1}^{p-1} p\big) \\ &\leqslant \mathbb{E}(X_{n+1} A_{n+1}^{p-1} p).\end{aligned}$$

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Here the last inequality follows from the following general estimate: If $p \geqslant 1$ and $A, D \geqslant 0$ reals, then by the mean value theorem we find a $\xi \in (A, A+D)$ such that

$$(A+D)^p - A^p = \xi^{p-1} pD \leqslant (A+D)^{p-1} pD.$$

We apply this inequality on $A := A_n$ and $D := \Delta P_n X_n$ and note that $A_{n+1} = A_n + P_n \Delta X_n$.

The first inequality of the lemma is shown. The second one follows immediately from the first one by applying Hoelder's inequality

$$\mathbb{E}(A_n^p) \leqslant p\mathbb{E}(X_n A_n^{p-1}) \leqslant p\mathbb{E}(X_n^p)^{1/p} \mathbb{E}((A_n^{p-1})^{p/(p-1)})^{1-1/p}.$$

□

Consider the lefthanded filtration $\mathcal{F}_{t-} = \overline{\bigcup_{s<t} \mathcal{F}_s}$. The corresponding conditional expectation operator we denote by

$$L_t(.) = \mathbb{E}(.\,|\,\mathcal{F}_{t-}) \quad \in B\big(L^p(\mathbb{P})\big).$$

Then we call a process $X : \mathcal{R} \to L^p(\mathbb{P})$ *predictable* if $L_t X_t = X_t$ for all $t \in \mathcal{R}$, or in other words, if X is adapted to the lefthanded filtration. A process X is *increasing* if $X_s \leqslant X_t$ (almost surely) for all $s \leqslant t \in \mathcal{R}$.

THEOREM 2.2. (Discrete decomposition.) *Let $\mathcal{R}$ be a totally ordered set, $p > 1$ real and $X : \mathcal{R} \to L^p(\Omega, \mathcal{F}, \mathbb{P})$ be a positive submartingale. Then we find functions $M, A : \mathcal{R} \to L^p(\Omega, \mathcal{F}, \mathbb{P})$ such that $X = M + A$, M is a martingale and A is predictable and increasing (in the above defined sense).*

PROOF: Consider the set of finite subsets of $\mathcal{R}$, that is,

$$\phi := \{\, u \subseteq \mathcal{R} \mid \operatorname{card}(u) < \infty \,\}.$$

Then consider the filter basis B consisting of all end pieces of ϕ, where we think of ϕ being ordered under the set inclusion, that is,

$$B = \big\{\alpha \subseteq \phi \mid \exists u_0 \in \phi : \ \alpha = \{\, u \mid u \supseteq u_0 \,\}\big\}.$$

Complete this filter basis to an ultrafilter $\mathcal{U}$ on ϕ. For fixed $t \in \mathcal{R}$ set

$$A_t : \phi \to \mathcal{H} \ : \ A_t(u) = 0 \text{ for } t \notin u \quad \text{and}$$

$$A_t(u) = \sum_{k=1}^{n} P_{t_{k-1}}(X_{t_k} - X_{t_{k-1}})$$

for $u = \{t_0 < \cdots < t_n = t < t_{n+1} < \cdots < t_m\}$. Set $M_t : \phi \to \mathcal{H} : M_t(u) = X_t - A_t(u)$. Using the previous Lemma 2.1 we get $\|A_t(u)\| \leqslant p\|X_t\|$ for $t \in u$ and trivially also for $t \notin u$. Thus

$$\|M_t(u)\| = \|X_t - A_t(u)\| \leqslant (p+1)\|X_t\|.$$

Note that the unit ball is compact under the weak topology. So we obtain weak limits along the ultrafilter, that is, set (here wlim denotes the weak limit)

$$M : \mathcal{R} \to L^p(\mathbb{P}) : M_t = \operatorname{wlim}_{\mathcal{U}} M_t(u)$$

and analogously define A. Note that the limits M_t respectively A_t are indeed in the subspace $\operatorname{Im}(P_t)$ respectively $\operatorname{Im}(L_t)$, since they are convex and norm closed and hence weakly closed. Now we get

$$X_t = \operatorname{wlim}_{\mathcal{U}} X_t - A_t(u) + A_t(u) = \operatorname{wlim}_{\mathcal{U}} M_t(u) + \operatorname{wlim}_{\mathcal{U}} A_t(u) = M_t + A_t.$$

Furthermore fix $s \leqslant t$ in $\mathcal{R}$. Note that for fixed $u \in \phi$ with $\{s,t\} \subseteq u$ we have $P_s M_t(u) = M_s(u)$. So we get

$$P_s M_t = P_s\big(\operatorname{wlim}_{\mathcal{U}} M_t(u)\big) = \operatorname{wlim}_{\mathcal{U}} P_s M_t(u) = \operatorname{wlim}_{\mathcal{U}} M_s(u) = M_s,$$

that is, M is a martingale. From the Doob-decomposition we have $A_t(u) - A_s(u) \geqslant 0$ for all $s \leqslant t$ and all $u \supseteq \{s,t\}$. Thus for all $0 \leqslant f \in L^\infty(\Omega)$ we get

$$\mathbb{E}\big((A_t - A_s)f\big) = \operatorname{wlim}_{\mathcal{U}} \mathbb{E}\big((A_t(u) - A_s(u))f\big) \geqslant 0,$$

that is, $A_t - A_s \geqslant 0$. ∎

Corollary 2.3. *Let $p > 1$ and $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a cadlag $L^p(\mathbb{P})$-valued, positive submartingale with respect to a right continuous, augmented filtration. Then we find a DM-decomposition $X = M + A$ into a cadlag $L^p(\mathbb{P})$-valued martingale $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, and a cadlag, pathwise increasing, $L^p(\mathbb{P})$-valued process $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ which is adapted to the lefthanded filtration.*

Remark. It is an open question for us, whether the process A is measurable with respect to the predictable σ-algbera (since it is adapted to the lefthanded filtration; the conjecture is relying on Lemma 3.2. Relevant to this question is also a criterion in [**2**] (or see the textbook [**5**, 22.12])).

Proof: Doob's regularisation theorem [**3**] (or see for example, [**5**, 6.27]) states that if the filtration is right continuous and X is a submartingale, then X has a cadlag version if and only if $t \mapsto \mathbb{E}(X_t)$ is right continuous.

We denote by $[X]$ the function $[X] : \mathbb{R}_+ \to L^1(\mathbb{P})$ canonically associated to X. According to 2.2 we choose a decomposition $[X] = [M] + [A]$ for certain $M, A : \mathbb{R}_+ \times \Omega \to \mathbb{R}$.

Due to Doob's regularisation, $\mathbb{E}(X_t)$ is right continuous, so we have $\mathbb{E}(A_t) = \mathbb{E}(X_t) - \mathbb{E}(M_t)$ is right continuous. Hence applying Doob's regularisation once more, we find a version A' of A which is cadlag.

The process A' is adapted to the lefthanded filtration since $L_t\big([A']_t\big) = [A']_t$ and the filtration is augmented. Furthermore, since $A'_s \leqslant A'_t$ almost everywhere ($s \leqslant t$), by a standard argument we see that A', restricted to the point of times $t \in \mathbb{Q}$, is pathwise increasing almost everywere. Hence A' itself is pathwise increasing almost everywhere, since A' is cadlag, and we can choose an everywhere increasing cadlag version A'' of A'.

Last but not least we choose M cadlag via $M = X - A''$ and we are done. ∎

Due to Theorem 2.2 we can generalise the Doléans-measure [**2**] to more general time scales than $\mathbb{R}$. This was already done in [**1**, 3.1] for the case $p = 2$. Here we can extend it now to $p > 1$ and in the proof of [**1**, 3.1] is really nothing to change, except using the above $L^p(\mathbb{P})$ variant 2.2 once where it is necessary. So we obtain

Theorem 2.4. (Doléans-measure.) *Let $\mathcal{R}$ be a totally ordered, sequentially right complete set and $p > 1$ real. Let $X : \mathcal{R} \to L^p(\Omega, \mathcal{F}, \mathbb{P})$ be a positive submartingale such that $t \mapsto \mathbb{E}(M_t)$ is sequentially right continuous. Then*

$$\mu\big(C \times (s,t]\big) = \mathbb{E}\big(1_C\,(X_t - X_s)\big)$$

for $C \in \mathcal{F}_s$ and $s < t$ in $\mathcal{R}$ defines a measure μ on the predictable σ-algebra $\mathcal{P}$.

3. The integrable decomposition

In this section let $\mathcal{R}$ be a totally ordered set which is endowed with the order topology and with a measure μ on its Borel algebra such that

(I) $\mathcal{R}$ is separable and has countable many successor elements.

(II) $\mu\big([s,t]\big) < \infty \qquad \forall s < t \in \mathcal{R}$.

We refer the reader to [**1**] for details and just note that any subset $\mathcal{R} \subseteq \mathbb{R}$ satisfies property (I). Now, the aim of this section is a Doob-Meyer-decomposition *in* the space

$$W^p := L^p(\mathcal{R} \times \Omega, \mathcal{B} \otimes \mathcal{F}, \mu \otimes \mathbb{P}).$$

The predictable σ-algebra $\mathcal{P} \subseteq \mathcal{B} \otimes \mathcal{F}$ let be induced by the predictable rectangles

$$\big\{(s,t] \times C \mid s < t \in \mathcal{R}, C \in \mathcal{F}_s\big\}.$$

We call then $L^p(\mathcal{R} \times \Omega, \mathcal{P}, \mu \otimes \mathbb{P}) \subseteq W^p$ the *space of predictable functions* in W^p.

The subsequent lemmas prepare the proof of the decomposition 3.4 and they are $L^p(\mathbb{P})$-variants of similar lemmas in [**1**].

Lemma 3.1. *Let $\mathcal{R}$ be a totally ordered set and $(\mathcal{F}_t)_{t\in\mathcal{R}}$ a filtration. We consider $\mathcal{F} = \bigcap_{t\in\mathcal{R}} \mathcal{F}_t$ and the conditional expectation $P(f) = \mathbb{E}(f \mid \mathcal{F})$.*

1. *For all $p \geqslant 1$ the operators P_t converges to P $(t \downarrow)$ in the weak operator topology in $B\big(L^p(\mathbb{P})\big)$.*
2. *If $1 \leqslant p \leqslant 2$, then P_t converges even strongly, that is, $\forall f \in L^p(\mathbb{P})$ $\lim_{t\downarrow} \big\|(P - P_t)f\big\|_{L^p(\mathbb{P})} = 0$.*

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PROOF: We denote the L^2-variant of P_t by $Q_t(.) = \mathbb{E}(. \mid \mathcal{F}_t) \in B(L^2(\mathbb{P}))$ and $Q(.) = \mathbb{E}(. \mid \mathcal{F}) \in B(L^2(\mathbb{P}))$. It is easy to check and shown in [1, 3.2] that

$$L^2(\Omega, \bigcap_{s\in\mathcal{R}} \mathcal{F}_s, \mathbb{P}) = \bigcap_{s\in\mathcal{R}} L^2(\Omega, \mathcal{F}_s, \mathbb{P}).$$

This and [1, 2.1] shows that $Q_t \to Q$ in the strong operator topology in $B(L^2(\mathbb{P}))$. Thus for $x, y \in L^\infty(\mathbb{P})$ we have

$$\langle (Q - Q_t)x, y\rangle = \langle (P - P_t)x, y\rangle \to 0 \quad (t \downarrow)$$

and it follows $P_t \to P$ $(t \downarrow)$ in the weak operator topology in $B(L^p(\mathbb{P}))$ by an immediate estimate.

If $1 \leqslant p \leqslant 2$, then we can sharpen this result: For arbitrary $\varepsilon > 0$ and $y \in L^p(\mathbb{P})$ we choose $x \in L^2(\mathbb{P})$ such that $\|x - y\|_{L^p(\mathbb{P})} \leqslant \varepsilon$ and $t_0 \in \mathcal{R}$ small enough such that for all $t \leqslant t_0$ we have

$$\begin{aligned}\|(P - P_t)y\|_{L^p(\mathbb{P})} &\leqslant \|(P - P_t)(y - x)\|_{L^p(\mathbb{P})} + \|(P - P_t)x\|_{L^p(\mathbb{P})} \\ &\leqslant 2\|x - y\|_{L^p(\mathbb{P})} + \|(P - P_t)x\|_{L^2(\mathbb{P})} \leqslant 3\varepsilon.\end{aligned}$$

∎

LEMMA 3.2. *Let $\mathcal{R}$ be a totally ordered set with measure μ and assume* (I)–(II). *Let $p \geqslant 1$ be real. Then we have a projection $L \in B(W^p)$ onto the predictable functions via*

$$L(X)(t) = L_t X_t \qquad X \in W^p,\ t \in \mathcal{R}.$$

If the filtration $(\mathcal{F}_t)_{t\in\mathcal{R}}$ is right continuous then we have a projection $P \in B(W_p)$ onto the adapted functions via

$$P(X)(t) = P_t X_t \qquad X \in W^p,\ t \in \mathcal{R}.$$

REMARK. Note that it follows from this lemma, that the space of predictable functions are just the processes $X \in W^p$ which are adapted to the filtration $\mathcal{F}_{t-}$ from the left.

PROOF: We have canonically

$$L^p(\mathcal{R} \times \Omega, \mathcal{B} \otimes \mathcal{F}, \mu \otimes \mathbb{P}) = L^p(\mathcal{R}, \mu, L^p(\Omega, \mathcal{F}, \mathbb{P})).$$

Considering here the right space, we see that the space of predictable functions is just the closure of the span

$$\{1_{(s,t]} f \mid s < t \in \mathcal{R}, f \in L^p(\Omega, \mathcal{F}_s, \mathbb{P})\}.$$

This is conform with the definition of the space of predictable functions for $p = 2$ in [1]. Hence we can apply lemma [1, 2.2], where the projection L for $p = 2$ is considered (and proved).

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More precisely, for bounded functions $X \in W^p$ the function LX is measurable (since in W^2) and indeed in W^p since

$$\|LX\|_{W^p}^p = \int \|L_t X_t\|_{L^p(\mathbb{P})}^p d\mu(t) \leqslant \|X\|_{W^p}.$$

Next we just extend L continuously on W^p and we easily check that L is a projection onto the predictable functions.

In the same fashion we may use Lemma [**1**, 2.3] for the assertion about the projection onto the adapted functions. ∎

Lemma 3.3. *Let $\mathcal{R}$ be a totally ordered set with property* (I) *and $Y \subseteq \mathcal{R}$. Then Y itself, as a totally ordered set, fulfills* (I).

Proof: Let $D \subseteq \mathcal{R}$ be countable, dense and contain all successor and predecessor elements of $\mathcal{R}$. For all $d \leqslant f \in D$ choose any $x \in [d, f] \cap Y$ if nonempty. The set of all that x forms a dense countable set X in Y. Indeed if $a < y < b$ are elements of Y then choose $d, f \in D$ with $a < d \leqslant y \leqslant f < b$. Then we find $x \in X$ such that $x \in [d, f] \cap Y \subseteq (a, b) \cap Y$.

Assume that Y has uncountable many successor elements $(s_i)_{i \in I} \subseteq Y$. Let $p_i \in Y$ be the predecessor of s_i in Y. Then (p_i, s_i), $i \in I$, forms disjoint open sets in $\mathcal{R}$ where uncountable of them are nonempty and $\mathcal{R}$ were not separable. ∎

We say a process $X \in W^p$ is *weakly right continuous* if $X : \mathcal{R} \to L^p(\mathbb{P})$ restricted to $\mathcal{R} \backslash N$ is right continuous with respect to the weak topology in $L^p(\mathbb{P})$ for some μ-nullset N. See also [**1**] for right continuity et cetera.

Say $X \in W^p$ is a *(sub)martingale* if this property holds for X restricted to $\mathcal{R} \backslash N$ in the common sense for some μ-nullset N; futhermore X is *increasing* if $X_s \leqslant X_t$ almost surely for all $s \leqslant t \in \mathcal{R} \backslash N$.

Theorem 3.4. (Integrable decomposition.) *Let $\mathcal{R}$ be a totally ordered set with finite measure μ and assume* (I). *Let the filtration $(\mathcal{F}_t)_{t \in \mathcal{R}}$ be right continuous. Let $p > 1$ and $X \in W^p$ be a weakly right continuous positive submartingale.*

Then we find weakly right continuous $M, A \in W^p$ such that $X = M + A$, M is a martingale and A is predictable and increasing.

Proof: The proof goes through as the proof of [**1**, Theorem 2.4] (which is divided into steps) beside minor and obvious adaption. We shall briefly discuss differences: At first reduce $\mathcal{R}$ to $\mathcal{R}' := \mathcal{R} \backslash N$, $\mu(N) = 0$, such that X is a weakly right continuous submartingale on $\mathcal{R}'$, see also Lemma 3.3.

Step 1a. Nothing to change.

Step 1b. In [**1**] the assumption

$$\sum_k \|P_{t_{k-1}} X_{t_k}\|_{L^2}^2 - \|X_{t_{k-1}}\|_{L^2}^2 \geqslant C$$

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appears. Indeed we have $C = 0$ since $\|P_s X_t\|^p_{L^p(\mathbb{P})} - \|X_s\|^p_{L^p(\mathbb{P})} \geqslant 0$ for $\forall s \leqslant t$, what is obvious since X is a positive submartingale.

STEP 2. Here the projection onto the adapted functions is used. We use here the $L^p(\mathbb{P})$-version Lemma 3.2. Furthermore we replace the estimate on the norm of the Doob-decomposition [**1**, 1.1] by the $L^p(\mathbb{P})$-variant Lemma 2.1. The constructed sequence $(A_n, M_n)_{n\geqslant 1}$ in $W^p \oplus W^p$ has then an accumulation point (M, A) with respect to the weak topology. So given some $f \in L^p(\mathbb{P})^*$ we find a subsequence such that $\lim_k < M_{n_k} + A_{n_k} - M - A, f >= 0$.

STEP 3. To see that the martingale is weakly right continuous, we replace [**1**, Remark 2.1] by the analogous $L^p(\mathbb{P})$-version 3.1. ∎

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