Can. J. Math., Vol. XXV, No. 4, 1973, pp. 842–855

ALGEBRAS OVER DEDEKIND DOMAINS

JOSEPH A. WEHLEN

Introduction. The purpose of this paper is two-fold: first, to show that Dedekind domains satisfy a generalization of the Wedderburn-Mal'cev Theorem and, secondly, to classify certain types of finitely generated projective algebras over a Dedekind domain.

With respect to the first problem, E. C. Ingraham has shown that a Dedekind domain R is an *inertial coefficient* ring (IC-ring) if and only if R has zero radical or R is a local Hensel ring. (R is an IC-ring in case R-algebras satisfy the Wedderburn Principal Theorem: If A is an R-algebra, finitely generated as an R-module, and A modulo its Jacobson radical is R-separable, then A is the sum—not necessarily direct—of that radical and some separable subalgebra of A.) We show that every Dedekind domain is a *weak inertial coefficient* ring (WIC-ring), i.e., any finitely generated algebra A over R with A modulo its Baer lower radical $L(A)$ separable over R satisfies the condition that A contains a separable subalgebra S such that $A = S + L(A)$. The idea was suggested by the fact (cf.[**10**, Proposition 3.15]) that for a finitely generated algebra A over a Dedekind domain R with zero radical, the Jacobson radical of A equals the lower radical of A.

In Section 1 we will deal with general properties of the lower radical of an algebra and in particular of separable algebras. Section 2 will develop the concept of the WIC-ring.

Secondly, we show in Section 3 that a finitely generated, projective algebra A over a Dedekind domain R is a residue algebra of a less than or equal to one-dimensional algebra if $A/L(A)$ is R-separable and A satisfies the idempotent conditions of S. U. Chase [**3**, Definition 4.1]. We further show that all such algebras are generalized triangular matrix algebras. This generalizes results of S. U. Chase and M. Harada.

Conventions. All rings have an identity and all ring homomorphisms and modules are unitary. We say that A is a finitely generated, projective R-algebra whenever A is an R-algebra which is finitely generated and projective as a module over R. N will denote the Jacobson radical of A and $L(A)$ will denote the Baer lower radical (equivalently, the prime radical) of the algebra A. The Hochschild dimension of the R-algebra A, R-dim A, is the left projective dimension of A as an $A^e = A \otimes_R A^{op}$ – module. R will always denote a commutative ring. All ideals will be two-sided unless otherwise specified.

Received April 10, 1972 and in revised form, October 5, 1972.

1. The lower radical of an algebra. We begin by listing several properties of the (Baer) lower radical $L(A)$ of a finitely generated algebra A. This involves several properties of prime ideals in algebras. (We will use here the fact that the lower radical is the intersection of the prime ideals of the ring [**6**, p. 61].) We list several general results on primes which are contained in Curtis [**4**].

THEOREM 1.1. *Let A be a finitely generated, faithful algebra over a Noetherian ring R. Then the following are true:*

(a) *If $\mathbf{P}$ is a prime in A, then $\mathbf{P} \cap R$ is a prime in R.*

(b) *If $\mathbf{p}$ is a prime in R, then there is a prime $\mathbf{P}$ in A such that $\mathbf{P} \cap R = \mathbf{p}$.*

(c) *If $\mathbf{p}$ is maximal in R, then any prime $\mathbf{P}$ such that $\mathbf{P} \cap R = \mathbf{p}$ is maximal in A and conversely.*

The proof follows by a careful application of the fact that R is in the center of A to the usual proofs of the Cohen-Seidenberg theory.

THEOREM 1.2. *The lower radical of any Noetherian ring is the largest nilpotent ideal of the ring.*

The proof of this theorem may be found in Divinsky [**6**, p. 53]. This fact makes the lower radical of an algebra over a Noetherian ring function in a manner analogous to the Jacobson radical of an algebra over a field. The following simple example indicates one area in which this is true: (Recall that an R-algebra A is R-separable if and only if R-dim $A = 0$.)

PROPOSITION 1.3. *Let R be a Noetherian domain. Let A be a finitely generated, torsion-free, separable R-algebra. Then $L(A) = (0)$.*

Proof. Since $L(A) \otimes Q$ is a nilpotent ideal in the separable Q-algebra $A \otimes Q$, $L(A) = (0)$, where Q denotes $R_{(0)}$.

PROPOSITION 1.4. *Let R be a Noetherian ring and A be an R-algebra.*

(a) *If A is semi-prime and is faithful as an R-module, then R is semi-prime.*

(b) *If R is semi-prime and A is a projective, separable, and finitely generated R-algebra, then A is semi-prime.*

Proof. For (a), we merely notice that since $L(R)A$ is a nilpotent ideal in A, it is contained in $L(A)$. Since A is faithful, $L(A)$ thus contains an isomorphic copy of $L(R)$. Hence $L(R) = (0)$.

For (b), if one assumes that A is R-free, then it is clear that $L(R)A = \cap (\mathbf{p}A)$. For each projective algebra A, there is a free R-algebra B containing A as an R-direct summand. Thus it follows that $L(R)A = \cap (\mathbf{p}A)$ for every finitely generated, projective R-algebra A.

Finally, one notices that $A/\mathbf{p}A$ is a finitely generated, projective (hence, torsion-free), separable algebra over the Noetherian domain $R/\mathbf{p}$ for every prime $\mathbf{p}$ of R. Hence $L(A) \subseteq (\mathbf{p}A)$ for each prime $\mathbf{p}$ by 1.3 and so $L(A) \subseteq \cap (\mathbf{p}A)$.

https://doi.org/10.4153/CJM-1973-087-8 Published online by Cambridge University Press

Hence $L(A) = L(R)A$; therefore $L(A) = (0)$, and so A is semi-prime.

It would be well to note that in 1.4(a), some condition is needed on the structure of A as an R-module. For if $R = \mathbf{Z}/4\mathbf{Z}$ ($\mathbf{Z}$ denotes the ring of rational integers) and A = the ring of 2×2-matrices over $Z/2\mathbf{Z}$, then A is semi-prime while R has lower radical $2\mathbf{Z}/4\mathbf{Z}$. Another sufficient hypothesis for A instead of faithful over R is that A be finitely generated, projective, and separable over R.

THEOREM 1.5. *Let A be an R-algebra.*

(a) *If A is a prime ring and faithful as an R-module, then R is a prime ring (i.e., a domain).*

(b) *If A is central separable and R is a prime ring, then so is A.*

(c) *If A is a finitely generated, projective, separable R-algebra and R is a Noetherian integrally closed domain, then A is a finite direct sum of prime rings.*

Proof. (a) If A is a prime ring and $\mathbf{a}$ and $\mathbf{b}$ are ideals in R with $\mathbf{ab} = 0$, but neither $\mathbf{a}$ nor $\mathbf{b}$ are zero, then $\mathbf{ab}A = \mathbf{a}A\mathbf{b}A = 0$. Since this is impossible, R is a prime ring.

(b) Suppose that B and C are two ideals in A such that $BC = 0$ but neither B nor C are zero. By the Morita theorems, [**5**, p. 54] $(B \cap R)$ and $(C \cap R)$ are two non-zero ideals in R such that $(B \cap R)(C \cap R) = 0$ which is impossible. Hence A is a prime ring.

(c) The center of A is finitely generated, projective (hence torsion-free) and separable over R. Under these conditions, Janusz [**13**, Theorem 4.3] proves that this center is a direct sum of domains. Therefore A is a direct sum of central separable algebras over domains. By (b), each of these components is a prime ring.

2. Weak inertial coefficient rings.

Definition 2.1. (a) A subalgebra S of an R-algebra A is called a weak inertial subalgebra if S is R-separable and $S + L(A) = A$ (sum not necessarily direct).

(b) A commutative ring R is called a weak inertial coefficient ring (a WIC-ring) if every finitely generated R-algebra A such that $A/L(A)$ is R-separable contains a weak inertial subalgebra. We say the uniqueness statement holds for R if, for any two weak inertial subalgebras S and S' of a finitely generated R-algebra A, there exists an m in $L(A)$ such that $(1 - m)S(1 - m)^{-1} = S'$.

(The IC-uniqueness statement requires that if S and S' are inertial subalgebras of A, then there is an m in the Jacobson radical of A such that $(1 - m)S(1 - m)^{-1} = S'$.)

We state several results without proof as they are easy generalizations of results contained in [**10**].

PROPOSITION 2.2. (a) *$R_1 \oplus R_2$ is a WIC-ring if and only if both R_1 and R_2 are WIC-rings.*

https://doi.org/10.4153/CJM-1973-087-8 Published online by Cambridge University Press

(b) *If R is a WIC-ring, then any finitely generated, commutative R-algebra S with $S/L(S)$ separable over R is a WIC-ring.*

(c) *Any homomorphic image of a WIC-ring is a WIC-ring.*

The proof follows exactly as in [**10**, Propositions 3.2 and 3.3] after noting that for $A = A_1 \oplus A_2$, $L(A) = L(A_1) \oplus L(A_2)$.

The relation between IC-rings and WIC-rings is given by:

PROPOSITION 2.3. *Every inertial coefficient ring for which the uniqueness statement holds is a WIC-ring for which the WIC-uniqueness statement holds. Hence, every Dedekind domain with zero radical (e.g., every field, the rational integers) and every semi-local ring which is a direct sum of local Hensel rings (e.g., every complete semi-local ring) is a WIC-ring.*

The proof is immediate from [**11**, Corollary to Proposition 1] and the fact that $L(A)$ is a quasi-regular ideal.

PROPOSITION 2.4. *Let A be a finitely generated R-algebra with $A/L(A)$ projective over R. If S is a weak inertial subalgebra of A, $A = S \oplus L(A)$ as R-modules.*

Proof. $A/L(A) \simeq S/(S \cap L(A))$. Therefore $S/(S \cap L(A))$ is R-projective and hence S-projective [**5**, Proposition 2.3, p. 48]. We then have that $S = (S \cap L(A)) \oplus T$ for some left ideal T of S. So $S \cap L(A)$ is an idempotent-generated ideal contained in the radical. Hence $S \cap L(A) = 0$.

Finally, E. C. Ingraham has shown the following result in [**11**, Theorem A].

PROPOSITION 2.5. *Let A be a finitely generated, commutative algebra over a Noetherian ring R with $A/L(A)$ R-separable. Then A contains a weak inertial subalgebra which is unique.*

THEOREM 2.6. *A Noetherian, integrally closed domain is a WIC-ring if and only if every proper homomorphic image is a WIC-ring.*

COROLLARY 2.6.1. *Every Dedekind domain is a WIC-ring.*

The corollary is easily proved by noting that every proper homomorphic image of a Dedekind domain is a direct sum of complete local rings and then by appealing to 2.3.

Before proceeding to the proof of Theorem 2.6, we must first obtain two lemmas.

LEMMA 2.7. *If A is a finitely generated, torsion-free algebra over a Noetherian domain R, then $A/L(A)$ is torsion-free.*

Proof. Suppose $ra \equiv 0 \bmod L(A)$ and $r \neq 0$. Then (ra) is a non-zero nilpotent ideal in A. But A is torsion-free. Hence (a) is nilpotent; whence $a \equiv 0 \bmod L(A)$.

LEMMA 2.8. *Let R be a Noetherian, integrally closed domain and let A be a*

https://doi.org/10.4153/CJM-1973-087-8 Published online by Cambridge University Press

finitely generated R-algebra with $A/L(A)$ separable over R. There exist mutually orthogonal idempotents e_P and e_T such that

$$A = e_PAe_P \oplus e_PAe_T \oplus e_TAe_P \oplus e_TAe_T.$$

Proof. Let C denote the center of $A/L(A)$, and $t(C)$ its torsion submodule. Then $C/t(C)$ is a torsion-free, finitely generated, separable R-algebra. Hence $C/t(C)$ is R-projective by [**1**, Proposition 4.3]. Thus there are central idempotents in C, $\bar{e}_P$ and $\bar{e}_T$, such that $C = C\bar{e}_P \oplus C\bar{e}_T$ and hence $A/L(A) = A/L(A)\bar{e}_P \oplus A/L(A)\bar{e}_T$. Then the idempotents can be lifted since $L(A)$ is nilpotent.

Proof of Theorem 2.6. Assume that every proper homomorphic image of R is a WIC-ring.

Let A be a finitely generated algebra over the Noetherian integrally closed domain R with the property that $A/L(A)$ is R-separable. By 2.8 there exist orthogonal idempotents e_T and e_P in A with $e_T + e_P = 1$, $\psi(e_T) = \bar{e}_T$ and $\psi(e_P) = \bar{e}_P$, where ψ is the natural homomorphism.

Since $e_TAe_P \oplus e_PAe_T \subseteq L(A)$, we need only consider the rings e_TAe_T and e_PAe_P. If we denote by ψ_P the homomorphism of e_PAe_P onto $(A/L(A))e_P$ induced by the restriction of ψ, then

$$\text{kernel}(\psi_P) = L(e_PAe_P) = e_PL(A)e_P.$$

The analogous statement holds for e_TAe_T.

Consider e_PAe_P. It is a finitely generated algebra with the algebra modulo its lower radical isomorphic to $(A/L(A))\bar{e}_P$ [**12**, Proposition 1, p. 48]. $(A/L(A))\bar{e}_P$ is separable and projective since a central separable algebra is projective over its center [**5**, Proposition 2.3] and the center is projective over R [**1**, Proposition 4.3]. Hence by [**10**, Theorem 3.13; **9**, Proposition 6.1] e_PAe_P contains a weak inertial subalgebra S_P.

For e_TAe_T, we can easily see that $e_TAe_T \subseteq t(A)$. Hence we can consider e_TAe_T as a finitely generated $R/\mathbf{a}$-algebra where $\mathbf{a} = \text{annih}_R\ (t(A)) \neq 0$. Moreover, $e_TAe_T/e_TL(A)e_T$ is $R/\mathbf{a}$-separable. But $R/\mathbf{a}$, being a proper homomorphic image of R, is a WIC-ring by hypothesis; so e_TAe_T contains a weak inertial subalgebra S_T.

It is clear that $S_P \oplus S_T = S$ is a weak inertial subalgebra. It only remains to show that the uniqueness statement holds for R. After noting that for any two weak inertial subalgebras S_P and S_P' of e_PAe_P, $L(S_P) = L(S_P') = 0$ by the proof of 1.4, we will omit the proof of this statement since it is identical with that of [**10**, Proposition 3.16].

A minor modification in the statement and proof of [**11**, Theorem B] yields an interesting result concerning WIC-rings. It gives a class of Noetherian WIC-rings which are not necessarily local, are not domains, but which are complete in their $L(R)$-adic topology.

THEOREM 2.9. *Let R be any commutative Noetherian ring and* $\mathbf{a}$ *an ideal contained in the lower radical of R. R is a WIC-ring if and only if $R/\mathbf{a}$ is a WIC-ring.*

COROLLARY 2.9.1. *Let R be a Noetherian ring, $f(X)$ a separable polynomial in $R[X]$, and n a natural number. $R[X]/(f(X))^n$ is a WIC-ring if and only if R is a WIC-ring. In particular, if R is a Dedekind domain, $R[X]/(f(X))^n$ is a WIC-ring for each separable polynomial $f(X)$ and for each natural number n.*

The author knows of no Noetherian ring which is not a *WIC*-ring.

3. Triangular algebras over a Dedekind domain. In this section, we wish to classify algebras over a Dedekind domain which satisfy the following definition which is a generalization of a definition given by S. U. Chase [**3**, Definition 4.1].

Definition 3.1. A is triangular if and only if every complete set of mutually orthogonal primitive idempotents $e_1, \ldots, e_n$ can be indexed so that $e_iL(A)e_j = 0$ whenever $i \geqq j$.

The goal of this section shall be to prove the following theorem:

MAIN THEOREM. *Let R be a Dedekind domain and A be a finitely generated, projective R-algebra with $A/L(A)$ separable over R. Then the following are equivalent:*

(a) *A is triangular.*

(b) *There exists a complete set of mutually orthogonal primitive idempotents $e_1, \ldots, e_n$ indexed so that $e_iL(A)e_j = 0$ whenever $i \geqq j$.*

(c) R-dim $A/I < \infty$ *for every ideal I such that A/I is R-projective and $A/(L(A) + I)$ is R-projective.*

(d) R-dim $A/L(A)^2 < \infty$.

(e) *There is a finitely generated, projective R-algebra B such that* R-dim $B \leqq 1$, *A is a homomorphic image of B, and $A/L(A)^2 \simeq B/L(B)^2$.*

The proof will be in stages: the first four equivalences being proved in Theorem 3.10 and the equivalence of (a) and (e) in Theorem 3.12.

We begin by giving several lemmas concerning algebras over a domain R. Let Q denote the field of quotients of the domain R for the remainder of the paper. $J(A)$ will denote the Jacobson radical of A.

LEMMA 3.2. *If A is a finitely generated, projective algebra over an integrally closed Noetherian domain R, then $L(A \otimes Q) = L(A) \otimes Q$.*

This lemma is an immediate consequence of Theorem 1.1 and basic properties of localization.

LEMMA 3.3. *If $A/L(A)$ is R-separable and R-projective, where A is a finitely generated algebra over a Noetherian domain R, then, for every maximal ideal* $\mathbf{m}$ *of R,*

$$J(A/\mathbf{m}A) = L(A/\mathbf{m}A) = (L(A) + \mathbf{m}A)/\mathbf{m}A.$$

Auslander and Goldman have shown the following result in [**2**, Proposition 2.8].

LEMMA 3.4. *Let R be a Dedekind ring. Let A be an hereditary R-algebra which is finitely generated and torsion-free as an R-module. A finitely generated, projective A-module E is indecomposable if and only if $E \otimes_R Q$ is a simple $A \otimes Q$-module.*

LEMMA 3.5. *Let A be a finitely generated, projective algebra over an integrally closed Noetherian domain R with $A/L(A)$ separable over R. Let e_1 and e_2 be idempotents in A. $e_1A \simeq e_2A$ if and only if $(e_1 \otimes 1)A \otimes Q \simeq (e_2 \otimes 1)A \otimes Q$.*

Proof. If $f : e_1A \simeq e_2A$, then $f \otimes 1$ is the desired isomorphism. On the other hand, if $f : (e_1 \otimes 1)A \otimes Q \simeq (e_2 \otimes 1)A \otimes Q$, set $g = f^{-1}$. Since $f(e_1 \otimes 1) = e_2b \otimes (1/d)$ and $g(e_2 \otimes 1) = e_1a \otimes (1/d_1)$, we set $f^*(e_1 \otimes 1) = e_2b \otimes 1$ and $h(e_2 \otimes 1) = e_1a \otimes (1/dd_1)$. Then $h^{-1} = f^*$. Clearly f^* restricted to $(e_1 \otimes 1)A \otimes 1$ defines an isomorphism of $(e_1 \otimes 1)A \otimes 1$ with $(e_2 \otimes 1)A \otimes 1$.

In order to simplify the proofs, with Chase [**3**, Definition 1.6], we define

$$\mathscr{T}(A', S, M) = \left\{\begin{bmatrix} a & m \\ 0 & s \end{bmatrix} : a \text{ is in } A',\ s \text{ is in } S, \text{ and } m \text{ is in } M\right\},$$

where A' and S are rings, and M is an (A', S)-bimodule, to be the ring with the usual matrix multiplication.

LEMMA 3.6. *Let R be a Dedekind domain. Let A' be a finitely generated, projective R-algebra with lower radical $L(A')$; let S be a finitely generated, projective, separable R-algebra; and let M be an (A', S)-bimodule which is R-projective. Suppose that $A'/L(A')$ is R-separable. Then $A = \mathscr{T}(A', S, M)$ is a finitely generated, projective R-algebra with lower radical*

$$\mathrm{L}(A) = \left\{\begin{bmatrix} a' & x \\ 0 & 0 \end{bmatrix} : a' \textit{ in } L(A'),\ x \textit{ in } M\right\}.$$

If R-dim $A' < \infty$ *then*

$$R\text{-dim } A = \max\{R\text{-dim } A',\ 1 + hd_{A'}(M)\} \leqq R\text{-dim } A' + 2.$$

Proof. A is clearly finitely generated and R-projective. Applying [**3**, Lemma 4.1] and [**16**, Theorem 2.1], we have that

$$\begin{aligned} R\text{-dim } A &= \sup\{R/\mathbf{m}\text{-dim } \mathscr{T}(A'/\mathbf{m}A', S/\mathbf{m}S, M/\mathbf{m}M)\} \\ &= \sup\{\text{gl dim } A/\mathbf{m}A\} \\ &= \sup\{\max[\text{gl dim } A'/\mathbf{m}A', 1 + hd_{A'/\mathbf{m}A'}(M/\mathbf{m}M)]\} \\ &= \max(R\text{-dim } A', 1 + hd_{A'}(M)) \\ &\leqq 1 + \text{gl dim } A' \\ &\leqq 2 + R\text{-dim } A'. \end{aligned}$$

where the supremum is taken over all maximal ideals $\mathbf{m}$ of R. The final two inequalities follow from [**17**, Theorem D].

If $f'\colon A' \to A'/L(A')$ is the canonical map, define

$$f:\left\{\begin{bmatrix} a' & x \\ 0 & s \end{bmatrix}\right\} = (f'(a'), s).$$

Clearly, f is an epimorphism with kernel

$$K = \left\{\begin{bmatrix} a' & x \\ 0 & 0 \end{bmatrix} : a' \text{ in } L(A'),\ x \text{ in } M\right\}.$$

One may easily verify that if $L(A')^r = 0$, then $K^{r+1} = 0$, and so K is the lower radical of A.

Lemma 3.6 generalizes a result [**3**, Theorem 4.1] of Chase. The following lemma is a generalization of [**13**, Lemma 1].

LEMMA 3.7. *Let A be a finitely generated, projective algebra over a Dedekind domain R with $A/L(A)$ separable. Let e and e' be primitive idempotents in $A/L(A)$ such that $L(A)e \neq 0$ and $eL(A)e' \neq 0$, and $L(A)^2 = 0$. Then $0 \leqq hd_A\, L(A)e < hd_A\, L(A)e'$ and hence*

$$hd_A\,(A/L(A))e < hd_A(A/L(A))e'.$$

Proof. Since $A/L(A)$ is separable, we know by [**17**, Theorem D] that $A/L(A)$ is hereditary. Hence, by Lemma 3.4, e and e' being primitive idempotents implies that $e \otimes 1$ and $e' \otimes 1$ are primitive. By Lemma 3.2, $L(A) \otimes Q = L(A \otimes Q) = J(A \otimes Q)$. Thus $L(A)e \neq 0$ implies that $L(A) \otimes Q(e \otimes 1) \neq 0$, since $L(A)$ is torsion-free over R. Similarly, $eL(A)e' \neq 0$ implies that $(e \otimes 1)L(A) \otimes Q(e' \otimes 1) \neq 0$. But this implies that

$$L(A) \otimes Q(e' \otimes 1) \simeq \sum (A \otimes Q/L(A) \otimes Q)(e_a \otimes 1),$$

since $L(A) \otimes Q(e' \otimes 1)$ is a non-zero, projective $A \otimes Q/L(A) \otimes Q$-module. Thus, by Lemmas 3.4 and 3.5, $L(A)e' \simeq \sum (A/L(A))e_a$.

Since for some a, $(e \otimes 1)(A \otimes Q/L(A) \otimes Q)(e_a \otimes 1) \neq 0$, by [**13**, Lemma 1] we have $e \otimes 1 \simeq e_a \otimes 1$ and so by Lemma 3.5, $e \simeq e_a$.

If $(A/L(A))e$ is not A-projective, then $hd_A\, L(A)\, e' \geqq hd_A(A/L(A))e = 1 + hd_A\, L(A)e \geqq 1$.

If $(A/L(A))e$ is A-projective, then $Ae = L(A)e \oplus I$ for some ideal I of A. Then $L(A)e = L(A)^2e \oplus L(A)I = L(A)I \subseteq I$. Hence $L(A)e = 0$, contrary to hypothesis. Thus this case cannot occur.

LEMMA 3.8. *Let A be a finitely generated, projective algebra over a Dedekind domain R with $A/L(A)$ separable. Let $e_1, \ldots, e_n$ be a complete set of mutually orthogonal primitive idempotents of A. Suppose there is an $s < n$ such that $e_iL(A) = 0$ for $i > s$, but $e_jL(A) \neq 0$ for $j \leqq s$. Set $e = e_{s+1} + \ldots + e_n$, $e' = 1 - e$, $A' = e'Ae'$, $S = eAe$ and $M = e'Ae$. Then $eAe' = 0$ and $A = \mathcal{T}(A', S, M)$ with S separable over R.*

https://doi.org/10.4153/CJM-1973-087-8 Published online by Cambridge University Press

Proof. Clearly, $eL(A)e \subseteq eL(A) = 0$. So $S = eAe$ is semiprime. S is therefore a homomorphic image of the separable algebra $A/L(A)$ and hence is separable. If $j \leqq s$ and $i > s$, $e_iL(A) = 0$ and $e_jL(A) \neq 0$. Whence $e_i \not\cong e_j$. By [**12**, Corollary to Proposition 4, p. 51], $Ae_i/L(A)e_i \not\cong Ae_j/L(A)e_j$. Thus by Lemma 3.5, $A/L(A) \otimes Q(e_i \otimes 1) \not\cong A/L(A) \otimes Q(e_j \otimes 1)$. Thus

$$(e_i \otimes 1)A \otimes Q(e_j \otimes 1) \subseteq L(A) \otimes Q.$$

Therefore, $e_iAe_j \subseteq L(A)$. Hence, we have shown that $eAe' \subseteq L(A)$ which implies that $eAe' \subseteq eL(A) = 0$. An application of [**3**, Theorem 2.3] completes the proof.

LEMMA 3.9. *Let A be a finitely generated, projective algebra over a Dedekind domain R. Then $A/L(A)^n$ is R-projective.*

Proof. We will show that $A/L(A)^n$ is torsion-free. Suppose that ra is contained in $L(A)^n$, where r is in R and a is in A. Then, since $L(A)^n \otimes Q = (L(A) \otimes Q)^n$, we have that $ra \otimes 1 = a_1 \cdot \ldots \cdot a_n \otimes 1$ or that $a \otimes 1 = a_1 \cdot \ldots \cdot a_n \otimes (1/r)$. Thus a is already in $L(A)^n$; so $A/L(A)^n$ is torsion-free.

Combining the preceding results, we obtain the first four equivalences of the main result.

THEOREM 3.10. *Let A be a finitely generated, projective algebra over a Dedekind domain R. Let $A/L(A)$ be R-separable. The following are equivalent:*

(a) *A is triangular.*

(b) *There exists a complete set of mutually orthogonal primitive idempotents $e_1, \ldots, e_n$ such that $e_iL(A)e_j = 0$ if $i \geqq j$.*

(c) R-dim *A/I is finite for every ideal I of A such that A/I is R-projective and $A/(L(A) + I)$ is R-projective.*

(d) R-dim *$A/\mathrm{L}(A)^2$ is finite.*

Proof. That (a) implies (b) is obvious. For (b) implies (c), it suffices to show that the result is true for A, since every homomorphic image with the properties of (b) contains a collection of mutually orthogonal primitive idempotents with the same property. Let r denote the number of isomorphism classes of primitive idempotents. Suppose $r = 1$. Then $L(A) = \sum_{i,j} e_iL(A)e_j = 0$. Thus A is semi-prime and hence separable by hypothesis. Thus R-dim $A = 0$ and A is hereditary. Assume the result is true for all $t < r$. By hypothesis, there is a complete set of mutually orthogonal primitive idempotents such that $e_iL(A)e_j = 0$ for all $i \geqq j$. Now

$$e_nL(A) = \sum_{j=1}^{n} e_nL(A)e_j = 0.$$

Thus we may assume that there is a $k < n$ such that $e_{k+1}L(A) = \ldots = e_nL(A) = 0$, but $e_iL(A) \neq 0$ for $k \geqq i$. Let $e = e_{k+1} + \ldots + e_n$ and $e' = 1 - e$. Then by Lemma 3.8, $eAe' = 0$, $S = eAe$ is a separable R-algebra, and

https://doi.org/10.4153/CJM-1973-087-8 Published online by Cambridge University Press

$A = \mathcal{T}(A', S, M)$ where $A' = e'Ae'$ and $M = e'Ae$. Note that A' now satisfies (b) and has fewer than r isomorphism classes of primitive idempotents and hence has finite Hochschild dimension by the inductive hypothesis. Now by Lemma 3.6, A has finite Hochschild dimension. That (c) implies (d) is obvious.

We must now show that (d) implies (a). Assume first that $L(A)^2 = 0$. Let $e_1, \ldots, e_n$ be any complete set of mutually orthogonal primitive idempotents of A and consider $A_i = (A/L(A))e_i$. If $e_iL(A)e_j \neq 0$, then by Lemma 3.7, $hd_A\, A_i < hd_A\, A_j$. Assume $\{e_i\}$ are indexed so that $hd_A\, A_i \geqq hd_A\, A_j$ if $i \geqq j$. Clearly $e_iL(A)e_j = 0$ for $i \geqq j$, whence A is triangular. We now proceed to the general case. Suppose that $L(A)^2 \neq 0$. By the above, any complete set of mutually orthogonal primitive idempotents can be indexed so that $e_iL(A)e_j \subseteq L(A)^2$ for $i \geqq j$. Suppose $e_iL(A)e_j \subseteq L(A)^s$ whenever $i \geqq j$ and for some $s > 1$. Then $e_iL(A)e_j = e_iL(A)^se_j$. This we may re-write:

$$e_iL(A)e_j = e_iL(A)^se_j = (e_iL(A)^{s-1})\left(\sum_k e_k\right)(L(A)e_j)$$

$$= \sum_{k=1}^{n} (e_iL(A)^{s-1}e_k)(e_kL(A)e_j).$$

But if $i \geqq k$, then $e_iL(A)^{s-1}e_k \subseteq L(A)^s$. Moreover if $k \geqq i$ and $k \geqq j$, $e_kL(A)e_j \subseteq L(A)^2$. Thus we have that $e_iL(A)e_j \subseteq L(A)^{s+1}$. Since $L(A)$ is nilpotent, by induction we have that $e_iL(A)e_j = 0$. Hence A is triangular.

LEMMA 3.11. *If the hypotheses of Theorem* 3.10 *hold, then* $L(A)$ *is* S*-isomorphic to* $P \oplus L(A)^2$, *where* $P \cong L(A)/L(A)^2$ *and* S *is the weak inertial subalgebra.*

Proof. From the short exact sequence

$$0 \rightarrow L(A)/L(A)^2 \rightarrow A/L(A)^2 \rightarrow A/L(A) \rightarrow 0,$$

and from Lemma 3.9, we obtain that $L(A)/L(A)^2$ is R-projective and hence S-projective [**5**, Proposition 2.3, p. 48]. Hence

$$0 \rightarrow L(A)^2 \rightarrow L(A) \rightarrow L(A)/L(A)^2 \rightarrow 0$$

splits as an S-sequence.

THEOREM 3.12. *Let* A *be a finitely generated, projective algebra over the Dedekind domain* R *with* $A/L(A)$ *separable.* A *is triangular if and only if there is a finitely generated, projective* R*-algebra* B *such that* A *is an epimorphic image of* B *and* $B/L(B)^2 \simeq A/L(A)^2$.

Proof. Suppose A is triangular. By Lemma 3.11, $A = S \oplus P \oplus L(A)^2$ as S-modules. Set $P^{(n)}$ equal to the n-fold tensor product of P with itself over S. Then $A \otimes Q = (S \otimes Q) \oplus (P \otimes Q) \oplus (L(A)^2 \otimes Q)$. By induction, one can check that $p^{(n)} \otimes Q \cong (P \otimes Q)^{(n)}$ using the middle-four-interchange. Set $B = S \oplus P \oplus P^{(2)} \oplus \ldots$. Since $(P \otimes Q)^{(r)} = 0$ for all r greater than some N, B is a finitely generated, projective R-algebra and A is an epimorphic image of B. $L(B)^2 = P^{(2)} \oplus \ldots$, and so $A/L(A)^2 \simeq B/L(B)^2$. That R-dim $B = 1$ follows from [**16**, Theorem 2.1] since for each maximal ideal m of R, $A/\mathbf{m}A$ is a triangular algebra over a field [**3**, Definition 4.1] and $B/\mathbf{m}B$ is clearly a less than or equal to one-dimensional algebra of which $A/\mathbf{m}A$ is a homomorphic image [**11**, Theorem 9].

To complete the proof of the theorem, one needs only show that every one-dimensional algebra is triangular since every R-projective homomorphic image of a triangular algebra is triangular. We will show that A satisfies the following definition due to M. Harada [**8**, pp. 465-6 and 472].

Definition 3.13. Let $A_1, \ldots, A_n$ be algebras over a fixed ring R and let M_{ij} be left A_i- and right A_j-bimodules with $M_{ij} = 0$ for $i > j$, and $M_{ii} = A_i$. Assume that R commutes with the M_{ij}. We consider a family of left A_i- and right A_j-bihomomorphisms which satisfy the following properties:

$$\phi_{ij}{}^t\colon M_{it} \otimes_{A_t} M_{tj} \to M_{ij}$$
$$\phi_{it}{}^t\colon M_{it} \otimes_{A_t} A_t \simeq M_{it}$$
$$\phi_{it}{}^i\colon A_i \otimes_{A_i} M_{it} \simeq M_{it}$$

with commutative diagrams:

$$\begin{array}{ccc} M_{ij} \otimes_{A_j} M_{jt} \otimes_{A_t} M_{tk} & \xrightarrow{id_{ij} \otimes \phi_{jk}{}^t} & M_{ij} \otimes_{A_j} M_{jk} \\ \downarrow{\scriptstyle \phi_{it}{}^j \otimes id_{tk}} & & \downarrow{\scriptstyle \phi_{ik}{}^j} \\ M_{it} \otimes_{A_t} M_{tk} & \xrightarrow{\phi_{ik}{}^t} & M_{ik} \end{array}$$

Set

$$T_n(A_i;\, M_{ij}/R) = \left\{ \begin{bmatrix} a_{11} & m_{12} \ldots m_{1n} \\ & a_{22} \ldots m_{2n} \\ & \cdot \\ & \cdot \\ \mathbf{0} & \quad a_{nn} \end{bmatrix} : a_{ii} \text{ in } A_i;\, m_{ij} \text{ in } M_{ij} \right\}.$$

$T_n(A_i;\, M_{ij}/R)$ is an R-algebra under the operations:

(a) $(m_{ij}) + (m_{ij}') = (m_{ij} + m_{ij}')$

(b) $(m_{ij}) \cdot (m'_{ij}) = (\sum_t \phi_{ij}{}^t(m_{it} \otimes m'_{tj}))$

(c) $r(m_{ij}) = (rm_{ij}) = (m_{ij}r) = (m_{ij})r$.

We call $T_n(A_i;\, M_{ij}/R)$ a generalized triangular matrix algebra over R.

If $A = T_n(A_i;\, M_{ij}/R)$ is a generalized triangular matrix algebra over R, then $B = T_n((A_i)_{s_i};\, \mathcal{M}_{ij}/R)$ is called the generalized triangular matrix algebra induced from A, where $\mathcal{M}_{ij}$ is the $s_i \times s_j$-matrices with entries from M_{ij} (cf. [**16**, pp. 131-133]).

LEMMA 3.14. *Let A be a finitely generated, projective algebra over a Dedekind domain R. Then any projective A-module P is a direct sum of copies of Ae_i where the e_i are in some collection of mutually orthogonal primitive idempotents.*

Proof. Clearly A can be written as a finite sum of indecomposable ideals. Apply [**12**, Proposition 2, p. 50] and the fact that a projective module is a summand of a free.

LEMMA 3.15. *Let A be a finitely generated, projective algebra over a Dedekind domain R. Let R-dim $A \leqq 1$ and $A/L(A)$ separable. Let $n(e)$ denote the minimal power of $L(A)$ such that $L(A)^{n(e)-1}e \neq 0$ and $L(A)^{n(e)} = 0$. If $n(e)$ is minimal among the $n(e_i)$ as e_i runs through a decomposition of 1, then $n(e) = 0$.*

Proof. Assume that $L(A)e \neq 0$ and $n(e)$ is minimal. Since R-dim $A \leqq 1$, we have that $L(A)$ is A-projective. This follows from Lemma 3.3, [**7**, Corollary to Theorem 3], and the fact that $hd_A\ L(A) = \sup\ hd_{A/\mathbf{m}A}\ L(A)/\mathbf{m}L(A)$ (cf. [**16**, the proof of Theorem 2.1]). Hence, by Lemma 3.14, $L(A)e \cong \sum (Ae_i)^{s_i}$. Then $0 = L(A)^{n(e)}e = L(A)^{n(e)-1}L(A)e = \sum (L(A)^{n(e)-1}e_i)^{s_i}$. Therefore $n(e) > n(e_i)$, a contradiction.

LEMMA 3.16. *Let A be a finitely generated, projective algebra over a Dedekind domain R with R-dim $A \leqq 1$. Let all of the idempotents $e_1, \ldots, e_n$ in a decomposition of 1 be pairwise non-isomorphic and indexed so that $n(e_i) \leqq n(e_{i+1})$. Then $e_iL(A)e_j = 0$.*

Proof. Since $n(e_1)$ is minimal, $L(A)e_1 = 0$. Hence $e_iL(A)e_1 = 0$ for all i. Assume that $e_iL(A)e_j = 0$ for all $i \geqq j$ and $j < k$. Then

$$L(A)e_k \cong \textstyle\sum_{t<k}(Ae_t)^{s_t}.$$

Thus

$$e_iL(A)e_k \cong \sum_{t<k} (e_iAe_t)^{s_t}.$$

But $e_i(A/L(A) \otimes Q)e_t = 0$, for $i \geqq k > t$. Therefore, $e_iA/L(A)e_t = 0$, and hence $e_iL(A)e_k = 0$.

THEOREM 3.17. *Let A be a finitely generated, projective algebra over a Dedekind domain R, R-dim $A \leqq 1$ and $A/L(A)$ R-separable.*

(a) *If no two idempotents in a complete set of mutually orthogonal primitive idempotents are isomorphic, then $A = T_n(e_iAe_i; e_iAe_j/R)$, where R-dim $e_iAe_i = 0$.*

(b) *In general, A is isomorphic to an induced generalized triangular matrix algebra.*

Proof. (a) follows directly from the preceding Lemma 3.16. The general case (b) follows by considering the algebra eAe where e is the sum of one idempotent from each isomorphism class. eAe can be shown to be a generalized triangular matrix algebra (cf. [**16**, Theorem 3.6; **18**]). It follows immediately that A is the generalized triangular matrix algebra induced from eAe.

This concludes the proof of Theorem 3.12 and hence of the main result of the section. We compare the concept of an algebra being triangular with the concept of almost one-dimensional introduced in [**18**]. We show that if R is a local Dedekind domain (a DVR) which is also a Hensel ring, the concepts of triangular and almost one-dimensional coincide.

Definition 3.18. Let R be a local Hensel ring and A be a finitely generated, projective R-algebra with A/N separable over R. A is almost one-dimensional if and only if every complete set of mutually orthogonal primitive idempotents $e_1, \ldots, e_n$ can be indexed so that $e_iNe_j \subseteq \mathbf{m}A$ whenever $i \geqq j$.

THEOREM 3.19. *Let A be a finitely generated, projective algebra over a regular local Hensel domain R with $A/L(A)$ separable over R. A is almost one-dimensional if and only if A is triangular.*

Proof. A triangular implies there is a complete set of mutually orthogonal primitive idempotents such that $e_iL(A)e_j = 0$ for $i \geqq j$. But $A/L(A)$ being R-separable implies that $N/\mathbf{m}A = J(A/\mathbf{m}A) = (L(A) + \mathbf{m}A)/\mathbf{m}A$, whence $e_iNe_j \subseteq \mathbf{m}A$ whenever $i \geqq j$. By [**18**, Theorem 3], A is almost one-dimensional.

On the other hand, if A is almost one-dimensional, then $J(A/\mathbf{m}A) \cong L(A)/\mathbf{m}L(A)$ since $A = S \oplus L(A)$. But

$$0 = \bar{e}_i(L(A)/\mathbf{m}L(A))\bar{e}_j = e_iL(A)e_j/e_i\mathbf{m}L(A)e_j,$$

whenever $i \geqq j$. Thus by Nakayama's lemma $e_iL(A)e_j = 0$, for $i \geqq j$. Hence A is triangular.

We conclude the paper with a generalization of a theorem of Chase [**3**, Theorem 4.2].

THEOREM 3.20. *Let A be a finitely generated, projective algebra over a Dedekind domain with $A/L(A)$ separable. If every principal left ideal of A is projective, then A is triangular.*

Proof. Let r be the number of isomorphism classes of primitive orthogonal idempotents. If $r = 1$, then $A/L(A) \otimes Q$—having one isomorphism class—is a simple ring. But $A/L(A) \otimes Q$ simple implies that $L(A) \otimes Q = 0$ and hence $L(A) = 0$. Thus A is separable and the result is obvious.

Assume the result is true for all $t < r$. If A has $L(A) = 0$, and hence A is separable by hypothesis, we are done. Suppose that $L(A) \neq 0$, and let there be an x in A such that $xL(A) = 0$. By assumption xA is projective and so by a theorem of Chase [**3**, Theorem 4.3], there is an e_0 in A such that $e_0^2 = e_0$; $xe_0 = x$ and $xa = 0$ implies $e_0a = 0$. Therefore, $xL(A) = 0$ implies that $e_0L(A) = 0$. Write $e_0 = \bar{e} + e_1$ where $\bar{e}$ and e_1 are orthogonal and $\bar{e}$ is a primitive idempotent. Then $\bar{e} = \bar{e}e_0$. Therefore, $\bar{e}L(A) = \bar{e}e_0L(A) = 0$. Let $e_1, \ldots, e_n$ be a complete set of mutually orthogonal primitive idempotents indexed so that $e_n = \bar{e}$. Since $e_nL(A) = \bar{e}L(A) = 0$, we may assume that there is some $k < n$ such that $e_{k+1}L(A) = \ldots = e_nL(A) = 0$, but $e_iL(A) \neq 0$ for $i \leqq k$.

Set $e = e_{k+1} + \ldots + e_n$ and $e' = 1 - e$. By Lemma 3.8, $S = eAe$ is separable and $A = \mathscr{T}(A', S, M)$ where $A' = e'Ae'$ and $M = e'Ae$.

Thus by [**3**, Lemma 4.4], every principal right ideal in A' is projective. Thus A' is triangular, by the inductive hypothesis. Hence it follows that A is triangular.

References

1. M. Auslander and D. Buchsbaum, *On ramification theory in Noetherian rings*, Amer. J. Math. *81* (1959), 749–765.

2. M. Auslander and O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc. *97* (1960), 1–24.

3. S. U. Chase, *A generalization of the ring of triangular matrices*, Nagoya Math. J. *18* (1961), 13–25.

4. C. W. Curtis, *Non-commutative extensions of Hilbert rings*, Proc. Amer. Math. Soc. *4* (1953), 945–955.

5. F. Demeyer and E. C. Ingraham, *Separable algebras over commutative rings*, Lecture Notes in Mathematics No. 181 (Springer-Verlag, New York, 1971).

6. N. J. Divinsky, *Rings and radicals* (Univ. of Toronto Press, Toronto, 1965).

7. S. Eilenberg, *Algebras of cohomologically finite dimension*, Comment. Math. Helv. *28* (1954), 310–319.

8. M. Harada, *Hereditary semi-primary rings and tri-angular matrix rings*, Nagoya Math. J. *27* (1966), 463–484.

9. G. Hochschild, *Cohomology groups of an associative algebra*, Ann. of Math. *46* (1945), 58–67.

10. E. C. Ingraham, *Inertial subalgebras of algebras over commutative rings*, Trans. Amer. Math. Soc. *124* (1966), 77–93.

11. ——— *Inertial subalgebras of complete algebras*, J. Algebra *15* (1970), 1–12.

12. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ. Vol. 37 (Amer. Math. Soc., Providence, R.I., 1964).

13. J. Jans and T. Nakayama, *On the dimension of modules and algebras VII, algebras with finite-dimensional residue algebras*, Nagoya Math. J. *11* (1957), 67–76.

14. G. J. Janusz, *Separable algebras over commutative rings*, Trans Amer. Math. Soc. *122* (1966), 461–479.

15. L. Silver, *Noncommutative localizations and applications*, J. Algebra *7* (1967), 44–76.

16. J. A. Wehlen, *Algebras of finite cohomological dimension*, Nagoya Math. J. *43* (1971), 127–135.

17. ——— *Cohomological dimension and global dimension of algebras*, Proc. Amer. Math. Soc. *32* (1972), 75–80.

18. ——— *Triangular matrix algebras over Hensel rings*, Proc. Amer. Math. Soc. *37* (1973), 69–74.

Ohio University,
Athens, Ohio