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Cohomology of Subregular Tilting Modules for Small Quantum Groups

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Abstract. Let U be a quantum group with divided powers at root of unity constructed from a root system R. Let $u \subset U$ be the small quantum group. The cohomology of u with trivial coefficients was computed by Ginzburg and Kumar. It turns out to be isomorphic to the functions algebra of the nilpotent cone of a semisimple algebraic group with root system R. In this note we calculate cohomology of u with coefficients in simplest reducible tilting module with nontrivial cohomology. It appears to be isomorphic to the functions algebra of the closure of the subregular nilpotent orbit.

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1. Introduction

Let R be an irreducible root system with the Coxeter number h. Let $l > h$ be an odd integer (we assume that l is not divisible by 3 if R is of type G_2). Let U be the quantum group of type 1 with divided powers associated to these data, see [10] (of type 1 means that the elements K_i^l are equal to 1). Let $u \subset U$ be the Frobenius kernel, see loc. cit. Let $\mathbf{1}$ be the trivial U-module. The cohomology $H^\bullet(u, \mathbf{1})$ was computed by Ginzburg and Kumar in [5], see also [8]. They proved that the odd cohomology $H^{\mathrm{odd}}(u, \mathbf{1})$ vanishes and the algebra of even cohomology $H^{2\bullet}(u, \mathbf{1})$ is isomorphic to the algebra $\mathbb{C}[\mathcal{N}]$ of functions on the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$, where $\mathfrak{g}$ is the semisimple Lie algebra associated to R. Moreover, this is an isomorphism of graded algebras with the grading on $\mathbb{C}[\mathcal{N}]$ corresponding to the natural $\mathbb{C}^*$-action on $\mathcal{N}$ by dilatations. This isomorphism is compatible with natural G-structures of both algebras where G is simply connected group associated to R.

Now let s_a be the simple affine reflection lying in the affine Weyl group associated to R, l, see, e.g., [2]. Let Θ_{s_a} be the corresponding wall-crossing functor, see, e.g., [12]. Let $T = \Theta_{s_a}\mathbf{1}$. It is easy to see that cohomology $H^\bullet(u, T)$ has a natural algebra structure; namely for any simple U-module L with highest weight lying on the affine

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wall of the fundamental alcove we have $H^\bullet(u, T) = \mathrm{Ext}^\bullet_u(L, L)$. Since T is a U-module the cohomology $H^\bullet(u, T)$ has a natural structure of G-module. Let $\mathcal{O} \subset \mathcal{N}$ be the subregular nilpotent orbit. The main result of this note is the following theorem:

MAIN THEOREM. *The odd cohomology $H^{\mathrm{odd}}(u, T)$ vanishes. The algebra $H^{2\bullet}(u, T)$ is isomorphic to the algebra $\mathbb{C}[\overline{\mathcal{O}}]$ of functions on the closure of $\mathcal{O}$. This is an isomorphism of graded algebras with the grading on $\mathbb{C}[\overline{\mathcal{O}}]$ corresponding to the action of $\mathbb{C}^*$ by dilatations. This isomorphism is compatible with natural G-structures of both algebras.*

Remark. One can prove the analogous theorem for the Frobenius kernel G_1 of an almost simple algebraic group G over an algebraically closed field of characteristic $p > h$.

We remark that $\mathbb{C}[\overline{\mathcal{O}}] = \mathbb{C}[\mathcal{O}]$ because of the normality of $\overline{\mathcal{O}}$, see [4, 9].

In [6], Hesselink computed the structure of $\mathbb{C}[\mathcal{N}]$ as graded G-module. It is easy to deduce the Hesselink theorem from the Ginzburg–Kumar Theorem (or rather from the Andersen–Jantzen vanishing Theorem, see [1]). In the same way we are able to compute the structure of $\mathbb{C}[\overline{\mathcal{O}}]$ as graded G-module, see Corollary 3 below.

For any dominant weight λ one defines the indecomposable tilting module $T(\lambda)$ with highest weight λ, see, e.g., [3]. For some time I believed that the cohomology of any $T(\lambda)$ has a parity vanishing property. In fact, this belief was the main motivation for this work. At the end of this note, I give an example when the cohomology of an indecomposable tilting module lives in both even and odd degrees.

Finally, I would like to mention that our Main Theorem is a particular case of recent results of R. Bezrukavnikov (private communication).

2. Proof of the Main Theorem

Recall that T has a unique trivial submodule $\mathbf{1}$ and $T/\mathbf{1} = H^0(s_a \cdot 0)$, see, e.g., [3]. Let $\phi : T \to H^0(s_a \cdot 0)$ be the quotient map.

LEMMA 1. *The map $\phi_* : H^\bullet(u, T) \to H^\bullet(u, H^0(s_a \cdot 0))$ is zero.*

Proof. The map ϕ_* is a map of $H^{2\bullet}(u, \mathbf{1}) = \mathbb{C}[\mathcal{N}]$- modules. It is known that the support of $H^\bullet(u, T)$ in $\mathcal{N}$ is equal to $\overline{\mathcal{O}}$, see [7, 11].

The cohomology $H^\bullet(u, H^0(s_a \cdot 0))$ was computed by Andersen and Jantzen in [1], 3.7. We reformulate their result as follows:

(a) Let π: $T^*(G/B) \to G/B$ be the cotangent bundle of the flag variety of the group G. Let s: $T^*(G/B) \to \mathcal{N}$ be the Springer resolution. Let L_θ be the line bundle on G/B corresponding to the root θ dual to the highest coroot of $\mathfrak{g}$ (more directly θ is the unique dominant short root). Then the even cohomology $H^{\mathrm{ev}}(u, H^0(s_a \cdot 0))$ vanishes;

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the odd cohomology is equal up to shift to $s_*\pi^*L_\theta$ (if we consider the cohomology as a coherent sheaf on $\mathcal{N}$).

In particular, if ϕ_* is nontrivial we obtain a section of the line bundle π^*L_θ supported on $s^{-1}(\overline{\mathcal{O}})$. Contradiction.

Remark. In fact, Andersen and Jantzen computed the cohomology of induced modules of an algebraic group over a field of characteristic $p > 0$. But their proof works in the quantum situation as well if we know some vanishing result. This vanishing theorem was proved in [1] in types A, B, C, D, G or for strongly dominant weights. In our case the weight θ is not strongly dominant. Broer proved the desired vanishing in case of characteristic 0 in [4]. In a recent work [9], all restrictions in the Andersen–Jantzen vanishing theorem were removed. This should be used in the above-mentioned generalization of our Main Theorem to characteristic p.

COROLLARY 1. *The odd cohomology $H^{\text{odd}}(u, T)$ vanishes. For any $i \geqslant 0$ we have an exact sequence*

$$0 \to H^{2i-1}(u, H^0(s_a \cdot 0)) \to H^{2i}(u, \mathbf{1}) \to H^{2i}(u, T) \to 0.$$

In particular, the natural map $H^\bullet(u, \mathbf{1}) \to H^\bullet(u, T)$ is surjective.

Proof. This follows easily from consideration of the cohomology long exact sequence associated with the short exact sequence

$$0 \to \mathbf{1} \to T \to H^0(s_a \cdot 0) \to 0.$$

Proof of the Main Theorem. The surjectivity of the map $\mathbb{C}[\mathcal{N}] = H^{2\bullet}(u, \mathbf{1}) \to H^{2\bullet}(u, T)$ implies that there exists a surjection $\psi: H^{2\bullet}(u, T) \to \mathbb{C}[\overline{\mathcal{O}}]$. Let $L(\lambda)$ be the simple G-module with highest weight λ. For any weight μ let $m_\lambda(\mu)$ be the multiplicity of the weight μ in $L(\lambda)$. It is known that the multiplicity of $L(\lambda)$ in $\mathbb{C}[\overline{\mathcal{O}}]$ is equal to $m_\lambda(0) - m_\lambda(\theta)$, see [4] 4.7. It is easy to deduce from Corollary 1 and (a) that the multiplicity of $L(\lambda)$ in $H^\bullet(u, T)$ also equals $m_\lambda(0) - m_\lambda(\theta)$ (we omit the proof since it is the same as the proof of Corollary 3 below). Hence, ψ is an isomorphism. The Theorem is proved.

Let $V = V(s_a \cdot 0)$ be the Weyl module with highest weight $s_a \cdot 0$.

COROLLARY 2. *The cohomology $H^\bullet(u, V)$ is given by*

$$H^{2i}(u, V) = H^{2i}(u, T), \qquad H^{2i+1}(u, V) = H^{2i}(u, \mathbf{1}).$$

Proof. It is enough to consider the cohomology long exact sequence associated with the short exact sequence

$$0 \to V \to T \to \mathbf{1} \to 0$$

and note that the map $H^\bullet(u, T) \to H^\bullet(u, \mathbf{1})$ is zero (this can be proved in the same way as Lemma 1).

Remark. One can easily compute the cohomology of the simple module $\mathbf{L} = \mathbf{L}(s_a \cdot 0)$ with highest weight $s_a \cdot 0$ using the short exact sequence

$$0 \to \mathbf{L} \to H^0(s_a \cdot 0) \to \mathbf{1} \to 0.$$

The answer is the following: $H^{2\bullet}(u, \mathbf{L}) = 0$ and for any $i \geqslant 0$ we have short exact sequence

$$0 \to H^{2i}(u, \mathbf{1}) \to H^{2i+1}(u, \mathbf{L}) \to H^{2i+1}(u, H^0(s_a \cdot 0)) \to 0.$$

Let R_+ be the set of positive roots and let W be the Weyl group. For any $w \in W$ let $(-1)^w = \det(w)$. Let ρ be the halfsum of positive roots. Let $w \cdot \lambda = w(\lambda + \rho) - \rho$. For any dominant weight λ, let $d_n(\lambda)$ (resp. $t_n(\lambda)$) be the multiplicity of the simple module $L(\lambda)$ in the component of degree n of $\mathbb{C}[\mathcal{N}]$ (resp. $\mathbb{C}[\overline{\mathcal{O}}]$). Let p_n be the function on the set X of weights, given by

$$\sum_{x\in X}\sum_{n\in\mathbb{Z}} p_n(x)t^n e^x = \prod_{\alpha\in R_+}\frac{1}{1-e^\alpha t}.$$

This function is essentially the Kostant–Lusztig partition function. Recall that Hesselink's theorem ([6]) states that $d_n(\lambda) = \sum_{w\in W}(-1)^w p_n(w \cdot \lambda)$. Let $2k-1$ be the length of reflection in θ.

COROLLARY 3 (cf. [4] 4.7). *We have*

$$t_n(\lambda) = \sum_{w\in W}(-1)^w (p_n(w \cdot \lambda) - p_{n-k}(w \cdot \lambda - \theta)).$$

Remark. (i) For types $A_l, B_l, C_l (l \geqslant 2), D_l (l \geqslant 3), G_2, F_4, E_6, E_7, E_8$ the number k equals to, respectively, $l, l, 2(l-1), 2l-3, 3, 8, 11, 17, 29$.

(ii) (J.Humphreys) Let $R^\vee$ be a root system dual to R. Wang proved (see [13]) that the number $k+1$ is equal to *the dual Coxeter number* $h^\vee(R^\vee)$ of the root system $R^\vee$ and the number $2k$ is equal to the dimension of a minimal nilpotent orbit of the group $G^\vee$ *Langlands dual* to the group G. It would be interesting to find an explanation of this connection.

Proof. Let B be the Borel subgroup of G. Let n be the nilpotent radical of the Borel subalgebra in $\mathfrak{g}$. Let $S^\bullet(n^*)$ be the algebra of functions on n. By [1, 4] we have

$$H^{2i}(u, \mathbf{1}) = \mathrm{Ind}_B^G(S^i(n^*)),\ R^{>0}\mathrm{Ind}_B^G(S^i(n^*)) = 0,$$
$$H^{2i-1}(u, H^0(s_a \cdot 0)) = \mathrm{Ind}_B^G(S^{i-k}(n^*) \otimes \theta),\ R^{>0}\mathrm{Ind}_B^G(S^{i-k}(n^*) \otimes \theta) = 0.$$

Now the Euler characteristic of $R^\bullet\mathrm{Ind}_B^G(?)$ is given by the Weyl character formula. The result follows.

EXAMPLE. Here we present an example when cohomology (over Frobenius kernel) of indecomposable tilting module lives in both odd and even degrees. Let R be of type A_2. Let s_1, s_2 be the simple reflections in Weyl group, and let s_0 be the affine reflection. Consider indecomposable tilting module $T = T(s_0 s_1 s_2 s_0 \cdot 0)$. It has a filtration with subquotients $H^0(s_0 s_1 s_2 s_0 \cdot 0)$, $H^0(s_0 s_1 s_2 \cdot 0)$, $H^0(s_0 \cdot 0)$ and $H^0(0)$. Let ω_1 and ω_2 be the fundamental weights. We have $s_0 s_1 s_2 s_0 \cdot 0 = (3l - 3)\omega_2$. By the Andersen–Jantzen theorem, the cohomology of $H^0(s_0 s_1 s_2 s_0 \cdot 0)$ equals to $\mathrm{Ind}_B^G(3\omega_2 \otimes S^\bullet(n^*))$ living in even degrees, the cohomology of $H^0(s_0 s_1 s_2 \cdot 0)$ or $H^0(s_0 \cdot 0)$ equals to $\mathrm{Ind}_B^G((\omega_1 + \omega_2) \otimes S^\bullet(n^*))$ living in odd degrees, finally the cohomology of $H^0(0)$ equals to $\mathrm{Ind}_B^G(S^\bullet(n^*))$ living in even degrees. Using the Kostant multiplicity formula, we obtain that multiplicity of $L(\lambda)$ in Euler characteristic of cohomology of T equals to $m_\lambda(3\omega_2) + m_\lambda(0) - 2m_\lambda(\omega_1 + \omega_2)$. In particular, multiplicity of $L(0)$ equals to 1 and multiplicity of $L(3\omega_1)$ equals to -1. This contradicts the parity vanishing.

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