

On infinitely many foliations by caustics in strictly convex open billiards

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(Received 4 July 2022 and accepted in revised form 19 April 2023)

Abstract. Reflection in a strictly convex bounded planar billiard acts on the space of oriented lines and preserves a standard area form. A *caustic* is a curve C whose tangent lines are reflected by the billiard to lines tangent to C . The famous Birkhoff conjecture states that the only strictly convex billiards with a foliation by closed caustics near the boundary are ellipses. By Lazutkin's theorem, there always exists a Cantor family of closed caustics approaching the boundary. In the present paper, we deal with an open billiard, whose boundary is a strictly convex embedded (non-closed) curve γ . We prove that there exists a domain U adjacent to γ from the convex side and a C^∞ -smooth foliation of $U \cup \gamma$ whose leaves are γ and (non-closed) caustics of the billiard. This generalizes a previous result by Melrose on the existence of a germ of foliation as above. We show that there exists a continuum of above foliations by caustics whose germs at each point in γ are pairwise different. We prove a more general version of this statement for γ being an (immersed) arc. It also applies to a billiard bounded by a closed strictly convex curve γ and yields infinitely many 'immersed' foliations by immersed caustics. For the proof of the above results, we state and prove their analogue for a special class of area-preserving maps generalizing billiard reflections: the so-called C^∞ -lifted strongly billiard-like maps. We also prove a series of results on conjugacy of billiard maps near the boundary for open curves of the above type.

Key words: billiard, caustic, foliation by caustics, conjugacy of billiard maps, symplectomorphism

2020 Mathematics Subject Classification: 37D50, 37J35, 37J10 (Primary)

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1. Introduction and main results

The billiard reflection from a strictly convex smooth planar curve $\gamma \subset \mathbb{R}^2$ (parameterized by either a circle or an interval) is a map \mathcal{T} acting on the subset in the space of oriented lines that consists of those lines that are either tangent to γ or intersect γ transversally at two points. (In general, the latter subset is not \mathcal{T} -invariant. In the case when γ is a closed curve, the latter subset is \mathcal{T} -invariant and called the *phase cylinder*.) Namely, if a line is tangent to γ , then it is a fixed point of the reflection map. If a line L intersects γ transversally at two points, take its last intersection point B with γ (in the sense of orientation of the line L) and reflect L from $T_B\gamma$ according to the usual reflection law: the angle of incidence is equal to the angle of reflection. By definition, the image $\mathcal{T}(L)$ is the reflected line oriented at B inside the convex domain adjacent to γ . The reflection map \mathcal{T} is called the *billiard ball map*. See Figure 1.

The space of oriented lines in Euclidean plane $\mathbb{R}_{x,y}^2$ is homeomorphic to a cylinder, and it carries the standard symplectic form

$$\omega = d\phi \wedge dp, \tag{1.1}$$

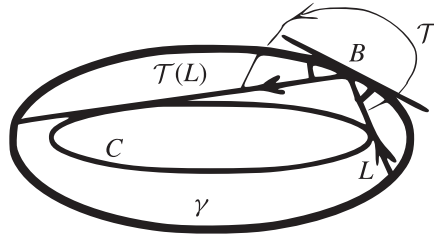


FIGURE 1. The billiard ball map and a caustic.

where $\phi = \phi(L)$ is the azimuth of the line L (its angle with the x -axis) and $p = p(L)$ is its signed distance to the origin O defined as follows. For each oriented line L that does not pass through O , consider the circle centered at O and tangent to L . We say that L is *clockwise* (*counterclockwise*), if it orients the latter circle clockwise (counterclockwise). By definition:

- $p(L) = 0$ if and only if L passes through the origin O ;
- $p = \text{dist}(L, O)$ if L is clockwise; otherwise $p = -\text{dist}(L, O)$.

It is well known that:

- the symplectic form ω is invariant under affine orientation-preserving isometries;
- the billiard reflections from all planar curves preserve the symplectic form ω .

Definition 1.1. A curve C is a *caustic* for the billiard on the curve γ if each line tangent to C is reflected from γ to a line tangent to C . See Figure 1. Here the tangent line is taken with arbitrary orientation, for which its reflection is well defined: the notion of caustic is orientation independent.

Remark 1.2. Consider the union \mathcal{U}_C of two families of oriented tangent lines to a curve C : lines taken with two possible orientations. The curve C is a caustic if and only if the intersection of the set \mathcal{U}_C with the definition domain of the billiard ball map is sent by the billiard map to \mathcal{U}_C . Note that a single family of tangent lines to a caustic need not be invariant. For example, in an elliptic billiard, each confocal hyperbola is a caustic, but its two families of tangent lines are permuted by the billiard map. However, each confocal ellipse is a caustic, with both tangent line families being invariant curves. If a caustic of a general strictly convex billiard is C^1 -close enough to the billiard boundary, then each individual family of its tangent lines is an invariant curve for the billiard map. For more details, see Remark 1.4.

The famous Birkhoff conjecture deals with a planar billiard bounded by a strictly convex closed curve γ . Recall that such a billiard is called *Birkhoff integrable* if there exists a topological annulus adjacent to γ from the convex side foliated by closed caustics, and γ is a leaf of this foliation. See Figure 2. It is well known that the billiard in an ellipse is integrable, since it has a family of closed caustics: confocal ellipses. The Birkhoff conjecture states the converse: *the only integrable planar billiards are ellipses*.

Remark 1.3. The condition of the Birkhoff conjecture stating that the caustics in question form a *foliation* is important: the famous result by Lazutkin [15] (1973) states that *each*

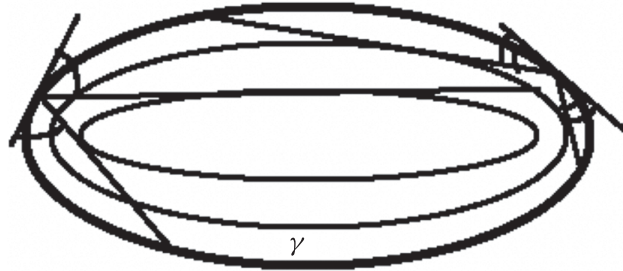


FIGURE 2. A Birkhoff integrable billiard.

strictly convex bounded planar billiard with boundary smooth enough has a Cantor family of closed caustics. Lazutkin proved it by using Kolmogorov–Arnold–Moser-type arguments. However, Lazutkin’s caustic family does not extend to a foliation in general.

The main result of the paper presented in §1.2 shows that the other condition of the Birkhoff conjecture stating that the caustics in question are *closed* is also important: the Birkhoff conjecture is false without closeness condition. Namely, we show that any open strictly convex C^∞ -smooth planar curve γ has an adjacent domain U (from the convex side) admitting a foliation by caustics of γ that extends to a C^∞ -smooth foliation of the domain with boundary $U \cup \gamma$ with γ being a leaf. (The exact sense of the notion ‘foliation by caustics’ for a strictly convex open billiard will be given in Definition 1.6 and Remark 1.7.) Moreover, we show that U can be chosen so that there exist infinitely many (continuum of) such foliations, and any two distinct foliations have pairwise distinct germs at every point in γ . We prove an analogous statement for a non-injectively immersed curve γ and ‘immersed foliations’ by immersed caustics. We state and prove an analogue of this statement in the special case, when γ is a closed curve.

The above-mentioned results extend a previous result of Melrose, who proved that for every C^∞ -smooth germ γ of a strictly convex planar curve, there exists a germ of C^∞ -smooth foliation by caustics of the billiard played on γ , with γ being a leaf [17, p. 184, Proposition 7.14].

Remark 1.4. Consider the map \mathcal{T} of billiard reflections from a strictly convex planar oriented C^∞ -smooth curve γ that is a one-dimensional submanifold in \mathbb{R}^2 parameterized by interval. Let $\widehat{\gamma}$ denote the family of its orienting tangent lines. Then the points of the curve $\widehat{\gamma}$ are fixed by \mathcal{T} . The map \mathcal{T} is a well-defined area-preserving map on an open subset adjacent to $\widehat{\gamma}$ in the space of oriented lines. The latter subset consists of those lines that intersect γ transversally and are directed to the concave side from γ at some intersection point. Each caustic close to γ corresponds to a \mathcal{T} -invariant curve (the family of its tangent lines chosen with appropriate orientation) and *vice versa*. Thus, a foliation by caustics induces a foliation by \mathcal{T} -invariant curves. In §2.7, we prove the converse: each C^∞ -smooth foliation by \mathcal{T} -invariant curves on a domain adjacent to $\widehat{\gamma}$ from an appropriate side (with $\widehat{\gamma}$ being a leaf) induces a C^∞ -smooth foliation by caustics (with γ being a leaf).

We show that *the billiard map has an infinite-dimensional family of C^∞ -smooth foliations by invariant curves (including $\widehat{\gamma}$) in an appropriate domain adjacent to $\widehat{\gamma}$ with pairwise distinct germs at each point of the curve $\widehat{\gamma}$* . This together with Remark 1.4 implies the existence of an infinite-dimensional family of foliations by caustics.

In §1.4, we state the generalization of the above result on foliations by invariant curves to a special class of area-preserving maps: the so-called C^∞ -lifted strongly billiard-like maps. Each one of the latter maps is defined on a domain $U \subset \mathbb{R} \times \mathbb{R}_{>0}$ taken together with an adjacent boundary interval $J \subset \mathbb{R} \times \{0\}$ consisting of fixed points, and satisfies appropriate twist and regularity conditions. The (weakly and strongly) billiard-like maps were introduced and studied in [10], where results on their dynamics were applied to curves with Poritsky property. For the C^∞ -lifted strongly billiard-like maps, we prove the existence of an infinite-dimensional family of C^∞ -smooth foliations by invariant curves with pairwise distinct germs at each point of the boundary segment. In §1.5, we describe one-to-one correspondence between germs of the latter foliations and germs at $S^1 \times \{0\}$ of C^∞ -smooth h -flat functions $\psi(t, h)$ on the cylinder $S^1 \times \mathbb{R}_{\geq 0}$ such that $\psi(0, h) \equiv 0$. This yields a one-to-one correspondence between foliations by caustics and the above germs of flat functions on a cylinder. Theorem 1.31 stated in §1.5 asserts that all the foliations by caustics (invariant curves) corresponding to a given billiard (map) have coinciding jets of any order at each point of the boundary curve.

The results of the paper mentioned below are motivated by the following open question attributed to Victor Guillemin:

Let two billiard maps corresponding to two strictly convex closed Jordan curves be conjugated by a homeomorphism. What can be said about the curves? Are they similar (that is, of the same shape)?

Theorem 1.27 presented in §1.4 states that each C^∞ -lifted strongly billiard-like map is C^∞ -smoothly symplectically conjugated near the boundary (and up to the boundary) to the normal form $(t, z) \mapsto (t + \sqrt{z}, z)$ restricted to $U \cup J$, where $J \subset \mathbb{R} \times \{0\}$ is an interval of the horizontal axis and $U \subset \mathbb{R} \times \mathbb{R}_+$ is a domain adjacent to J . In particular, this holds for the billiard map corresponding to each C^∞ -smooth strictly convex (immersed) curve. As an application, we obtain a series of results on (symplectic) conjugacy of billiard maps near the boundary for billiards with reflections from C^∞ -smooth strictly convex curves parameterized by intervals. These conjugacy results are stated in §1.6 and proved in §2.10. One of them (Theorem 1.39) states that for any two strictly convex open billiards, each of them being bounded by an infinite curve with asymptotic tangent line at infinity in each direction, the corresponding billiard maps are C^∞ -smoothly conjugated near the boundary.

The results of the paper are proved in §2. The plan of proofs is presented in §1.7. The corresponding background material on symplectic properties of a billiard ball map is recalled in §1.3.

1.1. Historical remarks. The Birkhoff conjecture was first stated in print by Poritsky [19], who proved it under an additional condition that for any two nested closed caustics, the smaller one is a caustic of the billiard played in the bigger one; the same result was later obtained in [1]. One of the most famous results on the Birkhoff conjecture is due to Bialy [4], who proved that if the phase cylinder of the billiard is foliated by non-contractible

invariant closed curves, then the billiard boundary is a circle; see also another proof in [22]. Recently, Kaloshin and Sorrentino proved that any integrable deformation of an ellipse is an ellipse [13]. Very recently, Bialy and Mironov proved the Birkhoff conjecture for centrally symmetric billiards having a family of closed caustics that extends up to a caustic tangent to four-periodic orbits [7]. For a detailed survey of the Birkhoff conjecture, see [5, 7–9, 13, 14, 21] and references therein.

Marvizi and Melrose have shown that the billiard ball map \mathcal{T} in a planar domain bounded by a C^∞ -smooth strictly convex closed curve γ always has an *asymptotic first integral* on a domain with boundary in the space of oriented lines: a domain adjacent to the family of tangent lines to γ . Namely, there exists a C^∞ -smooth function F on the closure of a domain as above such that the difference $F \circ \mathcal{T} - F$ is C^∞ -smooth there, and it is flat at the points of the family of tangent lines to γ ; see [16, Theorem 3.2]; see also the statement of their result in Theorem 2.1.

Kaloshin and Koudjina proved that for a non-integrable billiard bounded by a strictly convex closed curve, the Taylor coefficients of the normalized Mather β -function are invariant under C^∞ -conjugacies [12]. They also obtained a series of results on conjugacy of elliptic billiard maps, showing in particular that global topological conjugacy implies similarity of underlying ellipses.

1.2. *Main result: an open convex arc has infinitely many foliations by caustics.* Consider an open planar billiard: a convex planar domain bounded by a strictly convex C^∞ -smooth one-dimensional submanifold γ that is a curve parameterized by interval; it goes to infinity in both directions. Let U be a domain adjacent to γ from the convex side. Consider a foliation \mathcal{F} of the domain U by strictly convex smooth curves, with γ being a leaf. We consider that it is a foliation by (connected components of) level curves of a continuous function h on $U \cup \gamma$ such that $h|_\gamma = 0$, $h|_U > 0$ and h strictly increases as a function of the transversal parameter. We also consider that for every $x \in \gamma$ and every leaf \mathcal{L} of the foliation \mathcal{F} , there are at most two tangent lines to \mathcal{L} through x . One can achieve this by shrinking the foliated domain U , since for every $x \in \gamma$, the line $T_x\gamma$ is the only line through x tangent to γ . Indeed, if there were another line through x tangent to γ at a point $y \neq x$, then the total increment of azimuth of the orienting tangent vector to γ along the arc xy would be greater than π . However, the latter azimuth is monotonous, and its total increment along the curve γ is no greater than π , since γ is convex and goes to infinity in both directions. The contradiction thus obtained proves uniqueness of tangent line through x .

Remark 1.5. In the above conditions, for every compact subarc $\gamma' \subset \gamma$ and every leaf \mathcal{L} of the foliation \mathcal{F} close enough to γ for every $x \in \gamma'$, there exist exactly two tangent lines to \mathcal{L} through x . This follows from convexity.

Definition 1.6. We say that \mathcal{F} is a *foliation by caustics* of the billiard played on γ if its leaves are caustics, see Figure 3, in the following sense. Let $x \in \gamma$, and let \mathcal{L} be a leaf of the foliation \mathcal{F} . If there exist two tangent lines to \mathcal{L} through x , then they are symmetric with respect to the tangent line $T_x\gamma$.

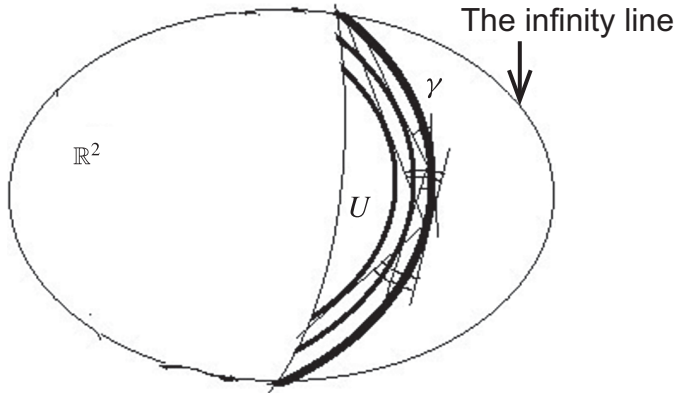


FIGURE 3. Foliation by caustics in an open strictly convex planar billiard. Here the ambient plane \mathbb{R}^2 is presented together with its boundary: the infinity line.

Remark 1.7. The above definition also makes sense in the case when γ is just a strictly convex arc that needs not go to infinity. *A priori*, in this case for some $x \in \gamma$, there may be more than two tangent lines through x to a leaf of the foliation, even for leaves arbitrarily close to γ . This holds, e.g., if there is a line through x tangent to γ at a point distinct from x . This may take place only in the case when the azimuth increment along γ of the orienting tangent vector to γ is bigger than π . In this case, we modify the above definition as follows. Let \mathcal{H} denote the space of triples (x, y, z) , where $x \in \gamma$ and y, z lie in the same leaf \mathcal{L} of the foliation \mathcal{F} , $y \neq z$, such that the lines xy and xz are tangent to \mathcal{L} at the points y and z respectively. Set

$$\overline{\mathcal{H}} := \mathcal{H} \cup \Delta, \Delta := \{(x, x, x) \mid x \in \gamma\}.$$

Let \mathcal{H}_0 denote the path-connected component of the space $\overline{\mathcal{H}}$ that contains Δ . We require that for every $(x, y, z) \in \mathcal{H}_0 \setminus \Delta$, the lines xy and xz be symmetric with respect to the line $T_x\gamma$.

Definition 1.8. Let $\gamma \subset \mathbb{R}^2$ be a smooth curve parameterized by an interval. Let $U \subset \mathbb{R}^2$ be a domain adjacent to γ . A collection of C^∞ -smooth foliations on $U \cup \gamma$ with γ being a leaf is said to be an *infinite-dimensional family of foliations with distinct boundary germs* if their germs at each point in γ are pairwise distinct, and if their collection contains a C^∞ -smooth N -parametric family of foliations for every $N \in \mathbb{N}$.

THEOREM 1.9.

- (1) Consider an open billiard bounded by a strictly convex C^∞ -smooth curve $\gamma \subset \mathbb{R}^2$: a one-dimensional submanifold parameterized by interval. There exists a simply connected domain U adjacent to γ from the convex side that admits a foliation by caustics of the billiard that extends to a C^∞ -smooth foliation on $U \cup \gamma$, with γ being a leaf. Moreover, U can be chosen to admit an infinite-dimensional family of foliations as above with distinct boundary germs. See Figure 3.

- (2) The above statements remain valid in the case when γ is just an arc: a strictly convex curve parameterized by an interval such that each its point has a neighborhood V whose intersection with γ is a submanifold in V .

Remark 1.10. It follows from Melrose’s result [17, p. 184, Proposition 7.14] that each point of the curve γ has an arc neighborhood $\alpha \subset \gamma$ for which there exists a domain U adjacent to α from the convex side such that $U \cup \alpha$ is C^∞ -smoothly foliated by caustics of the billiard played on γ . The new result given by Theorem 1.9 is the statement that the latter holds for the whole curve γ and there exist infinitely many foliations by caustics with distinct boundary germs.

Below we extend Theorem 1.9 to the case of immersed (or closed) curve γ .

Definition 1.11. Let $\gamma \subset \mathbb{R}^2$ be a strictly convex C^∞ -smooth curve that is the image of an interval $(0, 1)$ with coordinate x under an immersion $\psi : (0, 1) \rightarrow \gamma$. Let $V \subset (0, 1) \times \mathbb{R}_+ \subset \mathbb{R}^2$ be a domain adjacent to the interval $J := (0, 1) \times \{0\}$. Fix a C^∞ -smooth immersion $\Psi : V \cup J \rightarrow \mathbb{R}^2$ extending ψ as a map $J \rightarrow \gamma$, sending V to the convex side from γ . Let $U \subset V$ be a domain adjacent to J and equipped with a foliation \mathcal{F} by smooth curves parameterized by intervals, with J being a leaf. We consider that \mathcal{F} is a foliation by level curves of a continuous function $h : U \rightarrow \mathbb{R}$, $h|_J = 0$, $h|_U > 0$, such that h strictly increases as a function of the transversal parameter. We say that \mathcal{F} is a *foliation by lifted caustics* of the billiard played on γ if Ψ sends each its leaf $\mathcal{F}_t = \{h = t\}$ to a caustic of the billiard, see Figure 4. In more detail, let \mathcal{H} denote the space of triples (x, y, z) , where $x \in J$ and y, z lie in the same leaf \mathcal{L} of the foliation \mathcal{F} , $y \neq z$, such that the lines $\Psi(x)\Psi(y)$ and $\Psi(x)\Psi(z)$ are tangent to the curve $\Psi(\mathcal{L})$ at the points $\Psi(y)$ and $\Psi(z)$ respectively. Set

$$\overline{\mathcal{H}} := \mathcal{H} \cup \Delta, \Delta := \{(x, x, x) \mid x \in J\}.$$

Let \mathcal{H}_0 denote the path-connected component of the space $\overline{\mathcal{H}}$ that contains Δ . We require that for every $(x, y, z) \in \mathcal{H}_0 \setminus \Delta$, the lines $\Psi(x)\Psi(y)$ and $\Psi(x)\Psi(z)$ be symmetric with respect to the line tangent to γ at $\Psi(x)$.

THEOREM 1.12. *Let γ, ψ, Ψ, J, V be as above. There exists a domain $U \subset V$ adjacent to J on which there exists a foliation by lifted caustics that extends to a C^∞ -smooth foliation on $U \cup J$, with J being a leaf. The above U can be chosen so that it admits an infinite-dimensional family of foliations as above with distinct boundary germs. See Figure 4.*

THEOREM 1.13. *Let γ be a strictly convex closed curve bijectively parameterized by a circle. Fix a topological annulus \mathcal{A} adjacent to γ from the convex side. Let $\pi : \tilde{\mathcal{A}} = \mathbb{R} \times [0, \varepsilon) \rightarrow \mathcal{A}$ be its universal covering, set $J := \mathbb{R} \times \{0\}$; $\pi : J \rightarrow \gamma$ is the universal covering over γ . There exists a domain $U \subset \tilde{\mathcal{A}} \setminus J$ adjacent to J that admits a foliation by lifted caustics of the billiard in γ that extends to a C^∞ -smooth foliation on $U \cup J$, with J being a leaf. Moreover, one can choose U so that there exist an infinite-dimensional family of foliations as above with distinct boundary germs.*

Remark 1.14. In general, in Theorem 1.13, the projected leaves are caustics that need not be closed, may intersect each other, and may have self-intersections. Each individual caustic

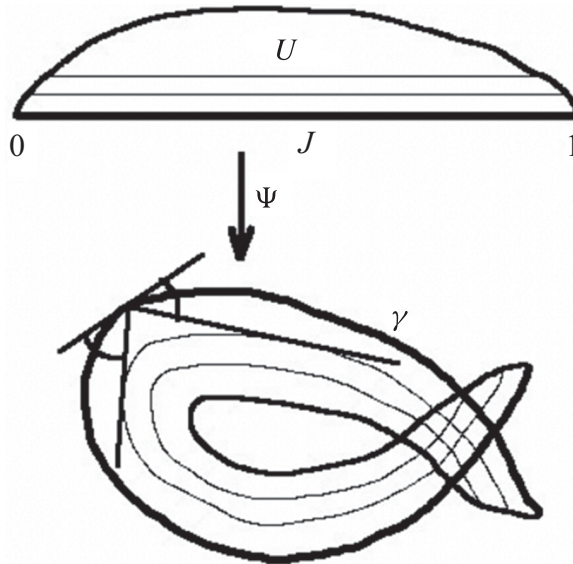


FIGURE 4. An immersed foliation by immersed caustics and a foliation by lifted caustics.

may have a finite length. However, the latter finite length tends to infinity, as the caustic in question tends to γ .

Example 1.15. Let γ be an ellipse. Then every annulus bounded by γ and a smaller confocal ellipse is analytically foliated by confocal elliptic caustics. The lifting of this foliation to the universal cover yields a special foliation from Theorem 1.13. Namely, its leaves are invariant under the covering action of the fundamental group of the annulus. However, a generic foliation from Theorem 1.13 need not satisfy the latter invariance statement.

Remark 1.16. Let $\gamma \subset \mathbb{R}^2$ be a strictly convex closed curve. Let J be its universal covering and let \mathcal{A}, U , be the same, as in Theorem 1.13. Let \mathcal{F} be a foliation of U by lifted caustics from the same theorem and let $b \subset U$ be its cross-section: a curve starting at a point $B \in J$ and transversal to \mathcal{F} . Let b', B' denote their images under the covering transformation defined by the generator of the fundamental group $\pi_1(\mathcal{A})$: the generator given by a counterclockwise circuit. The curve b' starts at B' . Both b and b' are projected to the same curve $\widehat{b} := \pi(b) \subset \mathcal{A} \cup \gamma$ starting at $\widehat{B} := \pi(B) = \pi(B') \in \gamma$. The foliation \mathcal{F} induces a germ of C^∞ -smooth holonomy diffeomorphism $(b, B) \rightarrow (b', B')$ sending a point of intersection of a leaf with b to its next intersection point with b' : next in the sense of orientation of the leaf J given by counterclockwise orientation of the curve γ . This together with the identification $b = b' = \widehat{b}$ yields a germ of holonomy map $g : (\widehat{b}, \widehat{B}) \rightarrow (\widehat{b}, \widehat{B})$. In general, the germ g depends on the choice of lifting b of the curve \widehat{b} , and so does its conjugacy class. A necessary condition of Birkhoff integrability of the billiard in γ with foliation by closed caustics being C^∞ -smooth up to the boundary is the existence of a foliation \mathcal{F} from Theorem 1.13 and a cross-section b for which the above germ g is the identity.

A generalization of Theorems 1.9 and 1.12 for the so-called C^∞ -lifted strongly billiard-like maps will be stated in §1.4.

1.3. *Background material: symplectic properties of billiard ball map.* Let γ be a C^∞ -smooth strictly convex oriented curve in \mathbb{R}^2 parameterized injectively either by an interval or by a circle. Let s be its natural length parameter respecting its orientation. We identify a point in γ with the corresponding value of the natural parameter s . Set

$$\Gamma = \{(q, u) \mid q \in \gamma, u \in T_q\mathbb{R}^2, \|u\| = 1\} \subset T\mathbb{R}^2_\gamma.$$

This is a two-dimensional surface parameterized diffeomorphically by $(s, \phi) \in \gamma \times S^1$; here $\phi = \phi(u)$ is the angle of a given unit tangent vector $u \in T_s\mathbb{R}^2$ with the orienting unit tangent vector $\dot{\gamma}(s)$ to γ . The curve

$$\tilde{\gamma} := \{\phi = 0\} = \{(s, \dot{\gamma}(s)) \mid s \in \gamma\}$$

is the graph of the above vector field $\dot{\gamma}$. For every $(q, u) \in \Gamma$, set

$$L(q, u) := \text{the oriented line through } q \text{ directed by the vector } u.$$

We treat the two following cases separately.

Case (I). The curve γ is either:

- parameterized by an interval and goes to infinity in both directions (for a convex curve, this means either the existence of asymptotic rays in both directions, or converging (in both directions) to one and the same point in the infinity line in the ambient projective plane \mathbb{RP}^2); or
- parameterized by a circle.

That is, γ bounds a strictly convex unbounded (respectively, bounded) planar domain. Let $\Gamma^0 \subset \Gamma$ denote the neighborhood of the curve $\tilde{\gamma}$ that consists of those $(q, u) \in \Gamma$ that satisfy the following conditions:

- (a) the line $L(q, u)$ either intersects γ at two points q and q' , or is the orienting tangent line to γ at q : $u = \dot{\gamma}(s)$; in the latter case, we set $q' := q$;
- (b) the angle between the oriented line $L(q, u)$ and any of the orienting tangent vectors to γ at q or q' is acute. (In the case under consideration, this condition implies that the line $L(q, u)$ has acute angle with the orienting tangent vector $\dot{\gamma}$ at each point of the arc qq' (for appropriately chosen arc qq' in the case, when γ is a closed curve).)

Let u' denote the directing unit vector of the line $L(q, u)$ at q' . Consider the two following involutions acting on Γ^0 and Γ respectively:

$$\beta : \Gamma^0 \rightarrow \Gamma^0, \quad \beta(q, u) = (q', u'); \quad \beta^2 = \text{Id};$$

$$I : \Gamma \rightarrow \Gamma \text{ is the reflection from } T_q\gamma : I(q, u) = (q, u^*),$$

where u^* is the vector symmetric to u with respect to the tangent line $T_q\gamma$. Let $\Gamma^0_+ \subset \Gamma^0$ denote the open subset of those pairs (q, u) in which the vector u is directed to the convex side from the curve γ .

Remark 1.17. The domain Γ^0 is β -invariant. It is a topological disk (cylinder) if γ is parameterized by an interval (circle). The domain Γ^0_+ is a topological disk (cylinder) adjacent to $\tilde{\gamma}$.

Let Π_γ denote the open subset of the space of oriented lines in \mathbb{R}^2 consisting of the lines $L(q, u)$ with $(q, u) \in \Gamma_+^0$. The mapping $\Lambda : (q, u) \mapsto L(q, u)$ is a diffeomorphism

$$\Lambda : \Gamma_+^0 \rightarrow \Pi_\gamma.$$

It extends to the set $\Gamma_+^0 \cup \tilde{\gamma}$ as a homeomorphism sending each point $(s, \dot{\gamma}(s)) \in \tilde{\gamma}$ to the tangent line $T_s\gamma$ directed by $\dot{\gamma}(s)$.

Remark 1.18. Let \mathcal{T} denote the billiard ball map given by reflection from the curve γ acting on oriented lines. It is well known that the billiard ball map \mathcal{T} restricted to Π_γ is conjugated by Λ to the product of two involutions

$$\tilde{\delta}_+ := I \circ \beta = \Lambda^{-1} \circ \mathcal{T} \circ \Lambda : \Gamma_+^0 \rightarrow \Gamma.$$

If the curve γ is C^∞ -smooth, then both involutions I and β are C^∞ -smooth on Γ and Γ^0 respectively. Their product is well defined and smooth on Γ^0 and fixes the points of the curve $\tilde{\gamma}$. Both involutions preserve the canonical symplectic form $\sin \phi ds \wedge d\phi$ on $\Gamma \setminus \tilde{\gamma}$, which is known to be the Λ -pullback of the standard symplectic form on the space of oriented lines. See [2, 3, 16–18, 20]; see also [10, §7.1].

Let us recall another representation of the billiard ball map \mathcal{T} in a chart where it preserves the standard symplectic form. To do this, consider the orthogonal projection $\pi_\perp : (T\mathbb{R}^2)|_\gamma \rightarrow T\gamma$ sending each vector $u \in T_q\mathbb{R}^2$ with $q \in \gamma$ to its orthogonal projection to the tangent line $T_q\gamma$. It projects the unit tangent bundle Γ to the unit ball bundle

$$T_{\leq 1}\gamma := \{(q, w) \mid q \in \gamma, w \in T_q\gamma, \|w\| \leq 1\}.$$

A tangent vector $w = w(\partial/\partial s) \in T_q\gamma$ will be identified with its coordinate $w = \pm\|w\|$ in the basic vector $\partial/\partial s$. Thus, $\pi_\perp(s, \phi) = (s, \cos \phi)$. Consider the following function and differential 2-form on $T\gamma$:

$$y := 1 - w; \quad \omega := ds \wedge dy. \tag{1.2}$$

The form ω coincides with the standard symplectic form on the tangent bundle $T\gamma$ of the curve γ (considered as a Riemannian manifold equipped with the metric $|ds|^2$ coming from the standard Euclidean metric on \mathbb{R}^2).

The curve $\tilde{\gamma} = \{(s, \dot{\gamma}(s)) \mid s \in \gamma\} = \{w = 1\} = \{y = 0\} \subset T\gamma$ is a component of the boundary $\partial T_{\leq 1}\gamma$. The projection π_\perp sends Γ_+^0 diffeomorphically to a domain in $T_{\leq 1}\gamma$ adjacent to $\tilde{\gamma}$. It extends homeomorphically to $\Gamma_+^0 \cup \tilde{\gamma}$ as the identity map $\text{Id} : \tilde{\gamma} \rightarrow \tilde{\gamma}$. Let $\mu_+ : \pi_\perp(\Gamma_+^0 \cup \tilde{\gamma}) \rightarrow \Gamma_+^0 \cup \tilde{\gamma}$ be the inverse to the restriction of the projection π_\perp to $\Gamma_+^0 \cup \tilde{\gamma}$. Set

$$\delta_+ := \pi_\perp \circ \tilde{\delta}_+ \circ \mu_+ = \pi_\perp \circ \Lambda^{-1} \circ \mathcal{T} \circ \Lambda \circ \mu_+. \tag{1.3}$$

THEOREM 1.19. [20, §1.5], [2, 3, 17, 18]; see also [10, Theorem 7.3] *The mapping $\delta_+ : \pi_\perp(\Gamma_+^0) \rightarrow T_{\leq 1}\gamma$ given by equation (1.3), is symplectic: it preserves the form $\omega = ds \wedge dy$.*

PROPOSITION 1.20. [10, Proposition 7.5] Let $\kappa(s)$ denote the (geodesic) curvature of the curve γ . The involutions I , β and the mappings $\tilde{\delta}_+$, δ_+ admit the following (asymptotic) formulas:

$$I(s, \phi) = (s, -\phi), \beta(s, \phi) = (s + 2\kappa^{-1}(s)\phi + O(\phi^2), -\phi + O(\phi^2)), \tag{1.4}$$

$$\tilde{\delta}_+(s, \phi) = (s + 2\kappa^{-1}(s)\phi + O(\phi^2), \phi + O(\phi^2)), \tag{1.5}$$

$$\delta_+(s, y) = (s + 2\sqrt{2}\kappa^{-1}(s)\sqrt{y} + O(y), y + O(y^{3/2})). \tag{1.6}$$

The asymptotics are uniform on compact subsets of points $s \in \gamma$, as $\phi \rightarrow 0$ (respectively, as $y \rightarrow 0$).

Case (2). Let γ be parameterized by an interval, but now it does not necessarily go to infinity or bound a region in the plane. Moreover, we allow γ to be an immersed curve that may self-intersect. In this case, some lines $L(q, u)$ may intersect γ at more than two points. Now the definition of the subset $\Gamma^0 \subset \Gamma$ should be modified to be the subset of those $(q, u) \in \Gamma$ for which either $u = \dot{\gamma}(q)$ or there exists a $q' \in \gamma \cap L(q, u)$ satisfying the condition (b) from Case (1) and such that the arc $qq' \subset \gamma$ is disjoint from the line $L(q, u)$, injectively immersed (that is, without self-intersections) and satisfies the statement that the orienting tangent vector $\dot{\gamma}$ at each point has acute angle with $L(q, u)$. (Here q and q' may not be the only points of intersection $\gamma \cap L(q, u)$. If $u = \dot{\gamma}(q)$, then we set $q' = q$.)

Remark 1.21. For any given $(q, u) \in \Gamma^0$, the point q' satisfying the conditions from the above paragraph exists whenever u is close enough to $\dot{\gamma}(q)$ (depending on q). Whenever it exists, it is unique. All the statements and discussion in the previous Case (1) remain valid in our Case (2). Now the mapping Λ is a local diffeomorphism but not necessarily a global diffeomorphism: an oriented line intersecting γ at more than two points (if any) may correspond to at least two different tuples $(q, u) \in \Gamma^0_+$.

1.4. Generalization to C^∞ -lifted strongly billiard-like maps. In this subsection and in what follows, we study the next class of area-preserving mappings introduced in [10] generalizing the billiard maps in equation (1.6).

Definition 1.22. (See [10, Definition 7.6]) Let (a, b) be a (may be (semi) infinite) interval in \mathbb{R} with coordinate s . Let $V \subset \mathbb{R} \times \mathbb{R}_+$ be a domain adjacent to the interval $J := (a, b) \times \{0\}$. A mapping $F : V \cup J \rightarrow \mathbb{R} \times \mathbb{R}_{\geq 0} \subset \mathbb{R}^2_{s,y}$ is called *billiard-like* if it satisfies the following conditions:

- (i) $F : V \cup J \rightarrow F(V \cup J)$ is a homeomorphism fixing the points in J ;
- (ii) $F|_V$ is a diffeomorphism preserving the standard area form $ds \wedge dy$;
- (iii) F has the asymptotics of the type

$$F(s, y) = (s + w(s)\sqrt{y} + O(y), y + O(y^{3/2})) \quad \text{as } y \rightarrow 0; \quad w(s) > 0, \tag{1.7}$$

uniformly on compact subsets in the s -interval (a, b) ;

(iv) the variable change

$$(s, y) \mapsto (s, z), \quad z = \sqrt{y} > 0$$

conjugates F to a smooth map $\tilde{F}(s, z)$ (called its *lifting*) that is also smooth at points of the boundary interval J ; thus, $w(s)$ is continuous on (a, b) .

If, in addition to conditions (i)–(iv), the latter mapping \tilde{F} is a product of two involutions I and β fixing the points of the line $z = 0$,

$$\tilde{F} = I \circ \beta, \quad I(s, z) = (s, -z),$$

$$\beta(s, z) = (s + w(s)z + O(z^2), -z + O(z^2)), \quad \beta^2 = \text{Id}, \tag{1.8}$$

then F will be called a (*strongly*) *billiard-like map*.

If F is strongly billiard-like, and the corresponding involution β (or equivalently, the conjugate map \tilde{F}) is C^∞ -smooth and also C^∞ -smooth at the points of the boundary interval J , then F is called C^∞ -*lifted*. The above definitions make sense for F being a germ of map at the interval J .

Example 1.23. The mapping δ_+ from equation (1.6) is strongly billiard-like in the coordinates (s, y) with $w(s) = 2\sqrt{2}\kappa^{-1}(s)$, see equations (1.4), (1.5), and (1.6). If the curve γ is C^∞ -smooth, then β and hence $\tilde{\delta}_+ = I \circ \beta$ are C^∞ -smooth, and hence, δ_+ is C^∞ -lifted.

PROPOSITION 1.24. *The class of (germs at J of) C^∞ -lifted strongly billiard-like maps is invariant under conjugacy by (germs at J of) C^∞ -smooth symplectomorphisms $G : V \cup J \rightarrow G(V \cup J) \subset \mathbb{R} \times \mathbb{R}_{\geq 0}$ sending J onto an interval in $\mathbb{R} \times \{0\}$. Here, $V \subset \mathbb{R} \times \mathbb{R}_+$ is a domain adjacent to J .*

Proof. Let F be a C^∞ -lifted strongly billiard-like map and $\tilde{F} = I \circ \beta$ be its lifting. Let $V \subset \mathbb{R} \times \mathbb{R}_+$ be a domain adjacent to J . Let F be defined on $V \cup J$, and let $G : V \cup J \rightarrow G(V \cup J) \subset \mathbb{R} \times \mathbb{R}_{\geq 0}$ be a C^∞ -smooth symplectomorphism as above. Let us denote $G(s, y) = (\widehat{s}(s, y), \widehat{y}(s, y))$. One has $\widehat{y}(s, 0) \equiv 0$, $\partial \widehat{s}(s, 0)/\partial s > 0$, $\partial \widehat{y}(s, 0)/\partial y > 0$, by definition and orientation-preserving property (symplecticity). Thus, $\widehat{y}(s, y) = yg(s, y)$, where $g(s, y)$ is a positive C^∞ -smooth function on a neighborhood of the interval J in $(\mathbb{R} \times \mathbb{R}_{>0}) \cup J$. The lifting \tilde{G} of the map G to the variables (s, z) , $z = \sqrt{y}$, acts as follows:

$$\tilde{G} : (s, z) \mapsto (\widehat{s}(s, z^2), \widehat{z}(s, z)); \quad \widehat{z} = \sqrt{\widehat{y}(s, z^2)} = z\sqrt{g(s, z^2)}. \tag{1.9}$$

The latter square root is well defined and C^∞ -smooth. This implies that the map \tilde{G} is a C^∞ -smooth diffeomorphism of domains with arcs of boundaries corresponding to $V \cup J$ and $G(V \cup J)$. Hence, the lifting $\tilde{G} \circ \tilde{F} \circ \tilde{G}^{-1}$ of the conjugate $F_G := G \circ F \circ G^{-1}$ is a C^∞ -smooth diffeomorphism that is the product of \tilde{G} -conjugates of the involutions I and β . One has $\tilde{G} \circ I \circ \tilde{G}^{-1} = I$, by equation (1.9); F_G is a symplectomorphism, since so are F and G ;

$$G(s, y) = (\widehat{s}, \widehat{y}) = (\widehat{s}(s, 0) + O(y), g(s, 0)y + O(y^2)), \tag{1.10}$$

by diffeomorphicity. Substituting equations (1.10) and (1.7) to the expression $F_G = G \circ F \circ G^{-1}$ and denoting $(s, 0) := G^{-1}(\widehat{s}, 0)$, we get

$$F_G(\widehat{s}, \widehat{y}) = \left(\widehat{s} + \frac{\partial \widehat{s}}{\partial s}(s, 0)w(s)g^{-(1/2)}(s, 0)(\widehat{y})^{1/2} + O(\widehat{y}), \widehat{y} + O(\widehat{y}^{3/2}) \right).$$

This implies that the conjugate F_G has type of equation (1.7) and hence is strongly billiard-like. This proves the proposition. \square

Convention 1.25. Let $J = (a, b) \times \{0\} \subset \mathbb{R}^2_{x,y}$. Let $U \subset \{y > 0\}$ be a domain adjacent to J . Let $F : U \cup J \rightarrow \mathbb{R} \times \mathbb{R}_{\geq 0}$ be a map fixing all the points of the interval J . Let $\widetilde{h} : U \cup J \rightarrow \mathbb{R}_{\geq 0}$ be a C^∞ -smooth F -invariant function, that is, $\widetilde{h}(z) = \widetilde{h} \circ F(z)$ whenever $z, F(z) \in U \cup J$, and let

$$\widetilde{h}|_J \equiv 0, \quad \frac{\partial \widetilde{h}}{\partial y} > 0. \tag{1.11}$$

Let $\widetilde{h} = \text{const.}$ denote the foliation by connected components of level curves of the function \widetilde{h} . This is a C^∞ -smooth foliation on $U \cup J$, with J being a leaf. It will be called a *foliation by F -invariant curves*.

THEOREM 1.26. *For every C^∞ -lifted strongly billiard-like map F , there exists a domain U adjacent to J such that $U \cup J$ admits a C^∞ -smooth F -invariant function \widetilde{h} satisfying equation (1.11); thus, $\widetilde{h} = \text{const.}$ is a foliation by F -invariant curves. Moreover, U can be chosen so that there is an infinite-dimensional family of foliations as above with distinct boundary germs.*

THEOREM 1.27. *For every function \widetilde{h} from Theorem 1.26, replacing it by its post-composition with a C^∞ -smooth function of one variable (which does not change the foliation $\widetilde{h} = \text{const.}$) one can achieve that there exists a C^∞ -smooth function $\tau = \tau(s, y)$ and a domain $U \subset \{y > 0\}$ adjacent to J such that (τ, \widetilde{h}) are symplectic coordinates on $U \cup J$ and in these coordinates,*

$$F(\tau, \widetilde{h}) = (\tau + \sqrt{\widetilde{h}}, \widetilde{h}). \tag{1.12}$$

Definition 1.28. Let $V \subset \mathbb{R} \times \mathbb{R}_+ \subset \mathbb{R}^2_{s,y}$ be a domain adjacent to an interval $J = (a, b) \times \{0\}$. A C^∞ -smooth function $f(s, y)$ on $V \cup J$ is *y-flat*, if it vanishes on J together with all its partial derivatives in (s, y) ; that is, f has trivial jets of all orders at each point in J . (In this case, the function $f(s, y)$ and each of its derivatives converge to zero uniformly on compact subsets in the s -interval (a, b) , as $y \rightarrow 0$, by continuity.)

Remark 1.29. In the conditions of the above definition, let (x, h) be new C^∞ -smooth coordinates on $V \cup J$ with $h(s, 0) \equiv 0$. Then each y -flat function is h -flat and *vice versa*. This follows from definition.

The proof of Theorem 1.26 uses the result of Marvizi and Melrose [16, Theorem 3.2] stating a formal analogue of Theorem 1.26: the existence of an F -invariant formal power series $\sum_{k=1}^{+\infty} h_k(s)y^k$, see Theorem 2.1. It implies that in appropriate coordinates (τ, h) , the map F takes the form $F(\tau, h) = (\tau + \sqrt{h} + \text{flat}(h), h + \text{flat}(h))$. Here, $\text{flat}(h)$ is an

h -flat function, see the above definition. In the coordinates (τ, ϕ) , $\phi = \sqrt{h}$, the lifted map \tilde{F} takes the form

$$\tilde{F}(\tau, \phi) = (\tau + \phi + \text{flat}(\phi), \phi + \text{flat}(\phi)). \tag{1.13}$$

We prove the existence of a C^∞ -smooth \tilde{F} -invariant function $\tilde{\phi}$ with $\tilde{\phi} - \phi = \text{flat}(\phi)$ (the next theorem), and then deduce the existence statements in Theorems 1.26, 1.9, and 1.12.

THEOREM 1.30. *Let $V \subset \mathbb{R}_\tau \times (\mathbb{R}_+)_\phi$ be a domain adjacent to an interval $J = (a, b) \times \{0\}$. Let $\tilde{F} : V \cup J \rightarrow \mathbb{R}_\tau \times (\mathbb{R}_{\geq 0})_\phi$ be a C^∞ -smooth mapping of the type in equation (1.13). (Here we do not assume any area-preserving property.)*

- (1) *There exists a domain $W \subset V$ adjacent to J and an \tilde{F} -invariant C^∞ -smooth function $\tilde{\phi}$ on $W \cup J$ of the type $\tilde{\phi}(\tau, \phi) = \phi + \text{flat}(\phi)$; $(\partial\tilde{\phi}/\partial\phi) > 0$.*
- (2) *For every function $\tilde{\phi}$ as above, one can shrink the domain W (keeping it adjacent to J) so that there exists a C^∞ -smooth function $\tilde{\tau}(\tau, \phi) = \tau + \text{flat}(\phi)$ such that the map $(\tau, \phi) \mapsto (\tilde{\tau}, \tilde{\phi})$ is a C^∞ -smooth diffeomorphism on $W \cup J$ that conjugates \tilde{F} to the map*

$$\hat{F} : (\tilde{\tau}, \tilde{\phi}) \mapsto (\tilde{\tau} + \tilde{\phi}, \tilde{\phi}). \tag{1.14}$$

- (3) *There exists a continuum of functions $\tilde{\phi}$ satisfying statement (1) such that the corresponding foliations $\tilde{\phi} = \text{const.}$ are C^∞ -smooth on the same subset $W \cup J$ and form an infinite-dimensional family of foliations with distinct boundary germs.*

1.5. *Unique determination of jets. Space of germs of foliations. Non-uniqueness.*

THEOREM 1.31. *All the germs of foliations satisfying the statements of any of Theorems 1.9, 1.12, 1.13, 1.26, and 1.30 at the corresponding boundary curve γ , J are flatly close to each other near the boundary. That is, they have the same n -jet for every n at each point of the boundary.*

The statement of Theorem 1.31 follows from the result of Marvizi and Melrose [16, Theorem 3.2] recalled below as Theorem 2.1. For completeness of presentation, we will give a proof of Theorem 1.31 in §2.8.

Let us now describe the space of jets at J of foliations satisfying the statements of Theorem 1.26. Recall that there exist coordinates (τ, \tilde{h}) in which $J = \{\tilde{h} = 0\}$ and F acts as in equation (1.12): $F : (\tau, \tilde{h}) \mapsto (\tau + \sqrt{\tilde{h}}, \tilde{h})$. Fix these coordinates (τ, \tilde{h}) . Without loss of generality, we consider that $0 \in J$.

Consider the foliation $\tilde{h} = \text{const.}$ Let \mathcal{F} be another C^∞ -smooth foliation by F -invariant curves defined on a domain V in the upper half-plane $\{\tilde{h} > 0\}$ adjacent to J that extends C^∞ -smoothly to J with J being a leaf. It is a foliation by level curves of an F -invariant function $g(\tau, \tilde{h}) = \tilde{h} + \text{flat}(\tilde{h})$, which follows from Theorem 1.31. We can and will normalize g so that

$$g(\tau, \tilde{h}) = \tilde{h} + \text{flat}(\tilde{h}), \quad g(0, \tilde{h}) \equiv \tilde{h}. \tag{1.15}$$

Remark 1.32. The above normalization can be achieved by replacing the function g by its post-composition with a function $\phi + \text{flat}(\phi)$ of one variable ϕ . Each foliation from Theorem 1.26 admits a unique F -invariant first integral g as in equation (1.15) and vice versa: for every F -invariant function g as in equation (1.15), the foliation $g = \text{const.}$ satisfies the statement of Theorem 1.26.

PROPOSITION 1.33. (1) For every C^∞ -smooth function g on a domain $V \subset \{\tilde{h} > 0\}$ adjacent to J that extends C^∞ -smoothly to J , satisfies equation (1.15), and is invariant under the mapping $F : (\tau, \tilde{h}) \mapsto (\tau + \sqrt{\tilde{h}}, \tilde{h})$, there exist a $\delta > 0$ and a unique C^∞ -smooth \tilde{h} -flat function $\psi(t, \tilde{h})$ on $S^1 \times [0, \delta)$, $S^1 = \mathbb{R}_t/\mathbb{Z}$ such that

$$\psi(0, \tilde{h}) = 0, \tag{1.16}$$

$$g(\tau, \tilde{h}) = \tilde{h} + \psi\left(\frac{\tau}{\sqrt{\tilde{h}}}, \tilde{h}\right). \tag{1.17}$$

Here we treat $\psi(t, \tilde{h})$ as a function of two variables that is 1-periodic in t . Conversely, for every $\delta > 0$, every \tilde{h} -flat function ψ on the cylinder $S^1 \times [0, \delta)$ satisfying equation (1.16) corresponds to some function g as above via equation (1.17), defined on $V \cup J$ with $J = \mathbb{R} \times \{0\}$ and $V = \{0 < \tilde{h} < \delta\}$.

(2) The analogous statements hold for the map $F : (\tau, \phi) \mapsto (\tau + \phi, \phi)$ and the function

$$g(\tau, \phi) = \phi + \psi\left(\frac{\tau}{\phi}, \phi\right). \tag{1.18}$$

THEOREM 1.34. Every germ at J of C^∞ -smooth foliation \mathcal{F} by invariant curves under a map F of one of the types in equation (1.12) or (1.14) is defined by a unique germ at $S^1 \times \{0\}$ of C^∞ -smooth h -flat function $\psi(t, h)$, $\psi : S^1 \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, $\psi(0, h) = 0$, so that \mathcal{F} is the foliation by level curves of the corresponding function g given by equation (1.17) or (1.18) respectively. Conversely, each germ of function ψ as above defines a germ of foliation as above at J .

THEOREM 1.35. The space of germs of foliations satisfy statements of any of Theorems 1.9, 1.12, 1.13, 1.26, and 1.30 at the corresponding boundary curve γ or J is isomorphic to the space of h -flat germs at $S^1 \times \{0\}$ of C^∞ -smooth functions $\psi(t, h)$ on $S^1 \times \mathbb{R}_{\geq 0}$ with $\psi(0, h) = 0$.

Theorems 1.31, 1.34, and 1.35 and Proposition 1.33 will be proved in §2.8. In §2.9, we deduce non-uniqueness statements of Theorems 1.9, 1.12, 1.13, 1.26, and 1.30 from Theorem 1.35 and the following proposition, which will be also proved there.

PROPOSITION 1.36. Let $J = (a, b) \times \{0\}$, $W \subset \mathbb{R}_\tau \times (\mathbb{R}_+)_\phi$ be a domain adjacent to J . Let $\tilde{F} : W \cup J \rightarrow \mathbb{R} \times \mathbb{R}_{\geq 0}$ be a map, as in equation (1.13). Any two \tilde{F} -invariant foliations (functions, line fields) on W having distinct germs at J have distinct germs at each point in J . The same statement holds for similar objects invariant under a C^∞ -lifted strongly billiard-like map.

1.6. *Corollaries on conjugacy of open billiard maps near the boundary.* The results stated below and proved in §2.10 concern (symplectic) conjugacy of billiard maps near the boundary.

Here we deal with a strictly convex oriented C^∞ -smooth curve $\gamma \subset \mathbb{R}^2$ that is not closed: parameterized by an interval. We consider that it is positively oriented as the boundary of its convex side. Let us first consider that γ goes to infinity in both directions and bounds a convex open billiard. By $\tilde{\gamma}$, we denote the family of its orienting unit tangent vectors; $\tilde{\gamma}$ lies in the space Γ , which is the unit tangent bundle of the ambient plane restricted to γ . Let s be a natural length parameter of the curve γ . Let $\Gamma_+^0 \subset \Gamma$ be the open subset adjacent to $\tilde{\gamma}$ defined in §1.3. It lies in the space of pairs (s, v) , where $s \in \gamma$ and $v \in T_s \mathbb{R}^2$ is a unit vector directed to the convex side from the curve γ . Recall that $\phi = \phi(v)$ denotes the angle between the vector v and the unit tangent vector $\dot{\gamma}(s)$. Let $(a, b) = (a_\gamma, b_\gamma) \subset \mathbb{R}$ denote the length parameter interval parameterizing γ . In the coordinates (s, ϕ) , the curve $\tilde{\gamma}$ is the interval $J = J_\gamma = (a, b) \times \{0\}$, and Γ_+^0 is a domain in $\mathbb{R} \times \mathbb{R}_+$ adjacent to J . Set $y = 1 - \cos \phi$, see equation (1.2). Recall that the billiard map \mathcal{T}_γ acting by reflection from γ of the above unit vectors is a C^∞ -smooth diffeomorphism defined on $\Gamma_+^0 \cup \tilde{\gamma}$. In the coordinates (s, y) , it is a symplectic map: a C^∞ -lifted strongly billiard-like map defined on $\mathcal{V} \cup J$, where $\mathcal{V} \subset \mathbb{R}_s \times (\mathbb{R}_+)_y$ is a domain adjacent to J .

The above statements remain valid in the case when the curve γ in question is a subarc (parameterized by interval but not necessarily infinite) of a strictly convex C^∞ -smooth curve; γ also may be an immersed curve.

Definition 1.37. Let $\gamma_1, \gamma_2 \subset \mathbb{R}^2$ be strictly convex C^∞ -smooth planar curves parameterized by intervals (they are allowed to be immersed), positively oriented as boundaries of their convex sides. Let $J_{\gamma_i} \subset \mathbb{R} \times \{0\}$, $i = 1, 2$, be the corresponding intervals defined above. We say that the billiard maps \mathcal{T}_{γ_i} are C^∞ -smoothly conjugated near the boundary in the (s, ϕ) - ((s, y) -) coordinates if there exist domains U_i in $\mathbb{R}_s \times (\mathbb{R}_+)_\phi$ (respectively, in $\mathbb{R}_s \times (\mathbb{R}_+)_y$) adjacent to J_{γ_i} and a C^∞ -smooth diffeomorphism $H : U_1 \cup J_{\gamma_1} \rightarrow U_2 \cup J_{\gamma_2}$ conjugating the billiard maps, $H \circ \mathcal{T}_{\gamma_1} \circ H^{-1} = \mathcal{T}_{\gamma_2}$. In the case when the billiard maps are conjugated in the (s, y) -coordinates, and the conjugating diffeomorphism H is a symplectomorphism, we say that they are C^∞ -smoothly symplectically conjugated near the boundary.

Remark 1.38. Smooth conjugacy of billiard maps near the boundary in the coordinates (s, y) implies their smooth conjugacy in the coordinates (s, ϕ) . This follows from the fact that for every two intervals $J_1, J_2 \subset \mathbb{R}_s \times \{0\}$ and every two domains $U_1, U_2 \subset \mathbb{R}_s \times (\mathbb{R}_+)_y$ adjacent to J_1 and J_2 respectively, every diffeomorphism $H : U_1 \cup J_1 \rightarrow U_2 \cup J_2$ lifts to a diffeomorphism of the corresponding domains in the (s, ϕ) -coordinates (taken together with adjacent intervals J_i). The latter statement follows from [10, Lemma 3.1] applied to the second component of the diffeomorphism H .

The results stated below on conjugacy of billiard maps near the boundary are corollaries of Theorems 1.27 and 1.30 on normal forms of C^∞ -lifted strongly billiard-like maps and their liftings.

THEOREM 1.39. *Let γ_1, γ_2 be strictly convex C^∞ -smooth one-dimensional submanifolds in \mathbb{R}^2 parameterized by intervals (thus, going to infinity in both directions) and positively oriented as boundaries of their convex sides. Let, in addition, the curves γ_i have finite asymptotic tangent lines at infinity: as $x \in \gamma_i$ tends to infinity (in each direction), the tangent line $T_x\gamma_i$ converges to a finite line. Then the corresponding billiard maps are C^∞ -smoothly conjugated near the boundary in (s, y) - (and hence, in (s, ϕ) -) coordinates.*

THEOREM 1.40. *The statement of Theorem 1.39 on conjugacy of billiard maps corresponding to C^∞ -smooth strictly convex curves γ_i remains valid in the case when each γ_j is either a submanifold going to infinity in both directions or a (may be immersed) subarc of an (immersed) C^∞ -smooth curve, and the two following statements hold:*

- (1) *as the length parameter s of the curve γ_j goes to an endpoint of the length parameter interval, the corresponding point of the curve γ_j tends either to a finite limit (endpoint of γ_j) where γ_j is C^2 -smooth, or to infinity;*
- (2) *in the latter case, when the limit is infinite, the tangent line $T_s\gamma_j$ has a finite limit: a finite asymptotic tangent line.*

Remark 1.41. Kaloshin and Koudjjan [12] proved continuous conjugacy near the boundary of two billiard maps corresponding to two arbitrary ellipses. For any two ellipses with two appropriate points deleted in each of them, they have also proved smooth conjugacy of the corresponding billiard maps on open domains adjacent to the corresponding boundary intervals J in the (s, ϕ) -coordinates.

Below we state a more general result and provide a sufficient condition of symplectic conjugacy of billiard maps in the coordinates (s, y) . To this end, let us recall the following definition.

Definition 1.42. Let γ be a C^2 -smooth oriented planar curve, and let s be its length parameter defining its orientation. Let $J_\gamma = (a_\gamma, b_\gamma) \subset \mathbb{R}_s$ denote the length parameter interval parameterizing γ . Let $\kappa = \kappa(s)$ denote the geodesic curvature of the curve γ as a function of s . The *Lazutkin length* of the curve γ is the integral

$$\mathcal{L}(\gamma) := \int_{a_\gamma}^{b_\gamma} \kappa^{2/3}(s) ds, \tag{1.19}$$

see [15, equation (1.3)]. (While the length parameter interval is defined up to translation, the integral $\mathcal{L}(\gamma)$ is uniquely defined.)

THEOREM 1.43. *Let γ_1, γ_2 be two strictly convex C^∞ -smooth (may be immersed) planar curves, parameterized by intervals and positively oriented as local boundaries of their convex sides. The corresponding billiard maps are C^∞ -smoothly conjugated near the boundary in (s, y) -coordinates, if and only if one of the two following conditions holds:*

- (i) *either both Lazutkin lengths $\mathcal{L}(\gamma_i)$ are finite;*
- (ii) *or both Lazutkin lengths $\mathcal{L}(\gamma_i)$ are infinite and the improper integrals in equation (1.19) defining them are*

- (a) either both infinite in both directions;
- (b) or both infinite in one and the same direction (with respect to the orientations of the curves γ_i) and both finite in the other direction.

The same criterion also holds for C^∞ -smooth conjugacy near the boundary in (s, ϕ) -coordinates.

THEOREM 1.44. *Let in the conditions of Theorem 1.43 some of conditions (i) or (ii) hold. Then the billiard maps are C^∞ -smoothly symplectically conjugated near the boundary if and only if the Lazutkin lengths of the curves γ_j are either both finite and equal, or both infinite and the above condition (ii) holds.*

Theorems 1.39 and 1.40 will be deduced from Theorem 1.43 using the following propositions on C^∞ -lifted strongly billiard-like maps and lemma on curves with asymptotic line at infinity.

PROPOSITION 1.45. *Let $F(s, y) = (s + w(s)\sqrt{y} + O(y), y + O(y^{3/2}))$ be a C^∞ -lifted strongly billiard-like map, see equation (1.7), defined on $U \cup J$, where $J = (a, b) \times \{0\}$ and $U \subset \mathbb{R} \times \mathbb{R}_+$ is a domain adjacent to J . Let $H(s, y) = (H_1(s, y), H_2(s, y))$ be a C^∞ -smooth diffeomorphism of the domain with boundary $U \cup J$ onto its image in $\mathbb{R} \times \mathbb{R}_{\geq 0}$ that conjugates F with its normal form $\Lambda : (t, z) \mapsto (t + \sqrt{z}, z)$, that is, $H \circ F \circ H^{-1} = \Lambda$. Fix a $s_0 \in (a, b)$.*

- (1) The diffeomorphism H is orientation-preserving, $H(J) \subset \mathbb{R} \times \{0\}$, and the restriction $H_1(s, 0)$ to J of its first component is an increasing function.
- (2) If H is symplectic, then, up to additive constant,

$$H_1(s, 0) = t_L(s) := \int_{s_0}^s w^{-(2/3)}(s) ds. \tag{1.20}$$

- (3) If H is not necessarily symplectic, then

$$H_1(s, 0) = \alpha t_L(s) + \beta \quad \text{for some } \alpha > 0 \quad \text{and } \beta \in \mathbb{R}. \tag{1.21}$$

PROPOSITION 1.46. *Let F, U, J be the same, as in Proposition 1.45. Let \tilde{F} be the lifting of the map F to the coordinates (s, ψ) , $\psi^2 = y$, which is a C^∞ -smooth diffeomorphism defined on $\tilde{U} \cup J$, $\tilde{U} = \{(s, \psi) \mid (s, \psi^2) \in U, \psi > 0\}$. Let \tilde{H} be a C^∞ -smooth diffeomorphism defined on $\tilde{U} \cup J$ conjugating \tilde{F} with the diffeomorphism $\tilde{\Lambda} : (t, \tilde{z}) \mapsto (t + \tilde{z}, \tilde{z})$: $\tilde{H} \circ \tilde{F} \circ \tilde{H}^{-1} = \tilde{\Lambda}$. Then $\tilde{H}(J) \subset \mathbb{R} \times \{0\}$, and the first component of the map \tilde{H} satisfies equation (1.21).*

LEMMA 1.47. *Let a C^∞ -smooth strictly convex planar curve γ go to infinity in some direction, and let it have a finite asymptotic tangent line at infinity in this direction (in the same sense as in Theorem 1.39). Then the improper integral in equation (1.19) defining the Lazutkin length converges in the given direction.*

Remark 1.48. For a C^∞ -smooth strictly convex planar curve going to infinity, the existence of a finite asymptotic tangent line is not a necessary condition for convergence of the improper integral in equation (1.19) defining the Lazutkin length. Namely, consider the graph $\{y = x^r\} \subset [1, +\infty) \times [1, +\infty)$, $r > 1$. One has

$$\int \kappa^{2/3}(s) ds < +\infty \quad \text{if and only if } r > 2. \tag{1.22}$$

Indeed, $ds = \sqrt{1 + r^2x^{2(r-1)}} dx$, $\kappa(s(x)) = r(r-1)x^{r-2}/(1 + r^2x^{2(r-1)})^{3/2}$, see equation (2.59),

$$\kappa^{2/3}(s(x)) ds = (r(r-1))^{2/3} \frac{x^{(2/3)(r-2)}}{\sqrt{1 + r^2x^{2(r-1)}}} dx \simeq (r(r-1))^{2/3} x^\nu dx, \quad \nu = -\frac{r+1}{3},$$

as $x \rightarrow +\infty$. Therefore, the integral in equation (1.22) converges if and only if $\nu < -1$, that is, $r > 2$. In the case of parabola $\{y = x^2\}$, the integral in equation (1.22) diverges.

1.7. *Plan of the proof of main results.* In §2.1, we recall the above-mentioned result of Marvizi and Melrose [16, Theorem 3.2] (with proof) yielding C^∞ -smooth coordinates in which a C^∞ -lifted strongly billiard-like map F takes the form $F(\tau, h) = (\tau + \sqrt{h} + \text{flat}(h), h + \text{flat}(h))$. It implies that the lifted map \tilde{F} , written in the coordinates (τ, ϕ) , $\phi = \sqrt{h}$, takes form of equation (1.13).

Statement (1) of Theorem 1.30 will be proved in §§2.2–2.4. To do this, first in §2.2, we construct a fundamental domain for the map \tilde{F} (a curvilinear sector Δ with vertex at a point in J) and an \tilde{F} -invariant function $\tilde{\phi}$ defined on a bigger sector that is ϕ -flatly close to ϕ on the latter bigger sector. Then in §2.3, we construct its \tilde{F} -invariant extension along the \tilde{F} -orbits and show that it is well defined on a domain adjacent to J . In §2.4, we prove that the extended function $\tilde{\phi}$ is C^∞ -smooth and ϕ -flatly close to ϕ . This will prove statement (1) of Theorem 1.30. Statement (2) on normal form will be proved in §2.5.

The existence statement in Theorem 1.26 will be deduced from statement (1) of Theorem 1.30 in §2.6, where we will also prove Theorem 1.27. Existence in Theorems 1.9, 1.12, and 1.13 will be proved in §2.7. The results from §1.5 on jets and space of germs of foliations will be proved in §2.8. Proposition 1.36 and non-uniqueness statements in the main theorems will be proved in §2.9.

The results of §1.6 on conjugacy of billiard maps near the boundary will be proved in §2.10.

2. Construction of foliation by invariant curves. Proofs of main results

2.1. *Marvizi and Melrose construction of an ‘up-to-flat’ first integral.* Here we recall the following theorem of Marvizi and Melrose with proof. Though it was stated in [16] for billiard ball maps, its statement and proof remain valid for C^∞ -lifted strongly billiard-like maps.

THEOREM 2.1. [16, Theorem 3.2]

- (1) *Let $V \subset (a, b) \times \mathbb{R}_{>0} \subset \mathbb{R}_{s,y}^2$ be a domain adjacent to the interval $J := (a, b) \times \{0\}$. Let $F : V \cup J \rightarrow \mathbb{R} \times \mathbb{R}_{\geq 0}$ be a C^∞ -lifted strongly billiard-like map. There exists a domain $W \subset V$ adjacent to J and a real-valued C^∞ -smooth function $h : W \cup J \rightarrow \mathbb{R}_{\geq 0}$, $h|_J \equiv 0$, $(\partial h / \partial y)|_J > 0$, such that the difference $h \circ F - h$ is C^∞ -smooth and y -flat. Moreover, one can normalize h as above so that the mapping F coincides, up to y -flat terms, with the time-1 map of the flow of the Hamiltonian vector field*

with the Hamiltonian function $\frac{2}{3}h^{3/2}$. This normalization determines the asymptotic Taylor series $h(s, y) = \sum_{k=1}^{+\infty} h_k(s)y^k$ of the function $h(s, y)$ uniquely.

- (2) The analogue of the above statement holds if J is replaced by the coordinate circle $S^1 = S^1 \times \{0\}$, $S^1 := \mathbb{R}_s/\mathbb{Z}$, lying in the cylinder $C := S^1 \times [0, \varepsilon)$ equipped with the standard area form and F is a strongly billiard-like map $C \rightarrow S^1 \times \mathbb{R}_{\geq 0}$. In this case, the coefficients $h_k(s)$ of the above normalized series are 1-periodic and C^∞ -smooth.
- (3) Let h be the function normalized as in statement (1). Let τ denote the time function for the Hamiltonian vector field with the Hamiltonian function h . In the coordinates (τ, h) (which are symplectic), the map F takes the form

$$F : (\tau, h) \mapsto (\tau + \sqrt{h} + \text{flat}(h), h + \text{flat}(h)). \tag{2.1}$$

Proof. The lifting $\tilde{F}(s, z)$, $z = \sqrt{y}$, of the map $F(s, y)$ is C^∞ -smooth and has the form

$$\tilde{F}(s, z) = \left(s + w(s)z + O(z^2), z + \frac{q(s)}{2}z^2 + O(z^3) \right), \tag{2.2}$$

where $q(s)$ is a C^∞ -smooth function on (a, b) . This follows from equation (1.7) and C^∞ -liftedness. The map $\tilde{F}(s, z)$ admits an asymptotic Taylor series in z . The map F has the form

$$F(s, y) = (s + w(s)\sqrt{y} + O(y), y + q(s)y^{3/2} + O(y^2)), \tag{2.3}$$

by equation (2.2), and it admits an asymptotic Puiseux series in y involving powers $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$. The coefficients of both series are C^∞ -smooth functions in s . Therefore, the mapping F acts by the formula $h \mapsto h \circ F$ not only on functions, but also on formal Puiseux series. It transforms each power series $h = \sum_{k=1}^{+\infty} h_k(s)y^k$ with coefficients being C^∞ -smooth functions on (a, b) to a Puiseux series of the above type. Our goal is to find an F -invariant power series (or equivalently, an \tilde{F} -invariant even power series $\sum_{k=1}^{+\infty} h_k(s)z^{2k}$) and then to choose its C^∞ -smooth representative. To do this, we use the following formula for the function $q(s)$ in equation (2.3), see [15, equation (1.2)], [10, equation (7.18)], which follows from the area-preserving property:

$$q(s) = -\frac{2}{3}w'(s). \tag{2.4}$$

Step 1. Constructing an even series $\sum_{k=1}^{+\infty} g_k(s)z^{2k}$ whose \tilde{F} -image is also an even series. We construct its coefficients g_k by induction as follows.

Induction base: $k = 1$. Let us find a function $g_1(s)$ such that the \tilde{F} -image of the function $g_1(s)z^2$ contains no z^3 -term. This is equivalent to the statement saying that the function $g_1(s + w(s)z)(z + (q(s)/2)z^2)^2$ contains no z^3 -term, which is, in its turn, equivalent to the differential equation

$$g_1'(s)w(s) + q(s)g_1(s) = 0, \quad q(s) = -\frac{2}{3}w'(s),$$

which has a unique solution $g_1(s) = w^{2/3}(s)$ up to constant factor. (Note that $w^{2/3}(s)y$ is a well-known function: the second Lazutkin coordinate [15, 16].)

Induction step in the case when $J = (a, b) \times \{0\}$ is an interval. We have already found an even Taylor polynomial $G_{n-1}(s, z) := \sum_{k=1}^{n-1} g_k(s)z^{2k}$, $n \geq 2$, such that the asymptotic

Taylor series in z of the function $G_{n-1} \circ \tilde{F}$ contains no odd powers of z of degrees no greater than $2n - 1$. Let us construct $g_n(s)$ and set $G_n(s, z) := \sum_{k=1}^n g_k(s)z^{2k}$ so that

$$G_n \circ \tilde{F} - G_n \text{ contains no } z^{2n+1}\text{-term.} \tag{2.5}$$

Note that $G_n \circ \tilde{F} - G_n$ obviously cannot contain odd powers of degrees less than $2n$. Let $b(s)z^{2n+1}$ denote the degree $2n + 1$ term in the Taylor series of the function $G_{n-1} \circ \tilde{F}$. Equation (2.5) is equivalent to the differential equation

$$g'_n(s)w(s) - \frac{2n}{3}w'(s)g_n(s) = -b(s), \tag{2.6}$$

which always has a solution $g_n(s)$ well defined on the interval (a, b) .

Step 2. Constructing an \tilde{F} -invariant series. The mapping \tilde{F} is the product $I \circ \beta$ of two involutions: $I(s, z) = (s, -z)$ and β . Let $g := \sum_{k=1}^{+\infty} g_k(s)z^{2k}$ be the series constructed in Step 1. One has

$$g \circ \tilde{F} = (g \circ I) \circ \beta = g \circ \beta, \tag{2.7}$$

since the series g is even. The series in equation (2.7) is even (Step 1). Hence, the series

$$t := g + g \circ \beta$$

is even and β -invariant by construction. Therefore, it is \tilde{F} -invariant. Its first coefficient is equal to $2g_1(s) = 2w^{2/3}(s) > 0$, by construction. We denote the \tilde{F} -invariant series thus constructed by $t := \sum_{k=1}^{+\infty} t_k(s)z^{2k}$.

Step 3. Symplectic coordinates and normalization. Let $t(s, y)$ be a function representing the series $\sum_{k=1}^{+\infty} t_k(s)y^k$, which is obtained from the latter series (given by Step 2) by the variable change $y = z^2$. It is defined on a domain W adjacent to J and C^∞ -smooth on $W \cup J$; $t|_J \equiv 0, (\partial t / \partial y)|_J > 0$. Let H_t denote the corresponding Hamiltonian vector field. Fix an arbitrary C^∞ -smooth function θ such that $d\theta(H_t) \equiv 1, \theta|_{s=0} = 0$: a time function for the vector field H_t . Then (θ, t) are symplectic coordinates for the form $\omega = dx \wedge dy$: $\omega = d\theta \wedge dt$. Shrinking W (keeping it adjacent to J), we can and will consider that they are global coordinates on $W \cup J$. The difference $t \circ F - t$ is t -flat, by construction, and hence, so is $dF(H_t) - H_t$. Therefore, in the coordinates (θ, t) , the symplectic map F takes the form

$$F : (\theta, t) \mapsto (\theta + \xi(t), t) + \text{flat}(t). \tag{2.8}$$

In the new coordinates (θ, t) , the map F is C^∞ -lifted strongly billiard-like, as in the old coordinates (s, y) , by Proposition 1.24.

CLAIM 1. *The function $\xi(t)$ in equation (2.8) has the form $\xi(t) = \sqrt{t}\psi(t)$, where $\psi(t)$ is a C^∞ -smooth function on a segment $[0, \varepsilon]$, $\varepsilon > 0, \psi \geq 0, \psi(0) > 0$.*

Proof. Let \tilde{F} denote the lifting of the map F to the coordinates $(\theta, \zeta), \zeta = \sqrt{t}$. One has $\tilde{F} = I \circ \beta$, where $I(\theta, \zeta) = (\theta, -\zeta)$ and β is an involution, $\beta(\theta, 0) \equiv (\theta, 0)$. The involution β takes the form

$$\beta(\theta, \zeta) = (\theta + r(\zeta), -\zeta) + \text{flat}(\zeta), \quad r(\zeta) = \xi(\zeta^2) \quad \text{for } \zeta > 0. \tag{2.9}$$

The function $r(\zeta)$ should be C^∞ -smooth, as is β , and $r'(0) > 0$ (strong billiard-likedness). The condition saying that β is an involution implies that $r(\zeta) + r(-\zeta) = \text{flat}(\zeta)$. This, in its turn, implies that $r(\zeta) = \zeta \psi(\zeta^2) + \text{flat}(\zeta)$, where ψ is a C^∞ -smooth function; $\psi(0) = r'(0) > 0$. This together with equation (2.9) implies the statement of the claim. \square

We have to find a function $h(s, y)$, $h(s, 0) \equiv 0$, such that the Hamiltonian vector field with the Hamiltonian function $\frac{2}{3}h^{3/2}$ coincides with $\xi(t)\partial/\partial\theta$. This function will satisfy the normalization statement of Theorem 2.1, part (1), by construction. We are looking for it as a function depending only on t : $h(s, y) = v(t)$. The above Hamiltonian vector field is then equal to $\sqrt{v(t)}v'(t)\partial/\partial\theta$. Thus, we have to solve the equation

$$v^{1/2}(t)v'(t) = \xi(t) = \sqrt{t}\psi(t), v(0) = 0.$$

Its solution $v(t)$ is given by

$$v(t) = \left(\frac{3}{2} \int_0^t \sqrt{p}\psi(p) dp \right)^{2/3}.$$

This is a C^∞ -smooth function, by construction and smoothness of the function $\psi(t)$. One has $(\partial h/\partial y)|_J > 0$, since $v'(0) = \psi^{2/3}(0) > 0$ and $\partial t/\partial y(s, 0) = 2g_1(s) = 2w^{2/3}(s) > 0$, by construction. Uniqueness of the Taylor series in y of the function $h(s, y)$ satisfying the above Hamiltonian vector field statement (up to flat terms) follows directly, as in [16, p. 383]. Statement (1) of Theorem 2.1 is proved. Statement (3) follows immediately from statement (1), since in the coordinates (τ, h) , see statement (3), the Hamiltonian field with the Hamiltonian function $\frac{2}{3}h^{3/2}$ is equal to $(\sqrt{h}, 0)$. Statement (2) (in the case when J is a circle and F is defined on a cylinder bounded by J) says that the Taylor coefficients of the series in y of the function $h(s, y)$ are well-defined functions on the circle J . This follows from the above uniqueness statement (which holds locally, in a neighborhood of every point $(s_0, 0) \in J$). Theorem 2.1 is proved. \square

2.2. *Step 1. Construction of an invariant function on a neighborhood of fundamental domain.* Here we give the first step of the proof of Theorem 1.30. We consider a fundamental sector Δ for the map \tilde{F} that is bounded by the segment $K = [0, \eta/2]$ of the ϕ -axis, by its \tilde{F} -image and by the straightline segment connecting their ends. We construct an \tilde{F} -invariant function $\tilde{\phi}$ that is ϕ -flatly close to ϕ on a sectorial neighborhood $S_{\chi, \eta}$ of $\bar{\Delta} \setminus \{(0, 0)\}$. See Figure 5.

Without loss of generality, we consider that the τ -interval contains the origin: $a < 0 < b$. Fix a number χ , $0 < \chi < \frac{1}{2}$. Consider the sectors

$$S_\chi = \{-\chi\phi < \tau < (1 + \chi)\phi\} \subset \mathbb{R}_\tau \times (\mathbb{R}_+)_\phi, \tag{2.10}$$

$$S_{\chi, \eta} := S_\chi \cap \{0 < \phi < \eta\}.$$

The domain $S_{\chi, \eta}$ will be the above-mentioned neighborhood of fundamental sector, where we construct an \tilde{F} -invariant function.

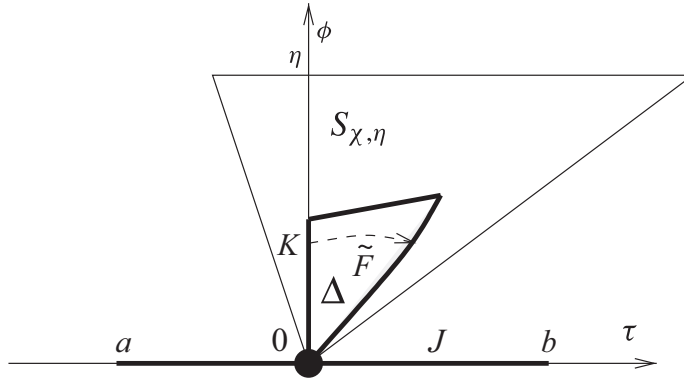


FIGURE 5. The fundamental domain Δ and its sectorial neighborhood $S_{\chi, \eta}$.

PROPOSITION 2.2. For every $\chi \in (0, \frac{1}{2})$ and $\eta > 0$ small enough depending on \tilde{F} and χ , the following statements hold.

- (i) The maps $\tilde{F}^{\pm 1}, \tilde{F}^{\pm 2}$ are well defined on $S_{\chi, 2\eta}$.
- (ii) The domains $S_{\chi, 2\eta}$ and $\tilde{F}^2(S_{\chi, 2\eta})$ are disjoint; the latter lies on the right from the former.
- (iii) The segment $K := \{0\} \times [0, \eta/2] \subset \mathbb{R}^2_{\tau, \phi}$ and its image $\tilde{F}(K)$ intersect just by the origin; $\tilde{F}(K)$ lies on the right from K . The domain $\Delta \subset S_{\chi, 2\eta}$ bounded by $K, \tilde{F}(K)$ and the straightline segment connecting the endpoints of the arcs K and $\tilde{F}(K)$ distinct from $(0, 0)$ is a fundamental domain for the map \tilde{F} . See Figure 5.

Proof. One has

$$d\tilde{F}(0, 0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.11}$$

The latter differential sends each line $\{\tau = \zeta\phi\}$ to the line $\{\tau = (\zeta + 1)\phi\}$. This implies that for every $\eta > 0$ small enough, statements (i)–(iii) hold. □

PROPOSITION 2.3. For every $\chi \in (0, \frac{1}{2})$ and $\eta > 0$ small enough depending on \tilde{F} and χ , there exists a C^∞ -smooth and \tilde{F} -invariant function $\tilde{\phi}(\tau, \phi)$ on $S_{\chi, \eta}$ such that the difference $\tilde{\phi}(\tau, \phi) - \phi$ is ϕ -flat on $S_{\chi, \eta}$: that is, tends to zero with all its partial derivatives, as $(\tau, \phi) \in S_{\chi, \eta}$ tends to zero.

Proof. Let $\nu : S_\chi \rightarrow \mathbb{R}$ denote the function

$$\nu := \frac{\tau}{\phi},$$

whose level curves are lines through the origin. The interval of values of the function ν on S_χ is $M := (-\chi, 1 + \chi)$. Fix a

$$\sigma > 0, \quad 2\sigma < \frac{1}{2} - \chi. \tag{2.12}$$

Consider the covering of the interval M by the intervals

$$(-\chi, \frac{1}{2} + \sigma), \quad (\frac{1}{2} - \sigma, 1 + \chi)$$

and a corresponding C^∞ -smooth partition of unity $\rho_1, \rho_2: M \rightarrow \mathbb{R}$,

$$\rho_1 \equiv 1 \text{ on } (-\chi, \frac{1}{2} - \sigma); \quad \rho_2 \equiv 1 \text{ on } (\frac{1}{2} + \sigma, 1 + \chi); \tag{2.13}$$

$$\rho_1, \rho_2 \geq 0, \quad \rho_1 + \rho_2 \equiv 1 \text{ on } M = (-\chi, 1 + \chi).$$

Set

$$\begin{aligned} \tilde{\phi}(x) &:= \rho_1(v(x))\phi(x) + \rho_2(v(x))\phi \circ \tilde{F}^{-1}(x) \\ &= \phi(x) + \rho_2(v(x))(\phi \circ \tilde{F}^{-1}(x) - \phi(x)). \end{aligned} \tag{2.14}$$

PROPOSITION 2.4. *For every fixed $\chi \in (0, \frac{1}{2})$, $\sigma \in (0, \frac{1}{2}(\frac{1}{2} - \chi))$ and every η small enough (depending on χ and σ), the function $\tilde{\phi}$ given by equation (2.14) is well defined on $S_{\chi,\eta}$ and \tilde{F} -invariant: if $x, \tilde{F}(x) \in S_{\chi,\eta}$, then $\tilde{\phi}(\tilde{F}(x)) = \tilde{\phi}(x)$. It is C^∞ -smooth, and the difference $\tilde{\phi}(x) - \phi(x)$ is ϕ -flat on $S_{\chi,\eta}$.*

Proof. Recall that \tilde{F} satisfies asymptotic equation (1.13):

$$\tilde{F}(\tau, \phi) = (\tau + \phi + \text{flat}(\phi), \phi + \text{flat}(\phi)).$$

Well-definedness and C^∞ -smoothness of the function $\tilde{\phi}$ on $S_{\chi,\eta}$ for small η are obvious. The difference $\tilde{\phi}(x) - \phi(x)$ is ϕ -flat on $S_{\chi,\eta}$, by equation (2.14), ϕ -flatness of the difference $\phi \circ \tilde{F} - \phi$, see equation (1.13), and the fact that the function $v(\tau, \phi) = \tau/\phi$ has partial derivatives of at most polynomial growth in ϕ , as $(\tau, \phi) \rightarrow 0$ along the sector $S_{\chi,\eta}$. Let us prove \tilde{F} -invariance of the function $\tilde{\phi}$ for η small enough. For every $\delta > 0$ and every $\eta > 0$ small enough (depending on δ), the inclusion $x, \tilde{F}(x) \in S_{\chi,\eta}$ implies that $x \in \{-\chi\phi < \tau < (\chi + \delta)\phi\}$, see equation (2.11). Choosing $\delta < \sigma$, we get that the latter sector lies in the sector $\{-\chi\phi < \tau < (\frac{1}{2} - \sigma)\phi\}$, since $\chi + \delta < \chi + \sigma < \frac{1}{2} - \sigma$, see equation (2.12). Thus, on the latter sector, $\rho_1 \circ v \equiv 1$ and $\rho_2 \circ v \equiv 0$, see equation (2.13). Hence, $\tilde{\phi}(x) = \phi(x)$, by equation (2.14). Similarly applying the above argument ‘in the inverse time’ yields that the inclusion $x, \tilde{F}(x) \in S_{\chi,\eta}$ implies that $\tilde{F}(x)$ lies in the sector $\{(1 - \chi - \delta)\phi < \tau < (1 + \chi)\phi\}$. The latter sector and hence $\tilde{F}(x)$ lie in the sector $\{(\frac{1}{2} + \sigma)\phi < \tau < (1 + \chi)\phi\}$, since

$$1 - \chi - \delta > 1 - \chi - \sigma = 1 - \chi + \sigma - 2\sigma > 1 - \chi + \sigma - \frac{1}{2} + \chi = \frac{1}{2} + \sigma.$$

Therefore, $\rho_2 \circ v(\tilde{F}(x)) = 1$, by equation (2.13), and $\tilde{\phi}(\tilde{F}(x)) = \phi \circ \tilde{F}^{-1}(\tilde{F}(x)) = \phi(x)$, by equation (2.14). Finally, we get that $\tilde{\phi}(x) = \tilde{\phi} \circ \tilde{F}(x)$ and hence $\tilde{\phi}$ is \tilde{F} -invariant. The proposition is proved. □

Proposition 2.4 immediately implies the statement of Proposition 2.3. □

2.3. Step 2. Extension by dynamics. Here we show that an \tilde{F} -invariant function $\tilde{\phi}$ constructed above on a neighborhood of the fundamental domain Δ extends along \tilde{F} -orbits to an \tilde{F} -invariant function on a domain W adjacent to $J = (a, b) \times \{0\} \subset \mathbb{R}_{\tau,\phi}^2$. The

fact that it is C^∞ -smooth on $W \cup J$ and coincides with ϕ up to ϕ -flat terms will be proved in the next subsection. It suffices to prove that the function $\tilde{\phi}$ extends as above to a rectangle $(a', b') \times [0, \eta']$ adjacent to an arbitrary relatively compact subinterval $J' = (a', b') \times \{0\} \subseteq J$. A union of the above rectangles corresponding to an exhaustion of J by a sequence of subintervals J' yields a domain W adjacent to all of J , where the extended function is defined. Therefore, we make the following convention.

Convention 2.5. Everywhere below, we identify the interval $J = (a, b) \times \{0\}$ with (a, b) and sometimes we denote $J = (a, b) \subset \mathbb{R}$. We consider that J is a finite interval: a, b are finite. We will consider that there exists a $\delta > 0$ such that $\tilde{F}^{\pm 1}$ are diffeomorphisms of the rectangle $J \times [0, \delta) \subset \mathbb{R}_{\tau, \phi}^2$ onto its images, and the ϕ -flat terms in asymptotic equation (1.13) are *uniformly ϕ -flat*: the difference $\tilde{F}(\tau, \phi) - (\tau + \phi, \phi)$ converges to zero uniformly in $\tau \in J$, and every partial derivative (of any order) also converges to zero uniformly, as $\phi \rightarrow 0$. *A priori*, ϕ -flatness of a C^∞ -smooth function on $J \times [0, \delta)$, which means triviality of all its jets at all points in $J \times \{0\}$, is weaker than its uniform ϕ -flatness. However, it implies uniform ϕ -flatness on compact subsegments in J , see Definition 1.28. Hence, one can achieve uniform flatness replacing J by its relatively compact subinterval. Under this assumption, the above difference and its differential are both uniformly $o(\phi^m)$ in $\tau \in J$ for each individual $m \in \mathbb{N}$.

The next proposition describes asymptotics of two-sided \tilde{F} -orbits.

PROPOSITION 2.6. *For every η small enough and $x := (\tau_0, \phi_0) \in J \times [0, \eta)$:*

- (a) *the iterates $\tilde{F}^j(x) = (\tau_j, \phi_j)$ are well defined for all $j \geq 0, j \leq N_+$, where $N_+ = N_+(x)$ is the maximal number j for which $\tau_j < b$;*
- (b) *the inverse iterates $\tilde{F}^{-j}(x) = (\tau_{-j}, \phi_{-j})$ are well defined for all $j \leq N_-$, where $N_- = N_-(x)$ is the maximal number j for which $\tau_{-j} > a$;*
- (c) *$\phi_j = \phi_0(1 + o(1))$ uniformly in τ_0 and $j \in [-N_-, N_+]$, as $\phi_0 \rightarrow 0$;*
- (d) *the points τ_j form an asymptotic arithmetic progression: $\tau_{j+1} - \tau_j = \phi_0(1 + o(1))$ uniformly in $\tau_0 \in J$ and in $j \in [-N_-, N_+ - 1]$, as $\phi_0 \rightarrow 0$.*

Proof. Consider two lines and segments through x :

$$L_\pm(x) := \{\phi = \phi_0 \pm \phi_0^4(\tau - \tau_0)\}, \lambda_\pm := L_\pm \cap (J \times [0, 2\eta)).$$

CLAIM 2. *For every $x = (\tau_0, \phi_0) \in J \times [0, 2\eta)$ with ϕ_0 small enough:*

- (e) *the image $\tilde{F}(L_\pm)$ is disjoint from λ_\pm and lies on its right;*
- (f) *the image $\tilde{F}^{-1}(L_\pm)$ is disjoint from λ_\pm and lies on its left;*
- (g) *the right sector $S_+(x)$ bounded by the right subintervals in λ_\pm with vertex x is \tilde{F} -invariant;*
- (h) *the left sector $S_-(x)$ bounded by the left subintervals in λ_\pm with vertex x is \tilde{F}^{-1} -invariant.*

Proof. If η is small enough, then $\tilde{F}^{\pm 1}$ are well defined on $J \times [0, 3\eta)$. If ϕ_0 is small enough, then each λ_\pm is projected to all of J , and the ϕ -coordinates of all its points are uniformly asymptotically equivalent to ϕ_0 , that is, their ratio with ϕ_0 tends uniformly to 1 (finiteness of J). The map \tilde{F} moves a point $z = (\tau, \phi) \in \lambda_\pm$ to $y := (\tau + \phi, \phi)$ up to a

ϕ -flat term, which is $o(\phi_0^m)$ for every $m \in \mathbb{N}$. However, the distance of the latter point y to the line L_{\pm} is equal to $\phi \simeq \phi_0$ times the $|\sin|$ of the azimuth of the line L_{\pm} . The latter $|\sin|$ is asymptotic to ϕ_0^4 , and hence, is greater than $\frac{1}{2}\phi_0^4$, whenever $\phi_0 < 1$ is small enough. Thus, $\text{dist}(y, L_{\pm}) \geq \frac{1}{3}\phi_0^5$. Therefore, adding a term $o(\phi_0^m)$, $m \geq 5$, to y will not allow to cross L_{\pm} , and we will get a point lying on the same, right side from the line L_{\pm} , as y . The cases of lines L_{\mp} and inverse iterates are treated analogously. Statements (e) and (f) are proved. They immediately imply statements (g) and (h). \square

Let $\eta \in (0, \frac{1}{8})$ be small enough so that \tilde{F} is defined on the rectangle $\Pi := J \times [0, 3\eta)$ and for every $x \in \Pi$ with $\phi_0 = \phi(x) \in [0, 2\eta]$, the sector $S_+(x)$ contains the points $x_j = \tilde{F}^j(x)$ until they go out of Π (Claim 2(g)). The intersection $S_+(x) \cap \partial\Pi$ is contained in the right lateral side $\{b\} \times [0, 3\eta)$. Therefore, the first j for which x_j goes out of Π is the one for which $\tau(x_j) \geq b$. This proves statement (a) of Proposition 2.6. The proof of statement (b) is analogous. For every $x \in \Pi$ with $\phi_0 = \phi(x)$ small enough, the above inclusion $x_j \in S_+(x)$ holds for $j = 1, \dots, N_+$. It implies statement (c) for the above j , by the definition of the sector S_+ . The proof of statement (c) for $j = -N_-, \dots, -1$ is analogous. Statement (d) follows from statement (c), since $\tau \circ \tilde{F}(x) - \tau(x) = \phi(x) + \text{flat}(\phi(x))$, see equation (1.13). Proposition 2.6 is proved. \square

COROLLARY 2.7.

- (1) For every $\eta > 0$ small enough, each point $x = (\tau_0, \phi_0) \in J \times [0, (2\eta)/3)$ has two-sided orbit lying in $J \times [0, \eta)$ and consisting of points x_j , $j \in [-N_-(x), N_+(x)]$, with $\phi_j \simeq \phi_0$, as $\phi_0 \rightarrow 0$; the latter asymptotics is uniform in the above j and in $\tau_0 \in J$.
- (2) Let Δ denote the fundamental domain (curvilinear triangle) for the map \tilde{F} from Proposition 2.2(iii). Let $\hat{\Delta}$ denote the complement of the closure $\bar{\Delta}$ to the union of its vertex $(0, 0)$ and the opposite side. If $\eta > 0$ is small enough, then the domain W saturated by the above two-sided orbits of points in $\hat{\Delta}$ lies in $J \times [0, (2\eta)/3)$ and contains the strip $J \times (0, \eta/4)$.
- (3) The orbit of each point in W contains either a unique point lying in the fundamental domain Δ , or two subsequent points lying in its lateral boundary curves (glued by \tilde{F}).
- (4) Each \tilde{F} -invariant function $\tilde{\phi}$ on $\hat{\Delta}$ extends to a unique \tilde{F} -invariant function on W as a function constant along the latter orbits.

The corollary follows immediately from Proposition 2.6. Step 2 is done.

2.4. Step 3. Regularity and flatness. End of proof of Theorem 1.30(1). Here we will prove the following lemma, which will imply statement (1) of Theorem 1.30.

LEMMA 2.8. Let in Corollary 2.7 the function $\tilde{\phi}$ on $\hat{\Delta}$ be the restriction to $\hat{\Delta}$ of a C^∞ -smooth \tilde{F} -invariant function defined on a neighborhood of $\hat{\Delta}$. Let the function $\tilde{\phi}(\tau, \phi) - \phi$ be flat on $\hat{\Delta}$: it tends to zero with all its partial derivatives, as $(\tau, \phi) \in \hat{\Delta}$ tends to zero. Consider its \tilde{F} -invariant extension to the above domain W from Corollary 2.7(4), and let us denote the extended function by the same symbol $\tilde{\phi}$. The difference $\tilde{\phi}(\tau, \phi) - \phi$ is C^∞ -smooth on $W \cup J$, and it is uniformly ϕ -flat (see Convention 2.5).

Proof. For every point $x = (\tau, \phi) \in W$, there exists an $N = N(x) \in \mathbb{Z}$ such that $\tilde{F}^N(x) \in \widehat{\Delta}$. The latter image $\tilde{F}^N(x)$ lies in the definition domain of the initial function $\tilde{\phi}$ (which is defined on a neighborhood of $\widehat{\Delta}$), and $\tilde{\phi}(x) = \tilde{\phi}_N(x) := \tilde{\phi}(\tilde{F}^N(x))$, by definition. This immediately implies C^∞ -smoothness of the extended function $\tilde{\phi}$ on W . Let us prove its ϕ -flatness. This will automatically imply C^∞ -smoothness at points of the boundary interval J . To do this, we use the asymptotics

$$d\tilde{F}(\tau, \phi) = A + \text{flat}(\phi), \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \tag{2.15}$$

$$N(x) = N(\tau, \phi) = O\left(\frac{1}{\phi}\right). \tag{2.16}$$

Here the flat term in equation (2.15) is uniformly flat, see Convention 2.5. Equation (2.15) follows from equation (1.13). Equation (2.16) holds, since $N \leq N_+ + N_- = O(1/\phi)$, which follows from Proposition 2.6(d).

We study the derivatives of the functions $\tilde{\phi}_N - \phi$, $N = N(x)$, at the point $x = (\tau, \phi)$, as functions in x with fixed N chosen as above for this concrete x . To prove uniform flatness, we have to show that all its partial derivatives tend to zero uniformly in $\tau \in J$, as $\phi \rightarrow 0$. We prove this statement for the first derivatives (Step 1) and then for the higher derivatives (Step 2).

Without loss of generality, everywhere below, we consider that $N \geq 1$, that is, x lies on the left from the sector Δ : for negative N , the proof is analogous.

Step 1. The first derivatives. The initial function $\tilde{\phi}$ defined on a neighborhood of the set $\widehat{\Delta}$ is already known to be ϕ -flat on $\widehat{\Delta}$. The differential of the composition $\tilde{\phi}_N = \tilde{\phi} \circ \tilde{F}^N$ at the point x , $N = N(x)$, is equal to

$$d(\tilde{\phi} \circ \tilde{F}^N)(x) = d\tilde{\phi}(\tilde{F}^N(x))d\tilde{F}(\tilde{F}^{N-1}(x)) \cdots d\tilde{F}(x). \tag{2.17}$$

PROPOSITION 2.9. *For every sequence of points $x(k) = (\tau_{0k}, \phi_{0k}) \in W$ with $\phi_{0k} \rightarrow 0$, as $k \rightarrow \infty$, and numbers $N_k = N(x(k)) \in \mathbb{N}$ with $\tilde{F}^{N_k}(x(k)) \in \widehat{\Delta}$, the difference $d(\tilde{\phi} \circ \tilde{F}^{N_k})(x(k)) - d\phi$ tends to zero, as $k \rightarrow \infty$.*

Proposition 2.9 implies uniform convergence to zero of the first derivatives.

In its proof (given below), we use the following asymptotics of differential $d\tilde{F}(\tilde{F}^j(x))$ and technical proposition on matrix products. We denote

$$M(\tau, \phi) := \text{the Jacobian matrix of the differential } d\tilde{F}(\tau, \phi).$$

PROPOSITION 2.10. *Let $x = (\tau_0, \phi_0) \in J \times (0, \eta/4)$, $x_j = (\tau_j, \phi_j) := \tilde{F}^j(x)$, $j = 0, \dots, N(x)$. For every $m \in \mathbb{N}$, one has*

$$M(\tau_j, \phi_j) = A + o(\phi_0^m), \quad \text{as } \phi_0 \rightarrow 0; \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.18}$$

uniformly in $j = 1, \dots, N(x)$ and in $\tau_0 \in J$ for each individual m .

Proof. Equation (2.18) follows from equation (2.15) and Proposition 2.6, part (c). □

PROPOSITION 2.11. Consider arbitrary sequences of numbers $\phi_{0k} > 0$, $N_k \in \mathbb{N}$, $\phi_{0k} \rightarrow 0$, $N_k = O(1/\phi_{0k})$, as $k \rightarrow \infty$, and matrix collections

$$M_k = (M_{1;k}, \dots, M_{N_k;k}), \quad M_{j;k} \in \text{GL}_2(\mathbb{R}),$$

$$M_{j;k} = A + o(\phi_{0k}^m) \quad \text{for every } m \in \mathbb{N}; \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.19}$$

Here the latter asymptotics is uniform in $j = 1, \dots, N_k$ for each individual m , as $k \rightarrow \infty$. Then the products of the matrices $M_{j;k}$ have the asymptotics

$$\widehat{M}_k := M_{N_k;k} \cdots M_{1;k} = \begin{pmatrix} 1 & N_k \\ 0 & 1 \end{pmatrix} + o(\phi_{0k}^m) \quad \text{for every } m \in \mathbb{N}. \tag{2.20}$$

Proof. Conjugation by the diagonal matrix $H_k := \text{diag}(1, \phi_{0k}^{-1})$ transforms the matrices $M_{j;k}$ and their product respectively to the following matrices:

$$\widetilde{M}_{j;k} = B_k + o(\phi_{0k}^m), \quad B_k = \begin{pmatrix} 1 & \phi_{0k} \\ 0 & 1 \end{pmatrix}; \quad \widetilde{M}_k := \widetilde{M}_{N_k;k} \cdots \widetilde{M}_{1;k}.$$

CLAIM 3. One has

$$\widetilde{M}_k = B_k^{N_k} + o(\phi_{0k}^m) = \begin{pmatrix} 1 & N_k \phi_{0k} \\ 0 & 1 \end{pmatrix} + o(\phi_{0k}^m). \tag{2.21}$$

Proof. Without loss of generality, we can and will consider that $N_k \phi_{0k} \rightarrow C \in \mathbb{R}_{\geq 0}$, passing to a subsequence, since $N_k = O(1/\phi_{0k})$, by assumption. Let $\mathcal{UT} \subset \text{GL}_2(\mathbb{R})$ denote the one-parametric subgroup of unipotent upper triangular matrices. Consider the tangent vector

$$V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in T_1 \mathcal{UT} \subset T_1 \text{GL}_2(\mathbb{R}).$$

Let us extend it to a left-invariant vector field on $\text{GL}_2(\mathbb{R})$, which is tangent to the \mathcal{UT} -orbits under right multiplication action. Take a small transverse section $S \subset \text{GL}_2(\mathbb{R})$ passing through the identity and consider the subset $\mathcal{W} \subset \text{GL}_2(\mathbb{R})$ foliated by arcs of phase curves of the field V starting in S and parameterized by time segment $[0, 2C + 1]$. The subset \mathcal{W} is a bordered domain (flowbox) diffeomorphic to the product $S \times [0, 2C + 1]$ via the diffeomorphism sending a point $y \in \mathcal{W}$ to the pair $(s(y), t(y))$ such that the orbit issued from the point $s(y) \in S$ arrives to y in time $t(y)$. Fix an arbitrary $m \geq 3$. In the new chart (s, t) , the multiplication by a matrix $\widetilde{M}_{j;k} = B_k + o(\phi_{0k}^m)$ from the right moves a point (s, t) to the point $(s, t + \phi_{0k})$ up to a small correction of order $o(\phi_{0k}^m)$. Therefore, the multiplication by $N_k \simeq (C + o(1))/\phi_{0k}$ similar matrices $\widetilde{M}_{j;k}$ with the $o(\phi_{0k}^m)$ in their asymptotics being uniform in j moves a point (s, t) to a point $(s, t + N_k \phi_{0k})$ up to a correction of order $N_k o(\phi_{0k}^m) = o(\phi_{0k}^{m-1})$. This implies equation (2.21) with m replaced by $m - 1$. Taking into account that m can be chosen arbitrary, this proves equation (2.21). \square

Conjugating equation (2.21) by the matrix H_k^{-1} and taking into account that $m \in \mathbb{N}$ is arbitrary yields equation (2.20) and proves Proposition 2.11. \square

Proof of Proposition 2.9. For $z \in \widehat{\Delta}$, set

$$St(z) := \left(\frac{\partial \tilde{\phi}}{\partial \tau}, \frac{\partial \tilde{\phi}}{\partial \phi} \right)(z).$$

The string of the first partial derivatives of the function $\tilde{\phi}_N = \tilde{\phi} \circ \tilde{F}^N(x)$, $N = N(x)$, is equal to the product

$$St(\tau_N, \phi_N)M(\tau_{N-1}, \phi_{N-1}) \cdots M(\tau_0, \phi_0), (\tau_j, \phi_j) = \tilde{F}^j(x), \quad j = 0, \dots, N,$$

$$St(\tau_N, \phi_N) = (0, 1) + o(\phi_0^m) \quad \text{for every } m \in \mathbb{N}, \tag{2.22}$$

by ϕ -flatness of the initial function $\tilde{\phi}$ on $\widehat{\Delta}$ and by the uniform asymptotics $\phi_j = \phi_0(1 + o(1))$, $j = 1, \dots, N$ (Proposition 2.6(c)).

Take arbitrary sequence of points $x(k) := (\tau_{0k}, \phi_{0k})$, $\tau_{0k} \in J$, $\phi_{0k} \rightarrow 0$, as $k \rightarrow \infty$. Set

$$(\tau_{jk}, \phi_{jk}) := \tilde{F}^j(x(k)), \quad N_k := N(x(k)).$$

The sequence of collections of Jacobian matrices $M_{j+1;k} := M(\tau_{jk}, \phi_{jk})$, $j = 0, \dots, N_k - 1$, satisfy the conditions of Proposition 2.11, by equations (2.16) and (2.18). Therefore, their product \widehat{M}_k , which is the Jacobian matrix of the differential $d\tilde{F}^{N_k}(x(k))$, has asymptotics in equation (2.20):

$$\widehat{M}_k := \text{the Jacobian matrix of } d\tilde{F}^{N_k}(x(k)) = \begin{pmatrix} 1 & N_k \\ 0 & 1 \end{pmatrix} + o(\phi_{0k}^m). \tag{2.23}$$

Thus, the matrix-string of the differential $d\tilde{\phi}_{N_k}(\tau_{0k}, \phi_{0k})$ is the product

$$St(\tau_{N_k}, \phi_{N_k})\widehat{M}_k = ((0, 1) + o(\phi_{0k}^m)) \begin{pmatrix} 1 & N_k \\ 0 & 1 \end{pmatrix} + o(\phi_{0k}^m) = (0, 1) + o(\phi_{0k}^{m-1}),$$

since $N_k = O(1/\phi_{0k})$, see equation (2.16). For $m = 2$, we get that the differential $d(\tilde{\phi}_{N_k}(\tau, \phi) - \phi)$ taken at the point $x(k)$ tends to zero, as $k \rightarrow \infty$. This proves Proposition 2.9. \square

Step 2. The higher derivatives. For a smooth function f defined on a neighborhood of a point x by $j_x^\ell(f)$, we will denote its ℓ -jet at x . Below we prove the following proposition.

PROPOSITION 2.12. *In the conditions of Proposition 2.9, for every $\ell \in \mathbb{N}$, the ℓ -jet at $x(k)$ of the difference $\tilde{\phi} \circ \tilde{F}^{N_k} - \phi$ tends to zero, as $k \rightarrow \infty$.*

Proposition 2.12 will imply C^∞ -smoothness and ϕ -flatness of the extended function $\tilde{\phi}$ at the points of the boundary interval $J \times \{0\}$.

For every $\ell \in \mathbb{N}$ and $x \in \mathbb{R}^2$, let J_x^ℓ denote the space of ℓ -jets of functions at the point x . The map \tilde{F} induces a transformation of functions, $g \mapsto g \circ \tilde{F}$. This induces linear operators in the jet spaces, $D_\ell \tilde{F}(x) : J_{\tilde{F}(x)}^\ell \rightarrow J_x^\ell$. We identify the space of ℓ -jets at each point in \mathbb{R}^2 with the ℓ -jet space at the origin, which, in its turn, is identified with the space $\mathcal{P}_{\leq \ell}$ of polynomials in two variables of degrees no greater than ℓ . Thus, we consider the operator $D_\ell \tilde{F}(x)$ as acting on the above space $\mathcal{P}_{\leq \ell}$. One has

$$D_\ell \tilde{F}^N(x) = D_\ell \tilde{F}(x) \cdots D_\ell \tilde{F}(F^{N-1}(x)). \tag{2.24}$$

Linear changes of variables (τ, ϕ) act on the space $\mathcal{P}_{\leq \ell}$ and induce an injective linear anti-representation $\rho : \text{GL}_2(\mathbb{R}) \rightarrow \text{GL}(\mathcal{P}_{\leq \ell})$. Let A denote the unipotent Jordan cell, see equation (2.19).

PROPOSITION 2.13. *For every sequence of points $x(k) = (\tau_{0k}, \phi_{0k}) \in W$ with $\phi_{0k} \rightarrow 0$, as $k \rightarrow \infty$, set $N_k := N(x(k))$, one has*

$$D_\ell \tilde{F}^{N_k}(x(k)) = \rho(A^{N_k}) + o(\phi_{0k}^m) \quad \text{for every } m \in \mathbb{N}. \tag{2.25}$$

Proof. One has

$$D_\ell \tilde{F}(\tau, \phi) = \rho(A) + \text{flat}(\phi), \tag{2.26}$$

by equation (2.15). Set $x_j(k) = (\tau_{jk}, \phi_{jk}) = \tilde{F}^j(x(k))$, $j = 0, \dots, N_k - 1$. One has

$$D_\ell \tilde{F}(x_j(k)) = \rho(A) + o(\phi_{0k}^m) \quad \text{for every } m \in \mathbb{N}, \tag{2.27}$$

by equation (2.26) and Proposition 2.6(c). We use equation (2.24) and the following multidimensional version of Proposition 2.11.

PROPOSITION 2.14. *Consider arbitrary sequences of numbers $\phi_{0k} > 0$, $N_k \in \mathbb{N}$, $\phi_{0k} \rightarrow 0$, $N_k = O(1/\phi_{0k})$, as $k \rightarrow \infty$, and matrix collections*

$$\begin{aligned} \mathcal{M}_k &= (M_{1;k}, \dots, M_{N_k;k}), \quad M_{j;k} \in \text{GL}(\mathcal{P}_{\leq \ell}), \\ M_{j;k} &= \rho(A) + o(\phi_{0k}^m) \quad \text{for every } m \in \mathbb{N}; \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.28}$$

Here the latter asymptotics is uniform in $j = 1, \dots, N_k$ for each individual m , as $k \rightarrow \infty$. Then the product of the matrices $M_{j;k}$ has the asymptotics

$$\widehat{M}_k := M_{N_k;k} \cdots M_{1;k} = \rho(A^{N_k}) + o(\phi_{0k}^m) \quad \text{for every } m \in \mathbb{N}. \tag{2.29}$$

Proof. Conjugating the matrices $M_{j;k}$ by $\rho(H_k)$, $H_k := \text{diag}(1, \phi_{0k}^{-1})$, transforms them to matrices

$$\tilde{M}_{j;k} = \rho(B_k) + o(\phi_{0k}^{m'}), \quad B_k = \begin{pmatrix} 1 & \phi_{0k} \\ 0 & 1 \end{pmatrix}, \quad m' = m - \ell - 1.$$

It suffices to show that the product of the matrices $\tilde{M}_{j;k}$ has asymptotics $\rho(B_k^{N_k}) + o(\phi_{0k}^m)$ for every $m \in \mathbb{N}$, as in Claim 3. This is done by applying the arguments from the proof of Claim 3 to the left-invariant vector field on $\text{GL}(\mathcal{P}_{\leq \ell})$ whose time t flow map acts by right multiplication by $\rho(A^t)$. □

Equation (2.25) is deduced from Proposition 2.14 and equations (2.24), (2.27), as equation (2.23). □

Proof of Proposition 2.12. The polynomial representing the ℓ -jet of the initial function $\tilde{\phi}$ at a point $z \in \widehat{\Delta}$ tends to the linear polynomial $P(\tau, \phi) = \phi$, as $z \rightarrow 0$, so that its distance to $P(\tau, \phi)$ is $o(\phi^m)$ for every $m \in \mathbb{N}$, by flatness of $\tilde{\phi}$ on $\widehat{\Delta}$. This together with Proposition 2.6(c) implies that the distance of its ℓ -jet at the point $\tilde{F}^{N_k}(x(k))$ to the polynomial ϕ is asymptotic to $o(\phi_{0k}^m)$. The image of the latter ℓ -jet under the operator $D_\ell \tilde{F}^{N_k}(x(k))$ is also

$o(\phi_{0k}^m)$ -close to ϕ for every $m \in \mathbb{N}$. This follows from the previous statement, equation (2.25), the fact that $\rho(A)$ fixes ϕ and the asymptotics $N_k = O(\phi_{0k}^{-1})$. Finally, we get that the difference of the ℓ -jet of the function ϕ at $x(k)$ and the ℓ -jet $j_{x(k)}^\ell(\tilde{\phi} \circ \tilde{F}^{N_k})$ of the extended function tends to zero, as $k \rightarrow \infty$. Proposition 2.12 is proved. \square

Lemma 2.8 follows from Proposition 2.12. It implies statement (1) of Theorem 1.30.

2.5. *Normal form. Proof of statement (2) of Theorem 1.30.* Let $\tilde{\phi}$ be a function from statement (1) of Theorem 1.30. The vector function $(\tau, \tilde{\phi})$ has non-degenerate Jacobian matrix at J . Hence, shrinking W , we can and will consider that $(\tau, \tilde{\phi})$ are C^∞ -smooth coordinates on $W \cup J$. In these coordinates,

$$\tilde{F} : (\tau, \tilde{\phi}) \mapsto (\tau + g(\tau, \tilde{\phi}), \tilde{\phi}), \quad g(\tau, \tilde{\phi}) = \tilde{\phi} + \text{flat}(\tilde{\phi}). \tag{2.30}$$

CLAIM 4. *Shrinking W , one can achieve that there exists a C^∞ -smooth function $\tilde{\tau}(\tau, \tilde{\phi}) = \tau + \text{flat}(\tilde{\phi})$ on $W \cup J$ such that $(\tilde{\tau}, \tilde{\phi})$ are C^∞ -smooth coordinates on $W \cup J$ in which \tilde{F} acts as in equation (1.14):*

$$\tilde{F} : (\tilde{\tau}, \tilde{\phi}) \mapsto (\tilde{\tau} + \tilde{\phi}, \tilde{\phi}). \tag{2.31}$$

Proof. Equation (2.31) is equivalent to the equation

$$\tilde{\tau} \circ F(x) = \tilde{\tau}(x) + \tilde{\phi}(x). \tag{2.32}$$

Shifting τ and shrinking W , we can and will consider that $(0, 0) \in J$,

$$g|_W > 0, \quad W \cap \{\tilde{\phi} = \zeta\} = (\alpha(\zeta), \beta(\zeta)) \times \{\zeta\} \quad \text{for all small } \zeta > 0, \tag{2.33}$$

$$\alpha(\zeta) \rightarrow a, \beta(\zeta) \rightarrow b \quad \text{as } \zeta \rightarrow 0; \quad \tilde{F}^{\pm 1} \text{ are well defined on } W \cup J.$$

Fix a small $\chi \in (0, \frac{1}{2})$ and a $\eta > 0$ (depending on χ) satisfying the statements of Proposition 2.2 and such that the second statement (set equality) in equation (2.33) holds for every $\zeta < 3\eta$. Consider the sector $S_{\chi,\eta}$, the segment $K = \{0\} \times [0, \eta/2]$, and the fundamental domain Δ bounded by K , $F(K)$, and the (now horizontal) straightline segment connecting their endpoints distinct from the origin. Set $\widehat{\Delta} := \overline{\Delta} \setminus \{(0, 0)\} \subset S_{\chi,\eta}$. First, we define the function $\tilde{\tau}$ on the sector $S_{\chi,\eta}$ so that equation (2.32) holds, whenever $x, \tilde{F}(x) \in S_{\chi,\eta}$. Afterward, we extend $\tilde{\tau}$ to all of W by dynamics.

Fix a $\sigma \in (0, \frac{1}{2}(\frac{1}{2} - \chi))$. Let ρ_1, ρ_2 be a partition of unity on the interval $(-\chi, 1 + \chi)$ subordinated to its covering by intervals $(-\chi, \frac{1}{2} + \sigma), (\frac{1}{2} - \sigma, 1 + \chi)$, see equation (2.13). Set $v := \tau/\tilde{\phi}$. For every $x \in S_{\chi,\eta}$, set

$$\tilde{\tau}(x) := \rho_1(v(x))\tau(x) + \rho_2(v(x))(\tau \circ \tilde{F}^{-1}(x) + \tilde{\phi}(x)). \tag{2.34}$$

The inclusion $x, \tilde{F}(x) \in S_{\chi,\eta}$ implies equation (2.32), whenever η is small enough, since then $\rho_1(v(x)) = 1$ and $\rho_2(v \circ F(x)) = 1$, as in the proof of Proposition 2.2, and by equation (2.34) and \tilde{F} -invariance of the function $\tilde{\phi}$. Recall that the height of the fundamental domain Δ is $\eta/2$. Let us now replace W by $W \cap \{\tilde{\phi} < (\eta/2)\}$. Then for every $x \in W$, there exists a $N = N(x) \in \mathbb{Z}$ such that $\tilde{F}^N(x) \in \widehat{\Delta}$; the latter N is unique, unless $\tilde{F}^N(x) \in \partial\widehat{\Delta}$. This follows from equation (2.33). Set

$$\tilde{\tau}(x) := \tilde{\tau}(\tilde{F}^N(x)) - N\tilde{\phi}(x). \tag{2.35}$$

The function $\tilde{\tau}(x)$ is well defined and C^∞ -smooth on all of W and satisfies equation (2.32) there. Indeed, it suffices to check smoothness on the boundary $\partial\widehat{\Delta}$ and on its images. If $x \in \partial\widehat{\Delta}$, then either $x, \tilde{F}(x) \in \widehat{\Delta}$ or $x, \tilde{F}^{-1}(x) \in \widehat{\Delta}$. In the first case, one can take $N = 0$ or $N = 1$. For both these values of N , the corresponding right-hand sides $\tilde{\tau}(x)$ and $\tilde{\tau} \circ F(x) - \tilde{\phi}(x)$ in equation (2.35) coincide, since equation (2.32) holds on $S_{\chi,\eta} \supset \widehat{\Delta}$. The second case is treated analogously. For the same reason, the function $\tilde{\tau}$ on $S_{\chi,\eta}$ given by equation (2.34) coincides with the corresponding expression in equation (2.35) (in which $N \in \{0, \pm 1\}$). This implies smoothness of the function in equation (2.35) on a neighborhood of the subset $\widehat{\Delta} \subset W$. One has $\tilde{\tau}(\tilde{F}(x)) = \tilde{\tau}(x) + \tilde{\phi}(x)$, by equation (2.35). This together with the above discussion implies that $\tilde{\tau}$ is C^∞ -smooth on W and satisfies equation (2.32) on all of W . The function $\tilde{\tau}$ extends to a C^∞ -smooth function on $W \cup J$, and the function $\tilde{\tau} - \tau$ is $\tilde{\phi}$ -flat. This is proved as in §2.4. Namely, first it is $\tilde{\phi}$ -flat on $S_{\chi,\eta}$, as in Proposition 2.3. Then we fix an arbitrary compact segment $[a', b'] \subset (a, b) = J$. The differential of the map $H : (\tau, \tilde{\phi}) \mapsto (\tilde{\tau}, \tilde{\phi})$ tends to the identity, and all its higher derivatives tend to zero, as $\tilde{\phi} \rightarrow 0$, uniformly in $\tau \in [a', b']$. In particular, $\partial\tilde{\tau}(\tau, \tilde{\phi})/\partial\tau \rightarrow 1$. Indeed, for every $x \in W$, set $N = N(x)$, one has $H(\tau, \tilde{\phi}) = (\tilde{\tau} \circ \tilde{F}^N(\tau, \tilde{\phi}) - N\tilde{\phi}, \tilde{\phi})$ on a neighborhood of x . This together with equations (2.23) and (2.25) applied to the differential and higher jet action of the iterates of the map \tilde{F} imply the above convergence statement. The restriction of the function $\tilde{\tau}$ to $S_{\chi,\eta}$ extends continuously to the origin as $\tilde{\tau}(0, 0) = 0$. Thus, $\tilde{\tau}(0, \tilde{\phi}) \rightarrow 0$, as $\tilde{\phi} \rightarrow 0$. This together with uniform convergence $\partial\tilde{\tau}(\tau, \tilde{\phi})/\partial\tau \rightarrow 1$ in $\tau \in [a', b']$, as $\tilde{\phi} \rightarrow 0$, implies uniform convergence $\tilde{\tau}(\tau, \tilde{\phi}) \rightarrow \tau$. Together with the above higher derivative convergence, this implies $\tilde{\phi}$ -flatness of the function $\tilde{\tau} - \tau$ and proves the claim. \square

The above claim immediately implies statement (2) of Theorem 1.30.

2.6. *Proof of existence in Theorem 1.26. Proof of Theorem 1.27.* Let F be a C^∞ -lifted strongly billiard-like map. Let (τ, h) be the coordinates from Theorem 2.1. Set $\phi = \sqrt{h}$. Let \tilde{F} denote the map F written in the coordinates (τ, ϕ) , which is C^∞ -smooth and takes the form $(\tau, \phi) \mapsto (\tau + \phi + \text{flat}(\phi), \phi + \text{flat}(\phi))$ (Theorem 2.1). There exists a \tilde{F} -invariant function $\tilde{\phi} = \phi + \text{flat}(\phi)$ (Theorem 1.30). The function $\tilde{h} := \tilde{\phi}^2$ is F -invariant, C^∞ -smooth, and $\tilde{h} = h + \text{flat}(h)$; hence, $(\partial\tilde{h}/\partial h) > 0$ on J and on some domain adjacent to J . The existence in Theorem 1.26 is proved. Non-uniqueness of the function \tilde{h} will be proved in §2.9.

Let us now prove Theorem 1.27. Let us fix a function \tilde{h} constructed above. Let θ denote the time function of the Hamiltonian vector field with the Hamiltonian function \tilde{h} normalized to vanish on the vertical axis $\{\tau = 0\}$. (We consider that $(0, 0) \in J$, shifting the coordinate τ .) The coordinates (θ, \tilde{h}) are symplectic. In these coordinates, $F(\theta, \tilde{h}) = (\theta + \xi(\tilde{h}), \tilde{h})$ for some function $\xi(\tilde{h}) = \sqrt{\tilde{h}}\psi(\tilde{h})$ in one variable, since F preserves the symplectic area; ψ is C^∞ -smooth and $\psi(0) > 0$, as in Claim 1 in §2.1. Afterward, modifying the functions \tilde{h} and θ , as at the end of §2.1, we get new coordinates (τ, \tilde{h}) (with new τ) in which F takes the form in equation (1.12). Theorem 1.27 is proved.

2.7. *Foliation by caustics. Proof of existence in Theorems 1.9, 1.12, and 1.13.* First, let us consider the case when γ is an unbounded submanifold: a strictly convex curve injectively parameterized by interval and bounding a domain in \mathbb{R}^2 (conditions of statement (1) of Theorem 1.9).

Let W denote the domain in the space of oriented lines that consists of lines intersecting γ twice and satisfying condition (b) from the beginning of §1.3. Let $\widehat{\gamma}$ denote the curve given by the family of orienting tangent lines of γ . The domain W is adjacent to $\widehat{\gamma}$. The billiard ball map is well defined on $W \cup \widehat{\gamma}$. Each line L close to a tangent line ℓ of γ carries a canonical orientation: the pullback of the orientation of the line ℓ under a projection $L \rightarrow \ell$ close to identity. The billiard ball map acting on thus oriented lines close to tangent lines of γ and intersecting γ twice will be treated as a map acting on non-oriented lines: we will just forget the orientation.

Let us fix a natural length parameter s on the curve γ that orients γ counterclockwise, that is, positively as the boundary of its convex side. Let us identify each point in γ with the corresponding length parameter value. Let us introduce the following tuples of coordinates on the domain W . For every line $L \in W$, let $s_1 = s_1(L)$ and $s_2 = s_2(L)$ denote the length parameter values of its intersection points with γ . Let ϕ_j denote the oriented angles between L and the tangent lines to γ at the points s_j . To each L , we put into correspondence the pair (s_1, ϕ_1) , where s_j are numerated so that $s_1 < s_2$. Set

$$y_1 = 1 - \cos \phi_1,$$

see equation (1.2). Any of the pairs (s_1, ϕ_1) or (s_1, y_1) defines L uniquely. Recall that (s_1, y_1) are symplectic coordinates on W , see the discussion after Remark 1.18. Let $V \subset \mathbb{R} \times \mathbb{R}_+$ denote the domain W represented in the coordinates (s_1, y_1) . It is adjacent to an interval $J = (a, b) \times \{0\}$ representing $\widehat{\gamma}$. Let $\widetilde{V} \subset \mathbb{R} \times \mathbb{R}_+$ denote the same domain represented in the coordinates (s_1, ϕ_1) .

PROPOSITION 2.15. *In the coordinates (s_1, y_1) , the billiard ball map is a C^∞ -lifted strongly billiard-like map F defined on $V \cup J$. In the coordinates (s_1, ϕ_1) , it is a C^∞ -smooth diffeomorphism \widetilde{F} defined on $\widetilde{V} \cup J$.*

Proof. The statements of the proposition follow from Proposition 1.20 and Example 1.23. □

PROPOSITION 2.16. *Shrinking V (without changing its boundary interval J), one can achieve that there exists a C^∞ -smooth F -invariant function $G(s_1, y_1)$ on $V \cup J$ such that*

$$G|_J \equiv 0, \quad \frac{\partial G}{\partial y_1} > 0.$$

Proof. The proposition follows from Theorem 1.26 (existence). □

From now on, by W , we denote the domain of those lines that are represented by points of the (shrunked) domain V from Proposition 2.16.

The level curves of the function G are F -invariant and form a C^∞ -smooth foliation. Lifting everything to the domain W in the space of lines, we get a foliation by invariant

curves under the billiard ball map. Each leaf is a smooth family of lines. Its enveloping curve is a caustic of the billiard in γ . To prove that γ and the caustics in question form a C^∞ -smooth foliation of a domain adjacent to γ , we use the following lemma.

LEMMA 2.17. *The above function G is C^∞ -smooth as a function on the domain with boundary $W \cup \widehat{\gamma}$ in the space of lines. It has non-zero differential on $W \cup \widehat{\gamma}$. Thus, its level curves form a C^∞ -smooth foliation of $W \cup \widehat{\gamma}$ with $\widehat{\gamma}$ being a leaf.*

Remark 2.18. The function $s_1(L)$ is smooth on W but not on $W \cup \widehat{\gamma}$: it is not C^1 -smooth at points of the curve $\widehat{\gamma}$. Therefore, *a priori*, a function smooth in (s_1, y_1) is not necessarily smooth on $W \cup \widehat{\gamma}$.

Proof of Lemma 2.17. The function G is C^∞ -smooth on W and has non-degenerate differential there, by Proposition 2.16. Let us prove that this also holds at points of the boundary curve $\widehat{\gamma}$. The function G lifts to an \widetilde{F} -invariant function

$$\widetilde{G}(s_1, \phi_1) = G(s_1, 1 - \cos \phi_1).$$

The map $(s_1, \phi_1) \rightarrow (s_1, s_2)$ is a diffeomorphism defined on $\widetilde{V} \cup J$. The analogous statement holds for the diffeomorphism $(s_2, \phi_2) \mapsto (s_1, s_2)$. One has

$$\widetilde{G}(s_1, \phi_1) = \widetilde{G}(s_1, -\phi_1) = \widetilde{G}(s_2, \phi_2) = \widetilde{G}(s_2, -\phi_2) \tag{2.36}$$

by invariance of the function \widetilde{G} under sign change at the second coordinate and by its invariance under the billiard ball map represented by $\widetilde{\delta}_+ : (s_1, \phi_1) \mapsto (s_2, -\phi_2)$. Given an unordered pair (s_1, s_2) , the tuples (s_1, ϕ_1) and (s_2, ϕ_2) are well defined up to permutation. Therefore, in the coordinates (s_1, s_2) on \widetilde{V} (where $a < s_1 \leq s_2 < b$ by definition), the function \widetilde{G} is C^∞ -smooth and extends C^∞ -smoothly to a neighborhood of the diagonal (identified with J) in $(a, b) \times (a, b)$ as a function invariant under coordinate permutation. This means that in the coordinates

$$(\alpha, \beta) := \left(\frac{s_1 + s_2}{2}, \frac{s_2 - s_1}{2} \right)$$

(which are C^∞ -diffeomorphic coordinates on $\widetilde{V} \cup J$), the function \widetilde{G} is invariant under sign change at β . Hence, \widetilde{G} is a C^∞ -smooth function in (α, β^2) ,

$$\widetilde{G}(s_1, \phi_1) = \widehat{G}(\alpha, \psi), \psi := \beta^2 :$$

the function \widehat{G} is C^∞ -smooth on the domain in $\mathbb{R}_\alpha \times (\mathbb{R}_+)_\psi$ adjacent to J and corresponding to W , and it is also smooth at points of the boundary J .

PROPOSITION 2.19. *The pair (α, ψ) forms C^∞ -smooth coordinates on the domain with boundary arc $W \cup \widehat{\gamma}$ in the space of lines.*

Proof. Consider the map $S\Lambda : (s_1, s_2) \rightarrow \{\text{lines}\}$ sending a pair of points (s_1, s_2) of the curve γ to the line through them. (For $s_1 = s_2 = s$, the image is the tangent line to γ at s .) The map $S\Lambda$ is C^∞ -smooth on $(a, b) \times (a, b)$. It is invariant under permutation of the coordinates s_1, s_2 , and its restriction to each connected component of the complement to the diagonal is a diffeomorphism, by convexity. Equivalently, it is C^∞ -smooth in the

coordinates (α, β) and invariant under sign change at β . Hence, it is smooth in (α, ψ) . Its differential is non-degenerate at those points, where $s_1 \neq s_2$, or equivalently, $\psi \neq 0$. It remains to check that it has non-degenerate differential at the points of the line $\{\psi = 0\}$. To do this, consider yet another tuple of coordinates (α^*, ψ^*) on $W \cup \widehat{\gamma}$ defined as follows: for every $L \in W \cup \widehat{\gamma}$:

- the point $\alpha^* = \alpha^*(L) \in (a, b)$ is the unique point in the curve γ where the tangent line to γ is parallel to L (it exists by Rolle theorem);
- the number $\psi^* = \psi^*(L)$ is the distance between the line L and the above tangent line.

PROPOSITION 2.20. *The coordinates (α^*, ψ^*) are C^∞ -smooth coordinates on $W \cup \widehat{\gamma}$.*

Proposition 2.20 follows from definition and strict convexity of γ .

Consider now (α^*, ψ^*) as functions of (α, ψ) . One obviously has

$$\psi^*(\alpha, 0) \equiv 0, \quad \alpha^*(\alpha, 0) \equiv \alpha, \quad \frac{\partial \alpha^*}{\partial \alpha}(\alpha, 0) \equiv 1, \quad \frac{\partial \psi^*}{\partial \alpha}(\alpha, 0) \equiv 0. \tag{2.37}$$

As $(\alpha, \psi) \rightarrow (\alpha_0, 0)$, one has

$$\psi^* \simeq \frac{1}{2}\kappa(\alpha_0)\psi. \tag{2.38}$$

Here κ is the curvature of the curve γ . Indeed, as $s_1, s_2 \rightarrow \alpha_0$, the line L through s_1 and s_2 is parallel to a line tangent to γ at a point α^* that is $o(s_1 - s_2)$ -close to $\alpha = (s_1 + s_2)/2$. The distance between the two lines is asymptotic to $\frac{1}{2}\kappa(\alpha)(\alpha^* - s_1)^2$, by [10, equation (2.1)]. This together with the equality $\alpha - s_1 = \beta \simeq \alpha^* - s_1$ implies equation (2.38), which in its turn implies that $\partial \psi^*(\alpha, 0)/\partial \psi > 0$. Together with equation (2.37), this implies non-degeneracy of the Jacobian matrix of the vector function $(\alpha^*(\alpha, \psi), \psi^*(\alpha, \psi))$ at the line $\{\psi = 0\}$. This proves Proposition 2.19. \square

The function $\widehat{G}(\alpha, \psi) = \widetilde{G}(s_1, \phi_1)$ is smooth in (α, ψ) , as was shown above. Hence, it is smooth on $W \cup \widehat{\gamma}$ (Proposition 2.19). It has non-zero differential on W , by Proposition 2.16. It remains to show that it has non-zero differential at each point $x \in \widehat{\gamma}$. Indeed, it is smooth in the coordinates (s_1, y_1) (in which $y_1|_{\widehat{\gamma}} \equiv 0$), and one has

$$\widetilde{G}(s_1, \phi_1) \simeq a(s_1)y_1(1 + o(1)) \quad \text{as } y_1 \rightarrow 0, \quad a(s) > 0, \tag{2.39}$$

by Proposition 2.16. However, as $s_1, s_2 \rightarrow s$, one has $y_1, \phi_1 \rightarrow 0$ and

$$y_1 = 1 - \cos \phi_1 = \frac{1}{2}\phi_1^2(1 + o(1)), \quad \psi = \left(\frac{s_2 - s_1}{2}\right)^2, \quad s_2 - s_1 \simeq 2(\kappa(s))^{-1}\phi_1.$$

Hence, $y_1 \simeq \frac{1}{2}\kappa^2(s)\psi$. This together with equation (2.39) implies that in the coordinates (α, ψ) , one has $\partial \widehat{G}(\alpha, 0)/\partial \psi > 0$. Together with the above discussion, this proves Lemma 2.17. \square

Proof of existence in Theorem 1.9. The function G defined on the set $W \cup \widehat{\gamma}$ in the space of lines is invariant under the billiard ball map. Therefore, its level curves are invariant families of lines. They form a C^∞ -smooth foliation of $W \cup \widehat{\gamma}$, with $\widehat{\gamma}$ being a leaf. Let us denote the latter foliation by \mathcal{F} . The enveloping curves of the curve $\widehat{\gamma}$ and of its leaves are respectively the curve γ and caustics of the billiard on γ . Let us show that they lie on its

convex side and there exists a domain $U \subset \mathbb{R}^2$ adjacent to γ from the convex side such that the latter caustics form a C^∞ -smooth foliation of $U \cup \gamma$, with γ being a leaf.

Fix a projective duality sending lines to points, e.g., polar duality with respect to the unit circle centered at a point O in the convex domain bounded by γ . Let us shrink W so that its points represent lines that do not pass through O . Then the duality represents the subset $W \cup \widehat{\gamma}$ in the space of lines as a domain in the affine chart $\mathbb{R}^2 \subset \mathbb{RP}^2$ with a boundary curve. The latter domain and curve will be also denoted by W and $\widehat{\gamma}$ respectively. The curve $\widehat{\gamma}$ is dual to γ .

PROPOSITION 2.21. *The curve $\widehat{\gamma}$ is strictly convex, and the domain W lies on its concave side.*

Proof. The curve $\widehat{\gamma}$ is strictly convex, being dual to the strictly convex curve γ . Each point $x \in W$ is dual to a line intersecting γ twice. Therefore, there are two tangent lines to $\widehat{\gamma}$ through x . Hence, x lies on the concave side from $\widehat{\gamma}$. □

For every $x \in W$, let $\mathcal{F}_x \subset \mathbb{R}^2 \subset \mathbb{RP}^2$ denote the leaf through x of the foliation \mathcal{F} (represented in the above dual chart), and let L_x denote its projective tangent line at x . The enveloping curve of the family of lines represented by the curve \mathcal{F}_x (treated now as a subset in the space of lines) is its dual curve \mathcal{F}_x^* . It consists of points L_y^* dual to the lines L_y for all $y \in \mathcal{F}_x$. Recall that the boundary curve $\widehat{\gamma}$ is a strictly convex leaf.

PROPOSITION 2.22. *Shrinking the domain W adjacent to $\widehat{\gamma}$ one can achieve that the map $x \mapsto L_x^*$ be a C^∞ -smooth diffeomorphism of the domain $W \cup \widehat{\gamma}$ onto a domain $U \subset \mathbb{R}^2 \subset \mathbb{RP}^2$ taken together with its boundary arc γ . The domain U lies on the convex side from the curve γ .*

Proof. The curve $\widehat{\gamma}$ is strictly convex. No tangent line passes through O , being dual to a point of the curve γ (which is a finite point), since the dual to O is the infinity line. Therefore, every compact arc in $\widehat{\gamma}$ has a neighborhood in \mathbb{R}^2 whose intersection with each leaf of the foliation \mathcal{F} is a strictly convex curve. Each tangent line to $\widehat{\gamma}$ intersects $\widehat{\gamma}$ only at one point: their unique tangency point. This follows from the fact that through each point $A \in \gamma$ passes a unique line tangent to γ , namely its tangent line at A , since γ bounds a strictly convex domain. Thus, shrinking W , we can and will consider that each leaf \mathcal{L} is strictly convex, no tangent line passes through O , and each tangent line intersects \mathcal{L} at one point: their unique tangency point. Hence, each line L tangent to \mathcal{L} is disjoint from the leaves lying on the convex side from \mathcal{L} . Thus, L is disjoint from $\widehat{\gamma}$ and $O \notin L$. Let U denote the set of points dual to lines tangent to leaves in W . In the dual picture, the latter statements mean that $U \subset \mathbb{R}^2$ and for every $A = L^* \in U$, there are no tangent lines to γ passing through A . The set U is path-connected, disjoint from γ , and it accumulates to all of γ . Therefore, it approaches γ from the convex side, by the previous statement. Hence, it lies entirely on its convex side. Let us now prove that shrinking W , one can achieve that the map $x \mapsto L_x^*$ be a diffeomorphism $W \cup \widehat{\gamma} \rightarrow U \cup \gamma$.

Fix a compact arc exhaustion

$$\widehat{\gamma}_1 \Subset \widehat{\gamma}_2 \Subset \dots = \widehat{\gamma}.$$

For every k , fix a flowbox $\Pi_k \subset W$ of the foliation \mathcal{F} adjacent to $\widehat{\gamma}_k$ whose leaves are strictly convex. We construct the flowboxes Π_k with decreasing heights, which means that for every k , each leaf of the flowbox Π_{k+1} crosses Π_k . Now replace W by the union $\bigcup_k \Pi_k$, which will be now denoted by W . The leaves of the foliation on $W \cup \widehat{\gamma}$ are strictly convex and connected, by construction. We claim that the map $x \mapsto L_x$, and hence $x \mapsto L_x^*$, is a C^∞ -smooth diffeomorphism. Indeed, it is a local diffeomorphism by strict convexity of leaves. It remains to show that $L_x \neq L_y$ for every distinct $x, y \in W$. Indeed, fix an $x \in W$, let \mathcal{L} denote the leaf of the foliation \mathcal{F} through x . Set $L = L_x$. Fix a k such that $x \in \Pi_k$. Every leaf in the flowbox Π_k that does not lie in its leaf through x either intersects L transversally or is disjoint from L , by convexity. Then the latter statement also holds for every other flowbox Π_ℓ , by construction and convexity. This implies that L can be tangent to no other leaf in W . It cannot be tangent to the same leaf \mathcal{L} at another point $y \neq x$: leaves have no bitangent lines, by construction. This proves diffeomorphicity of the map $x \mapsto L_x^*$. \square

The above C^∞ -smooth diffeomorphism $x \mapsto L_x^*$ sends W onto a domain $U \subset \mathbb{R}^2$ adjacent to γ . It sends leaves of the foliation \mathcal{F} to the corresponding caustics of the billiard on γ . Hence, the caustics together with the curve γ form a C^∞ -smooth foliation of $U \cup \gamma$. Constructing the above flowboxes Π_k narrow enough in the transversal direction (step by step), we can achieve that for every $x \in \gamma$ and every leaf \mathcal{L} of the foliation \mathcal{F} , there are at most two tangent lines through x to the caustic \mathcal{L}^* . Indeed, each leaf \mathcal{L} of the foliation \mathcal{F} is a leaf of some flowbox Π_k . Its dual caustic \mathcal{L}^* is convex, as is \mathcal{L} . It will satisfy the above tangent line statement if the total angle increment of its tangent vector is no greater than π . The latter angle increment statement holds for the curve γ . Hence, it remains valid for the caustics dual to the leaves of the flowbox Π_k if Π_k is chosen narrow enough. The existence statement (1) of Theorem 1.9 is proved.

The existence statement (2) of Theorem 1.9 (the case when γ is an arc) is proved analogously with small modifications. We deal with the billiard map $\widetilde{\delta}_+ = I \circ \beta$ acting on the restriction to γ of the unit tangent bundle to \mathbb{R}^2 . The map $I : (q', u) \mapsto (q', u^*)$ is the symmetry of a vector $u \in T_{q'}\mathbb{R}^2$, $q' \in \gamma$, with respect to the tangent line $T_{q'}\gamma$. The map $\beta(q, u) = (q', u')$ is defined for u close enough to $\dot{\gamma}(q)$, with q, q' being the same, as in §1.3, Case (2), Remark 1.21; the unit vectors $u \in T_q\mathbb{R}^2$, $u' \in T_{q'}\mathbb{R}^2$ being tangent to the line qq' and inducing the same orientation of this line. The corresponding map $F = \delta_+ : (s_1, y_1) \mapsto (s_2, y_2)$ from equation (1.3) with $s_1 = s(q) < s_2 = s(q')$ is a well-defined C^∞ -lifted strongly billiard-like map on the union of the natural parameter interval $J = (a, b) = (a, b) \times \{0\}$ of the curve γ and a domain in $\mathbb{R} \times \mathbb{R}_+$ adjacent to it. Now the map Λ sending (s_1, y_1) to the corresponding oriented line is a local C^∞ -smooth diffeomorphism. The modified Lemma 2.17 adapted to the case under consideration states that the F -invariant function G is locally C^∞ -smooth (up to the boundary) with non-zero differential as a function of a line. The above proof of Lemma 2.17 remains valid, since it uses only local arguments. The modified Lemma 2.17 implies (analogously to the above discussion) that the foliation $G = \text{const.}$ with leaves represented as smooth families of lines induces a C^∞ -smooth foliation by caustics (their enveloping curves) on an appropriate domain adjacent to γ from its convex side; smooth up to the boundary.

If γ is an arc, then we need not prove that for every $x \in \gamma$ and every leaf \mathcal{L}^* of the foliation by caustics, there are at most two lines through x tangent to \mathcal{L}^* . Indeed, the latter tangent line condition (which is required in the definition of foliation by caustics for γ being an unbounded submanifold) is not required for γ being an arc: see Remark 1.7. \square

The proof of the existence in Theorem 1.12 repeats the above proof of the existence in Theorem 1.9 with obvious changes. The existence statement of Theorem 1.13 follows from that of Theorem 1.12.

2.8. *Space of foliations. Proofs of Theorems 1.31, 1.34, and 1.35 and Proposition 1.33.*

Proof of Theorem 1.31. It suffices to prove the statement of Theorem 1.31 for foliations from Theorems 1.26 and 1.30, since the foliations in Theorems 1.9, 1.12, and 1.13 are obtained from foliations in Theorem 1.26 by duality, see the above subsection.

Case of Theorem 1.26. Consider a C^∞ -lifted strongly billiard-like map. We already know that in appropriate coordinates, it takes the form in equation (1.12):

$$F(\tau, h) = (\tau + \sqrt{h}, h). \tag{2.40}$$

The function h is F -invariant, and so are its level lines.

Suppose the contrary: there exists another C^∞ -smooth F -invariant function $G(\tau, h)$, $G(\tau, 0) \equiv 0$, without critical points on $J = \{h = 0\}$ and such that there exists an $x \in J$ where the foliations $G = \text{const.}$ and $h = \text{const.}$ have different k -jets for some k . Without loss of generality, we consider that $x = (0, 0)$, shifting the coordinate τ . Then the asymptotic Taylor series of the function G at x contains at least one monomial $a_{mn}\tau^m h^n$ with $m \geq 1$ and a non-zero coefficient a_{mn} . Set

$$\mathcal{N} := \{(m, n) \mid a_{mn} \neq 0, m \geq 1\}, \quad d := \min\{m + 2n \mid (m, n) \in \mathcal{N}\}.$$

Consider the lower (1, 2)-quasihomogeneous terms containing τ :

$$G_d(\tau, h) := \sum_{(m,n) \in \mathcal{N}, m+2n=d} a_{mn} \tau^m h^n.$$

One has $G \circ F(\tau, h) - G(\tau, h) \equiv 0$. However,

$$\begin{aligned} G \circ F(\tau, h) - G(\tau, h) &= G(\tau + \sqrt{h}, h) - G(\tau, h) \\ &= G_d(\tau + \sqrt{h}, h) - G_d(\tau, h) + \text{higher terms}. \end{aligned} \tag{2.41}$$

Here ‘higher terms’ means ‘a function that admits an asymptotic Taylor series in (τ, \sqrt{h}) at $(0, 0)$ that contains only terms $a_{mn}\tau^\alpha h^\beta$ of quasihomogeneous degrees $\alpha + 2\beta > d$ ’. Let m_0 denote the higher degree of τ in a monomial entering G_d . The difference $G_d(\tau + \sqrt{h}, h) - G_d(\tau, h)$ is quasihomogeneous of degree d . It contains the monomial $\tau^{m_0-1} h^\ell$, $m_0 - 1 + 2\ell = d$, $\ell \in \frac{1}{2} + \mathbb{Z}$, with non-zero coefficient, by construction. This monomial will not cancel out with other monomials in the asymptotic Taylor series of the difference $G \circ F(\tau, h) - G(\tau, h)$, by construction. Therefore, the latter difference cannot be identically equal to zero. The contradiction thus obtained proves the statement of Theorem 1.31 in the conditions of Theorem 1.26.

Case of Theorem 1.30. In this case, we know that the map in question is conjugated to $(\tau, \phi) \mapsto (\tau + \phi, \phi)$. The statement of Theorem 1.31 for the latter map is proved by the above arguments for lower homogeneous (that is, (1, 1)-quasihomogeneous) terms of the Taylor series of the function G . □

Proof of Proposition 1.33. Let $g(\tau, h)$ be a C^∞ -smooth function invariant under the map $F : (\tau, h) \mapsto (\tau + \sqrt{h}, h)$ that has type in equation (1.15):

$$g(\tau, h) = h + \text{flat}(h), \quad g(0, h) \equiv h. \tag{2.42}$$

Invariance is equivalent to the equality $g(\tau + \sqrt{h}, h) = g(\tau, h)$. This together with equation (2.42) implies that

$$\text{the function } \psi(s, h) := g(s\sqrt{h}, h) - h \text{ is 1-periodic in } s, \quad \psi(0, h) = 0. \tag{2.43}$$

Moreover, the function $\psi(s, h)$ is C^∞ -smooth and h -flat on a cylinder $S^1 \times [0, \delta)$, $S^1 = \mathbb{R}_s/\mathbb{Z}$, for a small $\delta > 0$. This follows from smoothness and h -flatness of the function $g(\tau, h) - h$. Conversely, consider an h -flat function $\psi(s, h)$ that is 1-periodic in s and such that $\psi(0, h) = 0$. Then the function

$$g(\tau, h) := \psi\left(\frac{\tau}{\sqrt{h}}, h\right) + h$$

is C^∞ -smooth, F -invariant, and its difference with h is h -flat, by construction. Statement (1) of Proposition 1.33 is proved. Statement (2) can be reduced to statement (1) and also can be proved analogously. □

Theorem 1.34 follows immediately from Proposition 1.33, and in its turn, it immediately implies Theorem 1.35.

2.9. Proof of Proposition 1.36 and non-uniqueness in main theorems.

Proof of Proposition 1.36. Let us prove the statement of Proposition 1.36 for a map \tilde{F} of the type in equation (1.13). We prove it for line fields: for other objects, the proof is analogous. Without loss of generality, we can and will consider that the map \tilde{F} takes the form $(\tau, \phi) \mapsto (\tau + \phi, \phi)$: see statement (2) of Theorem 1.30. Let G_1 and G_2 be two \tilde{F} -invariant line fields on W with distinct germs at J . This means that there exists a sequence of points $x(k) = (\tau(k), \phi(k))$ with $\phi(k) \rightarrow 0$ and $\tau(k)$ lying in a compact subset in J such that the lines $G_1(x(k)), G_2(x(k)) \subset T_{x(k)}\mathbb{R}^2$ are distinct. Taking a subsequence, we can and will consider that $x(k) \rightarrow x = (\tau_0, 0)$, as $k \rightarrow \infty$. The two-sided orbit of a point $x(k)$ with big k consists of points with ϕ -coordinate $\phi(k)$ whose τ -coordinates form an arithmetic progression with step $\phi(k)$ converging to zero. At each point of the orbit, the lines of the fields G_1 and G_2 are distinct, since this holds at $x(k)$ and by \tilde{F} -invariance. Therefore, passing to limit, as $k \rightarrow \infty$, we get that for every point $z \in J$, there exist points z' arbitrarily close to z with $G_1(z') \neq G_2(z')$. Hence, the germs at z of the line fields G_1 and G_2 are distinct. The first statement of Proposition 1.36, for a map \tilde{F} of the type in equation (1.13), is proved. Its second statement, for a C^∞ -lifted strongly billiard-like map $F : V \cup J \rightarrow F(V \cup J) \subset \mathbb{R}^2$, follows from its first statement and the fact that F

is conjugated to the map $(\tau, \phi) \mapsto (\tau + \phi, \phi)$ by a homeomorphism that is smooth on the complement to the boundary interval J . The latter conjugating homeomorphism is the composition of a diffeomorphism conjugating F to the map $(\tau, h) \mapsto (\tau + \sqrt{h}, h)$ (Theorem 1.27) and the map $(\tau, h) \mapsto (\tau, \phi), \phi = \sqrt{h}$. Proposition 1.36 is proved. \square

Proof of non-uniqueness in Theorems 1.30, 1.26, 1.9, 1.12, and 1.13. Existence of a continuum of distinct germs of foliations satisfying the statements of any of the above-mentioned theorems follows immediately from Proposition 1.36 and Theorem 1.35, which states that there are as many distinct boundary germs as flat functions on the cylinder $S^1 \times \mathbb{R}_{\geq 0}$ with distinct germs at $S^1 \times \{0\}$. It remains to show that there exists a domain adjacent to the boundary interval (or the curve γ) which admits an infinite-dimensional family of corresponding foliations with distinct germs.

Case of Theorem 1.30. Fix coordinates (τ, ϕ) in which $\tilde{F}(\tau, \phi) = (\tau + \phi, \phi)$. Recall that the coordinates (τ, ϕ) are defined on $W \cup J$, where $W \subset \mathbb{R} \times \mathbb{R}_+$ is a domain adjacent to the interval $J = (a, b) \times \{0\}$. Without loss of generality, we consider that $a < 0 < b$. Fix a C^∞ -smooth h -flat function ψ on the cylinder $S^1 \times \mathbb{R}_{\geq 0}$, $S^1 = \mathbb{R}/\mathbb{Z}$, with non-trivial germ at $S^1 \times \{0\}$:

$$\psi(s, h) = \psi(s + 1, h) = \text{flat}(h), \quad \psi(0, h) = 0, \quad |\psi| < \frac{1}{8}. \tag{2.44}$$

For every $\varepsilon > 0$, the function

$$g_\varepsilon(\tau, \phi) := \phi + \varepsilon\chi(\tau, \phi), \quad \chi(\tau, \phi) := \psi\left(\frac{\tau}{\phi}, \phi\right) \tag{2.45}$$

is \tilde{F} -invariant, C^∞ -smooth, and well defined on $W \cup J$, $g_\varepsilon(\tau, \phi) = \phi + \text{flat}(\phi)$. Let us show that shrinking the domain W one can achieve that the foliation $g_\varepsilon = \text{const.}$ is regular, that is, g_ε has no critical points on W , whenever ε is small enough.

CLAIM 5. *Replacing W by a smaller domain adjacent to J , one can achieve that each partial derivative of the function χ (of any order) be bounded on W . For any given m and every $\delta > 0$ shrinking W (depending on m and δ), one can achieve that all its order m partial derivatives have moduli less than δ .*

Proof. The modulus of each partial derivative of order m admits an upper bound by a quantity

$$\left| \frac{\partial^m \chi(\tau, \phi)}{\partial \tau^\nu \partial \phi^{m-\nu}} \right| \leq c_m (1 + |\tau|^m) (1 + \phi^{-2(m+1)}) \sum_{\ell, r=1}^m |\psi_{\ell r}|, \tag{2.46}$$

$$c_m = \text{const.} > 0, \quad \psi_{\ell r}(\tau, \phi) = \frac{\partial^{\ell+r} \psi}{\partial s^\ell \partial \phi^r} \left(\frac{\tau}{\phi}, \phi \right) = o(\phi^k) \quad \text{for every } k \in \mathbb{N}.$$

Here the latter $o(\phi^k)$ is uniform in τ , as $\phi \rightarrow 0$. Equation (2.46) follows from 1-periodicity and flatness of the function $\psi(s, h)$, and chain rule for calculating derivatives. Let us now replace the domain W by a smaller domain adjacent to J on which the right-hand side in equation (2.46) is bounded for each m and is less than δ for a given m . To do this, fix an arbitrary $\eta > 0$. First let us replace W by the connected component adjacent to J of its intersection with the rectangle $(a, b) \times (0, \eta)$. In the case when (a, b) is a

finite interval, the right-hand side in equation (2.46) is uniformly bounded on W and tends to zero uniformly in $\tau \in (a, b)$, as $\phi \rightarrow 0$. This follows from boundedness of derivatives of the function ψ on any compact cylinder $S^1 \times [0, \eta]$ and from the fact that the asymptotics $\psi_{\ell r}(\tau, \phi) = o(\phi^{3m+3})$ kills polynomial growth of the function $\phi^{-2(m+1)}$. Therefore, shrinking W , one can achieve that for given m and δ , the right-hand side in equation (2.46) be less than δ on W .

In the case when some (or both) of the boundary points a or b is infinity, take an exhaustion of the interval (a, b) by segments $[a_k, b_k]$. By the above argument, we can take a rectangle $\Pi_k = (a_k, b_k) \times (0, d_k) \subset W$ on which for all m , the right-hand sides in equation (2.46) be bounded, and for some given m , the same right-hand side be less than a given δ . Replacing W by $\bigcup_k \Pi_k$, we achieve that the two latter inequalities hold on $W \cup J$. □

Let W satisfy the statements of the above claim so that each first partial derivative of the function χ has modulus less than $\frac{1}{2}$. Then for every $\varepsilon \in [0, 1]$, the foliation $g_\varepsilon = \text{const.}$ is regular on $W \cup J$, since for those ε , one has $\partial g_\varepsilon / \partial \phi = 1 + \varepsilon(\partial \chi / \partial \phi) > \frac{1}{2}$ on $W \cup J$. All its leaves are \tilde{F} -invariant. For distinct values of the parameter ε the germs of the corresponding foliations are distinct at each point of the interval J , by Theorem 1.35, Proposition 1.36, and non-triviality of the germ at $S^1 \times \{0\}$ of the function ψ . This yields a one-dimensional family of foliations from Theorem 1.30 with pairwise distinct germs at each point in J .

Now let us apply the above argument with the expression $\varepsilon \chi$ in equation (2.45) being replaced by an arbitrary linear combination

$$\tilde{\chi}_\varepsilon(\tau, \phi) = \sum_{k=1}^N \frac{\varepsilon_k}{k! 4^k} \psi^k\left(\frac{\tau}{\phi}, \phi\right), \quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_N) \in [0, 1]^N. \tag{2.47}$$

Recall that $|\psi| < \frac{1}{8}$. This inequality together with the above assumption that the first partial derivatives of the function $\chi(\tau, \phi) = \psi(\tau/\phi, \phi)$ have moduli less than $\frac{1}{2}$ on W imply that for every ε as in equation (2.47), the module of each first partial derivative of the function $\tilde{\chi}_\varepsilon$ is less than $\frac{1}{2}$ on $W \cup J$. This implies that the foliation by level curves of the function $g_\varepsilon(\tau, \phi) = \phi + \tilde{\chi}_\varepsilon$ is a C^∞ -smooth foliation on $W \cup J$. We get an N -dimensional family of foliations on $W \cup J$ depending on $(\varepsilon_1, \dots, \varepsilon_N) \in [0, 1]^N$ with pairwise distinct germs at J , and hence, at each point of the curve J (Proposition 1.36). The non-uniqueness statement of Theorem 1.30 is proved.

Case of Theorem 1.26. Its non-uniqueness statement follows from that of Theorem 1.30 and also from the above arguments.

Case of Theorem 1.9(1). Here, $\gamma \subset \mathbb{R}^2$ is an unbounded one-dimensional submanifold. Let us consider the billiard ball map acting on lines as a C^∞ -lifted strongly billiard-like map F . Let us introduce new (symplectic) coordinates (τ, h) in which $F(\tau, h) = (\tau + \sqrt{h}, h)$, see equation (1.12). The map F is defined on $W \cup J$, where $J = (a, b) \times \{0\}$, $a < 0 < b$, parameterizes the family of lines tangent to γ and $W \subset \mathbb{R} \times \mathbb{R}_+$ is a domain adjacent to J . Representing lines as points in \mathbb{RP}^2 via a projective duality $\mathbb{RP}^{2*} \rightarrow \mathbb{RP}^2$ transforms J to a strictly convex curve $\gamma^* \subset \mathbb{RP}^2$ dual to γ , and W to a domain adjacent

to γ^* from the concave side. See §2.7. We can and will consider that γ^* and W lie in an affine chart \mathbb{R}^2 , as in the proof of the existence in Theorem 1.9 in §2.7. In what follows, we identify J with γ^* . Consider the foliation $h = \text{const.}$ by F -invariant curves. Let us first construct a one-dimensional family of foliations using a C^∞ -smooth h -flat function $\psi(s, h)$ on $S^1 \times \mathbb{R}_{\geq 0}$, $S^1 = \mathbb{R}_s/\mathbb{Z}$, with non-trivial germ at $S^1 \times \{0\}$, as in equation (2.44). Namely, set

$$g_\varepsilon(\tau, h) := h + \varepsilon\chi(\tau, h), \quad \chi(\tau, h) := \psi\left(\frac{\tau}{\sqrt{h}}, h\right), \quad \varepsilon \in [0, 1].$$

The functions g_ε are F -invariant and $g_\varepsilon(\tau, h) = h + \text{flat}(h)$. The germs of any two foliations $g_{\varepsilon_1} = \text{const.}$, $g_{\varepsilon_2} = \text{const.}$, $\varepsilon_1 \neq \varepsilon_2$, are distinct at each point in J , by Theorem 1.35 and Proposition 1.36. It remains to prove their regularity and regularity of the dual foliations by caustics on one and the same domain. To do this, we use the following claim.

CLAIM 6. *Shrinking the domain W adjacent to $J = \gamma^*$, one can achieve that for every $\varepsilon \in [0, 1]$, the foliation $g_\varepsilon = \text{const.}$ is regular on $W \cup J$, its leaves are strictly convex curves, as is γ^* , and the map $\Lambda_\varepsilon : x \rightarrow L_{x,\varepsilon}$ sending a point $x \in W$ to the projective line $L_{x,\varepsilon}$ tangent to the level curve $\{g_\varepsilon = g_\varepsilon(x)\}$ at x is a diffeomorphism on W .*

Proof. Consider the function h and the above function χ as functions on $W \cup \gamma^*$ as on a subset in $\mathbb{R}^2 \subset \mathbb{R}\mathbb{P}^2$. The curve $\gamma^* = \{h = 0\}$ is strictly convex. Hence, shrinking W , we can and will consider that each level curve $\{h = \text{const.}\} \cap W$ is strictly convex. Consider the rectangles $\Pi_k \subset W$ from the proof of the above Claim 5 (in the coordinates (τ, h)) with decreasing heights. They are represented as curvilinear quadrilaterals in $\mathbb{R}^2 \subset \mathbb{R}\mathbb{P}^2$. Choosing them with heights small enough, we can achieve that $\|\nabla\chi\| < \frac{1}{2}\|\nabla h\|$ on Π_k . Let us now replace W by $\bigcup_k \Pi_k$. Then, $\nabla g_\varepsilon \neq 0$ on W , and hence, the foliation $g_\varepsilon = \text{const.}$ is regular for all $\varepsilon \in [0, 1]$. Choosing Π_k with heights small enough (step by step), one can also achieve that each level curve $\{g_\varepsilon = \text{const.}\} \cap \Pi_k$ be strictly convex for every $\varepsilon \in [0, 1]$, by strict convexity of the boundary curve γ^* and h -flatness of the function ψ . In more detail, let (x, y) be coordinates on the ambient affine chart \mathbb{R}^2 . Strict convexity of level curves $\{g_\varepsilon = \text{const.}\}$ is equivalent to non-vanishing of the Hessian $H(g_\varepsilon)$:

$$H(g_\varepsilon) \neq 0, \quad H(g) := \frac{\partial^2 g}{\partial x^2} \left(\frac{\partial g}{\partial y}\right)^2 + \frac{\partial^2 g}{\partial y^2} \left(\frac{\partial g}{\partial x}\right)^2 - 2 \frac{\partial^2 g}{\partial x \partial y} \left(\frac{\partial g}{\partial x}\right) \left(\frac{\partial g}{\partial y}\right).$$

(The Hessian $H(g)$ of a function g was introduced by Tabachnikov in his paper [21], where he used it to study his conjecture stating that every polynomially integrable outer billiard is an ellipse (later this conjecture was solved in [11]). The Hessian was also used by Bialy, Mironov, and later the author in the solution of Bolotin's polynomial version of the Birkhoff conjecture, which is the result of papers [5, 6, 9].)

The Hessian $H(g_\varepsilon)$ is the sum of the Hessian $H(h)$ (which is non-zero on $W \cup \gamma^*$, since the curves $\{h = \text{const.}\}$ are strictly convex) and a finite sum of products; each product contains ε , at least one derivative of the function χ , at most two derivatives of the function h , at most two derivatives of the function τ ; each derivative is of order at most two. Choosing the rectangles Π_k with heights small enough, we can achieve that the module of

the latter sum of products be no greater than $\frac{1}{2}\varepsilon|H(h)|$ for $\varepsilon \in [0, 1]$. This follows from convexity of the curve γ^* and h -flatness of the function ψ : shrinking W , one can achieve that all the first and second derivatives of the function χ have moduli bounded by arbitrarily small δ (Claim 5). Then $H(g_\varepsilon) \neq 0$ on W , hence, the curves $\{g_\varepsilon = \text{const.}\} \cap W$ are strictly convex.

Now for every k , we choose smaller rectangles $\tilde{\Pi}_k \subset \Pi_k$ with decreasing heights and with the lateral (that is, vertical) sides lying in the lateral sides of the bigger rectangles Π_k that satisfy the following additional statement. For every $\varepsilon \in [0, 1]$, let $\Pi_{k,\varepsilon}$ denote the minimal flowbox for the foliation $g_\varepsilon = \text{const.}$ with lateral (that is, transversal) sides lying in the lateral sides of Π_k that contains $\tilde{\Pi}_k$. This is the union of arcs of leaves that go from one lateral side of Π_k to the other one and cross $\tilde{\Pi}_k$. For every k , we can and will subsequently choose $\tilde{\Pi}_k$ with heights small enough (that is, narrow enough in the transversal direction) so that for every $\varepsilon \in [0, 1]$, the flowbox $\Pi_{k,\varepsilon}$ exists, lies in Π_k , and the heights of the flowboxes $\Pi_{k,\varepsilon}$ be decreasing in k : more precisely, for every k , each local leaf in $\Pi_{k+1,\varepsilon}$ crosses $\Pi_{k,\varepsilon}$, as in the proof of Proposition 2.22. Then the map $\Lambda_\varepsilon : x \mapsto L_{x,\varepsilon}$ is a diffeomorphism on $W_\varepsilon := \bigcup_k \Pi_{k,\varepsilon}$ for every $\varepsilon \in [0, 1]$, as at the end of the proof of Proposition 2.22. Hence, it is a diffeomorphism on

$$\tilde{W} := \bigcup_k \tilde{\Pi}_k. \tag{2.48}$$

The claim is proved. □

CLAIM 7. *Consider the foliation by caustics of the billiard on γ that is dual to the foliation $g_\varepsilon = \text{const.}$. There exists a domain $U \subset \mathbb{R}^2$ adjacent to γ from the convex side where the above foliation by caustics is C^∞ -smooth (and also smooth at the points of the curve γ) for every $\varepsilon \in [0, 1]$. Moreover, shrinking U , one can achieve that for every $x \in \gamma$ and every $\varepsilon \in [0, 1]$, there are at most two tangent lines through x to any given leaf of the corresponding foliation by caustics on U .*

Proof. Let \tilde{W} be the domain in equation (2.48) constructed above. For every $\varepsilon \in [0, 1]$, the map $\Lambda_\varepsilon^* : x \mapsto L_{x,\varepsilon}^*$ sending x to the point dual to the corresponding line $L_{x,\varepsilon}$ is a diffeomorphism, since so is Λ_ε . It sends the domain \tilde{W} foliated by level curves of the function g_ε onto a domain U_ε adjacent to γ and foliated by their dual curves: caustics of the billiard on γ . They form a C^∞ -smooth foliation on $U_\varepsilon \cup \gamma$. For the proof of the first statement of Claim 7, it remains to show that there exists a domain U adjacent to γ that lies in the intersection $\bigcap_\varepsilon U_\varepsilon$ (and hence, for each ε , it is smoothly foliated by the corresponding caustics). To do this, we construct the above \tilde{W} and a smaller domain $W' \subset \tilde{W}$ adjacent to γ^* so that the following statement holds:

(*) for every $p \in W'$ and every $\varepsilon \in [0, 1]$, there exists a $q = q(p, \varepsilon) \in \tilde{W}$ such that the projective line $L_{p,0}$ tangent to the curve $\{h = h(p)\}$ at p is tangent to the leaf of the foliation $g_\varepsilon = \text{const.}$ at q .

Statement (*) implies that the image $U = \Lambda_0(W')$ is contained in all the domains U_ε and regularly foliated by caustics dual to level curves of the function g_ε for every $\varepsilon \in [0, 1]$.

Take the rectangles Π_k and $\tilde{\Pi}_k$ from the proof of Claim 6. Let us call their sections $h = \text{const.}$ horizontal and transversal sections $\tau = \text{const.}$ vertical. For every k , fix two vertical sections $\ell_{1,k}$ and $\ell_{2,k}$ crossing the interior $\text{Int}(\Pi_k)$ that lie in the $1/2^k$ -neighborhoods of the corresponding lateral sides of the rectangle Π_k . We can and will choose a rectangle $\Pi'_k \subset \tilde{\Pi}_k$ with lateral sides lying on $\ell_{1,k}$ and $\ell_{2,k}$, and height small enough so that for every $\varepsilon \in [0, 1]$ and every $p \in \Pi'_k$, there exists a $q = q(p, \varepsilon) \in \tilde{\Pi}_k$ satisfying statement (*). This is possible by flatness of the function ψ and strict convexity of the curve γ^* . Then statement (*) holds for the domain $W' = \bigcup_k \Pi'_k \subset W$. This together with the above discussion proves the first statement of Claim 7. One can achieve that its second statement (on tangent lines) holds as well by choosing the above rectangles Π'_k with height small enough, as in the proof of the existence in Theorem 1.9 at the end of §2.7. □

Claim 7 implies non-uniqueness statement (1) of Theorem 1.9 (the case when γ is an unbounded one-dimensional submanifold), with one-dimensional family of foliations with distinct germs. Modifying the above arguments as in the proof of non-uniqueness statement of Theorem 1.30 (see equation (2.47) and the discussion after it), we get an N -dimensional family of foliations with distinct germs for every $N \in \mathbb{N}$. Non-uniqueness statement (2) of Theorem 1.9 is proved analogously, with small modifications analogous to those mentioned at the end of §2.7. Non-uniqueness statements of Theorems 1.12 and 1.13 are proved analogously. □

2.10. *Conjugacy of billiard maps and Lazutkin length. Proof of Theorems 1.39, 1.40, 1.43, and 1.44, Propositions 1.45 and 1.46, and Lemma 1.47.*

Proof of Proposition 1.45. The map

$$(s, y) \mapsto (X, Y) := (t_L(s), w^{2/3}(s)y), \quad t_L(s) = \int_{s_0}^s w^{-(2/3)}(u) du$$

is symplectic and conjugates F to a C^∞ -lifted strongly billiard-like map of the type

$$\Phi : (X, Y) \mapsto (X + \sqrt{Y} + O(Y), Y + o(Y^{3/2})), \tag{2.49}$$

see [10, Theorem 7.11] and Proposition 1.24. (This statement is implicitly contained in [15, 16].) Thus, without loss of generality, we can and will consider that F has the form in equation (2.49), hence, $w(s) \equiv 1$. Then, $t_L(s) = s$ up to additive constant. Thus, we have to show that

$$H_1(s, 0) = \alpha s + \beta, \tag{2.50}$$

$$H_1(s, 0) = s + \beta \quad \text{if } H \text{ is symplectic.} \tag{2.51}$$

By definition, $H(s, y) = (H_1(s, y), H_2(s, y))$ conjugates F to $\Lambda : (t, z) \mapsto (t + \sqrt{z}, z)$. Hence, it sends the fixed point line $\{y = 0\}$ of the map F to the fixed point line $\{z = 0\}$ of the map Λ , thus, $H_2(s, 0) \equiv 0$. Writing conjugacy equation on the first components yields

$$\begin{aligned}
 H_1 \circ F(s, y) &= H_1(s + \sqrt{y} + O(y), y + o(y^{3/2})) = H_1(s, 0) + \frac{\partial H_1}{\partial s}(s, 0)\sqrt{y} + O(y) \\
 &= \Lambda_1 \circ H(s, y) = H_1(s, y) + \sqrt{H_2(s, y)} = H_1(s, 0) \\
 &\quad + \sqrt{\frac{\partial H_2}{\partial y}(s, 0)\sqrt{y} + O(y)}.
 \end{aligned}$$

This yields

$$\frac{\partial H_1}{\partial s}(s, 0) \equiv \sqrt{\frac{\partial H_2}{\partial y}(s, 0)} > 0. \tag{2.52}$$

The Jacobian matrix of the map H at points $(s, 0)$ is equal to

$$\text{Jac}(s, 0) = \frac{\partial H_1}{\partial s}(s, 0) \frac{\partial H_2}{\partial y}(s, 0) = \left(\frac{\partial H_1}{\partial s}(s, 0)\right)^3 > 0, \tag{2.53}$$

by equation (2.52) and since $H_2(s, 0) \equiv 0$, which yields $\partial H_2/\partial s(s, 0) = 0$. This proves orientation-preserving property of the diffeomorphism H and increasing of the function $H_1(s, 0)$.

Now let H be symplectic, that is, $\text{Jac}(s, 0) \equiv 1$. Then $\partial H_1(s, 0)/\partial s \equiv 1$, by equation (2.53). This means that $H_1(s, 0) = s + \beta$ for some $\beta \in \mathbb{R}$. This proves equation (2.51), that is, equation (1.20).

Now let H be not necessarily symplectic. Let us prove equation (2.50), that is, equation (1.21). Suppose the contrary: there exist two points $s_0 < s_0^* \in (a, b)$ such that

$$\ell := \frac{\partial H_1}{\partial s}(s_0, 0) \neq \ell^* := \frac{\partial H_1}{\partial s}(s_0^*, 0).$$

Fix small $\varepsilon, \delta > 0$ such that

$$s_0 - \varepsilon, s_0^* + \varepsilon \in (a, b), \quad [\ell - \delta, \ell + \delta] \cap [\ell^* - \delta, \ell^* + \delta] = \emptyset.$$

Fix a small $\eta > 0$ and a $y_0 \in (0, \eta/4)$, set $q_0 = (s_0, y_0)$. Let $q_{-N_-}, \dots, q_{-1}, q_0, q_1, \dots, q_{N_+}$, $q_j = (s_j, y_j)$, denote the F -orbit of the point q_0 in the rectangle $[s_0 - \varepsilon, s_0^* + \varepsilon] \times [0, \eta]$. Here, $N_{\pm} = N_{\pm}(q_0)$. It is known that the s -coordinates of its points form an asymptotic arithmetic progression $s_j = s(q_j)$, and their y -coordinates are asymptotically equivalent:

$$s_{j+1} - s_j \simeq \sqrt{y_0}, y_j \simeq y_0 \quad \text{as } y_0 \rightarrow 0, \tag{2.54}$$

uniformly in $j \in [-N_-(y_0), N_+(y_0) - 1]$,

$$s_{-N_-} < s_0, \quad s_{N_+} > s_0^*, \tag{2.55}$$

whenever y_0 is small enough (depending on ε). See [10, Lemma 7.13]. The image of the above orbit under the map H should be an orbit of the map $\Lambda : (t, z) \mapsto (t + \sqrt{z}, z)$. The abscissas of its points, $t_j := H_1(q_j)$, form an arithmetic progression: $t_{j+1} - t_j = \sqrt{z_0}$,

$z_0 = z(H(q_0))$. We claim that this yields a contradiction to the inequality $\ell \neq \ell^*$ and equation (2.54). Indeed, one has

$$t_1 - t_0 = H_1(q_1) - H_1(q_0) \simeq \ell(s_1 - s_0) \simeq \ell\sqrt{y_0}, \tag{2.56}$$

by equation (2.54) and the Lagrange increment theorem. However, take a family of indices $k = k(y_0)$ such that $s_k = s_k(y_0) \rightarrow s_0^*$, as $y_0 \rightarrow 0$: it exists, since the asymptotic progression s_j has steps uniformly decreasing to 0, it starts on the left from s_0 and ends on the right from $s_0^* > s_0$, see equation (2.55). Repeating the above argument for t_k and t_{k+1} yields

$$t_{k+1} - t_k \simeq \ell^* \sqrt{y_0}, \quad t_1 - t_0 \simeq \ell \sqrt{y_0} \quad \text{as } y_0 \rightarrow 0.$$

Hence, $t_{k+1} - t_k \neq t_1 - t_0$, whenever y_0 is small enough, since $\ell \neq \ell^*$. The contradiction thus obtained to the equality of the above differences proves that $\partial H_1(s, 0)/\partial s \equiv \text{const}$. This proves equation (2.50) and Proposition 1.45. □

Proof of Proposition 1.46. This repeats the proof of Proposition 1.45(2). □

Proof of Theorem 1.44. Let us fix an arbitrary point in each curve γ_i and parameterize it by natural length parameter so that the given point corresponds to zero parameter value. For every $i = 1, 2$, let \mathcal{T}_{γ_i} denote the billiard map corresponding to the curve γ_i . It is a C^∞ -lifted strongly billiard-like map in the coordinates (s, y) defined on a domain in $\mathbb{R} \times \mathbb{R}_+$ adjacent to an interval $J_{\gamma_i} = (a_i, b_i) \times \{0\}$; here (a_i, b_i) is the length parameter interval parameterizing γ_i . The corresponding function $w = w_i(s)$ is equal to $2\sqrt{2}\kappa_i^{-1}(s)$, where κ_i is the curvature of the curve γ_i . There exists a domain $U_i \subset \mathbb{R}_s \times (\mathbb{R}_+)_y$ adjacent to J_{γ_i} such that there exists a C^∞ -smooth symplectomorphism $H_i = (H_{1i}, H_{2i})$ on $U_i \cup J_{\gamma_i}$ conjugating \mathcal{T}_{γ_i} to its normal form $\Lambda : (t, z) \mapsto (t + \sqrt{z}, z)$, $H_i \circ \mathcal{T}_{\gamma_i} \circ H_i^{-1} = \Lambda$ (Theorem 1.27). The restriction to the s -axis of the first component H_{1i} is given by the Lazutkin parameter:

$$H_{1i}(s, 0) = t_L(s) := \int_0^s w_i^{-(2/3)}(u) du + \text{const.} = \frac{1}{2} \int_0^s \kappa_i^{2/3}(u) du + \text{const.}, \tag{2.57}$$

by Proposition 1.45. Therefore, the image $H_{1i}(J_{\gamma_i})$ is the interval $\tilde{J}_i \times \{0\}$ equipped with the coordinate t_L , whose length is thus equal to $\frac{1}{2}\mathcal{L}(\gamma_i)$. The image domain $H_i(U_i) \subset \mathbb{R} \times \mathbb{R}_+$ is adjacent to \tilde{J}_i . Thus, (symplectic) C^∞ -conjugacy of the billiard maps near the boundary is equivalent to the existence of a (symplectic) C^∞ -diffeomorphism Φ commuting with Λ , defined on a domain $V_1 \subset \mathbb{R}_t \times (\mathbb{R}_+)_z$ adjacent to \tilde{J}_1 and sending it onto a domain $V_2 \subset \mathbb{R}_t \times (\mathbb{R}_+)_z$ adjacent to \tilde{J}_2 that extends as a C^∞ -diffeomorphism to \tilde{J}_1 , $\Phi(\tilde{J}_1) = \tilde{J}_2$. The latter diffeomorphism Φ exists in the class of symplectomorphisms if and only if $\mathcal{L}(\gamma_1) = \mathcal{L}(\gamma_2)$ and one of the conditions (i) or (ii) of Theorem 1.43 holds. Indeed, if Φ is a symplectomorphism commuting with Λ and smooth on $V_1 \cup \tilde{J}_1$, then its restriction to \tilde{J}_1 should be a translation (Proposition 1.45). This implies that the lengths of the intervals \tilde{J}_1 and $\tilde{J}_2 = \Phi(\tilde{J}_1)$ are equal (that is, the Lazutkin lengths of the curves γ_1, γ_2 are equal) and at least one of the conditions (i) or (ii) holds. Conversely, if $\mathcal{L}(\gamma_1) = \mathcal{L}(\gamma_2)$ and one of the conditions (i) or (ii) holds, then we can and will consider

that $\tilde{J}_1 = \tilde{J}_2$, applying a translation. Then $\Phi = \text{Id}$ has the required properties. This proves Theorem 1.44. \square

Proof of Theorem 1.43. Let us repeat the above argument, where now the above diffeomorphism $\Phi : V_1 \cup \tilde{J}_1 \rightarrow V_2 \cup \tilde{J}_2$ commuting with Λ is not necessarily symplectic. Let such a Φ exist. Then its restriction to \tilde{J}_1 is an affine map in the first coordinate, $t \mapsto \alpha t + \beta$ (Proposition 1.45), and $\tilde{J}_2 = \Phi(\tilde{J}_1)$. This implies that $\mathcal{L}(\gamma_2) = \alpha \mathcal{L}(\gamma_1)$, and one of the conditions (i) or (ii) holds. Conversely, let one of the conditions (i) or (ii) hold. Then, without loss of generality, we can and will consider that the interval \tilde{J}_2 is obtained from \tilde{J}_1 by a homothety $t \mapsto \alpha t$, applying a translation. The latter homothety extends to the linear map $\Phi : (t, z) \mapsto (\alpha t, \alpha^2 z)$ commuting with Λ and thus having the required properties. The statement of Theorem 1.43 on conjugacy in (s, y) -coordinates is proved. Together with Remark 1.38, it implies that if one of the conditions (i) or (ii) holds, then the billiard maps are C^∞ -smoothly conjugated near the boundary in (s, ϕ) -coordinates. Let us prove the converse: C^∞ -smooth conjugacy in (s, ϕ) -coordinates implies that one of the conditions (i) or (ii) holds.

Let \tilde{H} be a C^∞ -smooth diffeomorphism conjugating the billiard maps \mathcal{T}_{γ_1} and \mathcal{T}_{γ_2} near the boundary in (s, ϕ) -coordinates, $\tilde{H} \circ \mathcal{T}_{\gamma_1} \circ \tilde{H}^{-1} = \mathcal{T}_{\gamma_2}$. Let H_i be symplectomorphisms conjugating the billiard maps \mathcal{T}_{γ_i} in (s, y) -coordinates with the map $\Lambda : (t, z) \mapsto (t + \sqrt{z}, z)$, see the above proof of Theorem 1.44. The variable changes

$$y \mapsto \phi = \arccos(1 - y), \quad z \mapsto \tilde{z} := \sqrt{z}$$

lift each diffeomorphism H_i to a diffeomorphism $\hat{H}_i(s, \phi) = (\hat{H}_{1i}(s, \phi), \hat{H}_{2i}(s, \tilde{\phi}))$ conjugating the corresponding billiard map with the map $\tilde{\Lambda} : (t, \tilde{z}) \mapsto (t + \tilde{z}, \tilde{z})$. Then the restriction of its first component $\hat{H}_{1i}(s, 0) = H_{1i}(s, 0)$ to the s -axis coincides with the Lazutkin parameter t_L of the curve γ_i up to post-composition with affine transformation, by Proposition 1.45. Set $\hat{J}_i := H_{1i}(J_{\gamma_i})$. The diffeomorphism $\Phi := \hat{H}_2 \circ \tilde{H} \circ \hat{H}_1^{-1}$ commutes with $\tilde{\Lambda}$, sends the interval \hat{J}_1 onto \hat{J}_2 , and sends a domain in $\mathbb{R} \times \mathbb{R}_+$ adjacent to \hat{J}_1 onto a domain in $\mathbb{R} \times \mathbb{R}_+$ adjacent to \hat{J}_2 . Therefore, the restriction of its first component to \hat{J}_1 is an affine map $t \mapsto \alpha t + \beta$, by Proposition 1.46. Thus, \hat{J}_2 is a rescaled image of the interval \hat{J}_1 up to translation. Recall that the lengths of the intervals \hat{J}_i are equal to the Lazutkin lengths of the corresponding curves γ_i divided by 2, see equation (2.57). This implies that the improper integrals defining the Lazutkin lengths of the curves γ_i converge or diverge simultaneously, and one of the conditions (i) or (ii) holds. Theorem 1.43 is proved. \square

Proof of Lemma 1.47. Consider a strictly convex C^2 -smooth planar curve γ going to infinity that has an asymptotic tangent line at infinity. Without loss of generality, we can and will consider that the latter tangent line is the horizontal x -axis in $\mathbb{R}_{x,y}^2$, γ lies above it, and γ is the graph of a C^2 -smooth function f defined on $[1, +\infty)$, $\gamma = \{y = f(x)\}$,

$$f(x), f'(x) \rightarrow 0 \quad \text{as } x \rightarrow +\infty, \quad f, f'' > 0, \quad f' < 0. \tag{2.58}$$

One can achieve this by applying rotation sending the asymptotic line to the x -axis, restricting ourselves to a subarc of the curve γ with the same asymptotic line and applying

symmetry with respect to the x -axis, if necessary. The improper integral defining the Lazutkin length of the curve γ takes the form

$$\int_1^{+\infty} \kappa^{2/3}(s) ds, \quad \kappa(s(x)) = \frac{f''(x)}{(1 + (f'(x))^2)^{3/2}}, \quad ds = \sqrt{1 + (f'(x))^2} dx. \quad (2.59)$$

Its convergence is equivalent to the convergence of the integral

$$\int_1^{+\infty} (f''(x))^{2/3} dx, \quad (2.60)$$

since $f'(x) \rightarrow 0$, thus, $1 + (f'(x))^2 \rightarrow 1$, as $x \rightarrow +\infty$.

CLAIM 8. For every C^2 -smooth function f as in equation (2.58), the improper integral in equation (2.60) converges.

Proof. The integral in equation (2.60) is estimated from above by Hölder inequality:

$$\begin{aligned} \int_1^{+\infty} (f''(x))^{2/3} dx &= \int_1^{+\infty} (xf''(x))^{2/3} x^{-(2/3)} dx \\ &\leq \left(\int_1^{+\infty} xf''(x) dx \right)^{2/3} \left(\int_1^{+\infty} \frac{dx}{x^2} \right)^{1/3}. \end{aligned} \quad (2.61)$$

Therefore, it remains to prove that the integral $\int_1^{+\infty} xf''(x) dx$ converges. Integrating by parts yields

$$\int_1^{+\infty} xf''(x) dx = xf'(x)|_1^{+\infty} - \int_1^{+\infty} f'(x) dx = xf'(x)|_1^{+\infty} - f(1). \quad (2.62)$$

Suppose the contrary: the integral on the left-hand side diverges. Then it is equal to $+\infty$, since $f''(x) > 0$. Therefore, $xf'(x) \rightarrow +\infty$, as $x \rightarrow +\infty$. Hence, $f'(x) > (1/x)$, whenever x is greater than some constant $N > 1$. Integrating the latter inequality along the semi-interval $[N, +\infty)$ yields $f(N) = +\infty$. The contradiction thus obtained proves convergence of the integral on the left-hand side in equation (2.62), and hence, of the integral in equation (2.60). The claim is proved. \square

The claim together with equation (2.59) imply convergence of the improper integral defining the Lazutkin length. This proves Lemma 1.47. \square

Proof of Theorem 1.39. The Lazutkin lengths of both curves γ_1 and γ_2 are finite, since they have asymptotic tangent lines at infinity in both directions and by Lemma 1.47. This together with Theorem 1.43 implies C^∞ -smooth conjugacy of the corresponding billiard maps near the boundary and up to the boundary. Theorem 1.39 is proved. \square

The proof of Theorem 1.40 is analogous to the above proof of Theorem 1.39.

Acknowledgements. I am grateful to Marie-Claude Arnaud, Sergei Tabachnikov, Alexander Plakhov, Ivan Beschastnyi, Stefano Baranzini, Vadim Kaloshin, and Comlan Koudjina for helpful discussions. I am grateful to the referees for careful reading the paper and for very helpful remarks. This work was partly supported by the Laboratory of Dynamical

Systems and Applications, HSE University, of the Ministry of Science and Higher Education of RF, grant agreement No 075-15-2019-1931 and by RFBR according to the research project 20-01-00420.

REFERENCES

- [1] E. Amiran. Caustics and evolutes for convex planar domains. *J. Differential Geom.* **28** (1988), 345–357.
- [2] V. Arnold. *Mathematical Methods of Classical Mechanics*. Springer-Verlag, New York, NY, 1978.
- [3] V. Arnold. *Contact Geometry and Wave Propagation (Monographs of L'Enseignement Mathématique, 34)*. Université de Genève, Genève, 1989.
- [4] M. Bialy, Convex billiards and a theorem by E. Hopf. *Math. Z.* **214**(1) (1993), 147–154.
- [5] M. Bialy and A. Mironov. Angular billiard and algebraic Birkhoff conjecture. *Adv. Math.* **313** (2017), 102–126.
- [6] M. Bialy and A. Mironov. Algebraic Birkhoff conjecture for billiards on Sphere and Hyperbolic plane. *J. Geom. Phys.* **115** (2017), 150–156.
- [7] M. Bialy and A. E. Mironov. The Birkhoff–Poritsky conjecture for centrally-symmetric billiard tables. *Ann. of Math. (2)* **196**(1) (2022), 389–413.
- [8] A. V. Bolsinov, V. S. Matveev, E. Miranda and S. Tabachnikov. Open problems, questions and challenges in finite-dimensional integrable systems. *Philos. Trans. Roy. Soc. A* **376**(2131) (2018), 20170430.
- [9] A. Glutsyuk. On polynomially integrable Birkhoff billiards on surfaces of constant curvature. *J. Eur. Math. Soc. (JEMS)* **23** (2021), 995–1049.
- [10] A. Glutsyuk. On curves with Poritsky property. *J. Fixed Point Theory Appl.* **24** (2022), 35; doi:10.1007/s11784-022-00948-7.
- [11] A. Glutsyuk and E. Shustin. On polynomially integrable planar outer billiards and curves with symmetry property. *Math. Ann.* **372** (2018), 1481–1501.
- [12] V. Kaloshin and C. E. Koudjinar. On some invariants of Birkhoff billiards under conjugacy. *Regul. Chaotic Dyn.* **27**(5) (2022), 525–537.
- [13] V. Kaloshin and A. Sorrentino. On local Birkhoff Conjecture for convex billiards. *Ann. of Math. (2)* **188**(1) (2018), 315–380.
- [14] V. Kaloshin and A. Sorrentino. On the integrability of Birkhoff billiards. *Philos. Trans. Roy. Soc. A* **376**(2131) (2018), 20170419.
- [15] V. F. Lazutkin. The existence of caustics for a billiard problem in a convex domain. *Math. USSR Izvestija* **7**(1) (1973), 185–214.
- [16] S. Marvizi and R. Melrose. Spectral invariants of convex planar regions. *J. Differential Geom.* **17** (1982), 475–502.
- [17] R. Melrose. Equivalence of glancing hypersurfaces. *Invent. Math.* **37** (1976), 165–192.
- [18] R. Melrose. Equivalence of glancing hypersurfaces 2. *Math. Ann.* **255** (1981), 159–198.
- [19] H. Poritsky. The billiard ball problem on a table with a convex boundary: an illustrative dynamical problem. *Ann. of Math. (2)* **51** (1950), 446–470.
- [20] S. Tabachnikov. Billiards. *Panor. Synthèses* **1** (1995), vi+ 142.
- [21] S. Tabachnikov. On algebraically integrable outer billiards. *Pacific J. Math.* **235**(1) (2008), 101–104.
- [22] M. P. Wojtkowski. Two applications of Jacobi fields to the billiard ball problem. *J. Differential Geom.* **40**(1) (1994), 155–164.