5

Classical and Quantum Registers

5.1 Introduction

In this chapter we extend the discussion of the previous chapter, from one detector to an apparatus with an arbitrary number of nodes at any given stage. Our labeling convention is followed throughout this book: subscripts always label stages whereas superscripts always label nodes and modules (discussed in Chapter 11). The number of nodes at stage Σ_n will be called the *rank* of the apparatus at that stage and denoted r_n. The ith node at stage Σ_n will be denoted by i_n, which should not be confused with labstate vectors such as $\boldsymbol{i}_n$, which are always denoted in bold font.

Whenever we are using classical mechanics (CM), the collection of classical nodes at stage Σ_n will be called a *classical binary register*, denoted $\mathcal{R}_n$. In that case, i_n is represented by classical bit B_n^i, so $\mathcal{R}_n$ is the *Cartesian product* of all the bits at that stage, that is,

$$\mathcal{R}_n \equiv B_n^1 \times B_n^2 \times \cdots \times B_n^{r_n}. \tag{5.1}$$

The cardinality (number of elements) of the rank r classical binary register $\mathcal{R}_n$ is $d_n \equiv 2^{r_n}$. If we wish to indicate the rank of $\mathcal{R}_n$, we write $\mathcal{R}_n^{[r_n]}$.

On the other hand, whenever we are using quantum mechanics (QM), the collection of nodes at stage Σ_n will be called a *quantum register* and denoted $\mathcal{Q}_n$ or $\mathcal{Q}_n^{[r_n]}$. In the quantum case, i_n is represented by qubit Q_n^i. The quantum register $\mathcal{Q}_n$ is the *tensor product* of all the qubits at that stage, that is,

$$\mathcal{Q}_n \equiv Q_n^1 \otimes Q_n^2 \otimes \cdots \otimes Q_n^{r_n}. \tag{5.2}$$

Such a tensor product space is a Hilbert space of dimension $d_n \equiv 2^{r_n}$.

5.2 Labels versus Ordering

Superscripts are used in quantized detector networks (QDN) to identify individual nodes. Therefore, the standard left-right ordering in Cartesian products such

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as (5.1) or the left-right ordering in tensor products such as (5.2) is redundant. Provided we retain superscripts, we can drop the $\times$ symbol in Cartesian products and the $\otimes$ symbol in tensor products. Moreover, we can dispense with the left-right ordering rule in such products. Henceforth, we adopt the convention that the Cartesian product $B^1 \times B^2$ can be represented unambiguously by either $B^1 B^2$ or $B^2 B^1$. Moreover, the ordered element $(\boldsymbol{i}^1, \boldsymbol{j}^2)$ of $B^1 \times B^2$ can unambiguously be represented by the notation $\boldsymbol{i}^1 \boldsymbol{j}^2$ or $\boldsymbol{j}^2 \boldsymbol{i}^1$. A similar convention will be applied to tensor products and to dual spaces.

Example 5.1 If $\boldsymbol{\Psi}^1$ is a vector in qubit Q^1 and $\boldsymbol{\Phi}^2$ is a vector in Q^2, then $\boldsymbol{\Psi}^1 \boldsymbol{\Phi}^2 = \boldsymbol{\Phi}^2 \boldsymbol{\Psi}^1 \equiv \boldsymbol{\Psi}^1 \otimes \boldsymbol{\Phi}^2$ is an element of the tensor product space $Q^1 Q^2 = Q^2 Q^1 \equiv Q^1 \otimes Q^2$.

Example 5.2 The rank-two classical register $\mathcal{R}^{[2]} \equiv B^1 B^2$ contains four classical states:

$$\mathcal{R}^{[2]} \equiv \{\mathbf{0}^1\mathbf{0}^2, \mathbf{1}^1\mathbf{0}^2, \mathbf{0}^1\mathbf{1}^2, \mathbf{1}^1\mathbf{1}^2\}. \tag{5.3}$$

These elements cannot be added together.

On the other hand, the corresponding rank-two quantum register $\mathcal{Q}^{[2]} \equiv Q^1 Q^2$ has a preferred basis $B^{[2]} \simeq \mathcal{R}^{[2]}$ corresponding to the above four classical states. Now, however, arbitrary linear combinations of the form $\alpha \boldsymbol{i}^1 \boldsymbol{j}^2 + \beta \boldsymbol{k}^1 \boldsymbol{l}^2$, for complex α, β and $i, j, k, l = 0, 1$, are allowed, giving new elements in $\mathcal{Q}^{[2]}$. The four elements $\mathbf{0}^1\mathbf{0}^2, \mathbf{1}^1\mathbf{0}^2, \mathbf{0}^1\mathbf{1}^2, \mathbf{1}^1\mathbf{1}^2$ form the preferred basis for $\mathcal{Q}^{[2]}$, while their duals $\overline{\mathbf{0}^1\mathbf{0}^2}, \overline{\mathbf{1}^1\mathbf{0}^2}, \overline{\mathbf{0}^1\mathbf{1}^2}, \overline{\mathbf{1}^1\mathbf{1}^2}$ form the preferred basis for the dual space $\overline{\mathcal{Q}^{[2]}}$.

5.3 The Signal Basis Representation

For any given rank, there are infinitely many more possible quantum states than possible classical states. Representing these quantum states efficiently therefore requires suitable notation. In this section, we introduce a representation that is based on observational context (what we can see). By this we mean it utilizes the specific details of the detectors involved in the experiments being discussed.

Throughout the rest of this chapter we discuss a collection of qubits seen at a single *stage*, or instant of the observer's time, so in this chapter we suppress any reference to time. However, time will not be overlooked in general. In the dynamical theory given in the next chapter, a temporal subscript n will be introduced that is associated with every labstate and with other quantities such as the rank of the quantum register. Our ultimate aim is to construct a general theory of observation in which apparatus itself becomes a dynamical quantity.

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Consider a rank-r quantum register $\mathcal{Q}^{[r]} \equiv Q^1 Q^2 \dots Q^r$. Observational context will inform us about the *preferred basis* $B^{[r]}$ for $\mathcal{Q}^{[r]}$. This consists of all possible *signal basis states* of the apparatus. These will represent all possible classical yes-no configurations of the detectors involved. Each element of this basis set is a tensor product of the form $\boldsymbol{i}^1\boldsymbol{i}^2\boldsymbol{i}^3\dots\boldsymbol{i}^r$, where the $\boldsymbol{i}^a$ are all either the $\mathbf{0}^a$ (ground state) or $\mathbf{1}^a$ (signal state) elements of the preferred basis B^a for the ath detector qubit Q^a. When written in this form, $B^{[r]}$ will be referred to as the *signal basis representation* (SBR).

By inspection, we see there are 2^r distinct elements in $B^{[r]}$ and together they constitute an orthonormal basis for the quantum register $\mathcal{Q}^{[r]}$. We can identify the elements in the classical register $\mathcal{R}^{[r]}$ with the elements of the preferred basis $B^{[r]}$ of the quantum register $\mathcal{Q}^{[r]}$.

Example 5.3 The SBR for a rank-three quantum register has $2^3 = 8$ elements and is given by

$$B^{[3]} \equiv \left\{\mathbf{0}^1\mathbf{0}^2\mathbf{0}^3, \mathbf{1}^1\mathbf{0}^2\mathbf{0}^3, \mathbf{0}^1\mathbf{1}^2\mathbf{0}^3, \mathbf{1}^1\mathbf{1}^2\mathbf{0}^3, \mathbf{0}^1\mathbf{0}^2\mathbf{1}^3, \mathbf{1}^1\mathbf{0}^2\mathbf{1}^3, \mathbf{0}^1\mathbf{1}^2\mathbf{1}^3, \mathbf{1}^1\mathbf{1}^2\mathbf{1}^3\right\}. \tag{5.4}$$

To define orthonormality, we introduce the dual basis $\overline{B^{[r]}}$, consisting of elements of the form $\overline{\boldsymbol{i}^1\boldsymbol{i}^2\boldsymbol{i}^3\dots\boldsymbol{i}^r}$, where $\overline{\boldsymbol{i}^a} = \overline{\mathbf{0}^a}$ or $\overline{\boldsymbol{i}^a} = \overline{\mathbf{1}^a}$ for $a = 1, 2, \dots, r$. Inner products in $\mathcal{Q}^{[r]}$ are defined by the action of elements of $\overline{B^{[r]}}$ on elements of $B^{[r]}$, plus linearity:

$$\begin{aligned}\overline{\boldsymbol{i}^1\boldsymbol{i}^2\boldsymbol{i}^3\dots\boldsymbol{i}^r}\,\boldsymbol{j}^1\boldsymbol{j}^2\boldsymbol{j}^3\dots\boldsymbol{j}^r &= (\overline{\boldsymbol{i}^1}\boldsymbol{j}^1)(\overline{\boldsymbol{i}^2}\boldsymbol{j}^2)(\overline{\boldsymbol{i}^3}\boldsymbol{j}^3)\dots(\overline{\boldsymbol{i}^r}\boldsymbol{j}^r)\\ &= \delta^{i^1 j^1}\delta^{i^2 j^2}\delta^{i^3 j^3}\dots\delta^{i^r j^r}.\end{aligned} \tag{5.5}$$

This rule is interpreted as follows. If the right-hand side of (5.5) is zero, then the elements $\boldsymbol{i}^1\boldsymbol{i}^2\boldsymbol{i}^3\dots\boldsymbol{i}^r$ and $\boldsymbol{j}^1\boldsymbol{j}^2\boldsymbol{j}^3\dots\boldsymbol{j}^r$ of the Hilbert space $\mathcal{Q}^{[r]}$ are orthogonal; otherwise, they are the same element and it has length (norm) of one.

Example 5.4 According to our notation, indices keep track of factor qubit spaces in a quantum register. Hence for a rank-four register, for example, we would find

$$\overline{\mathbf{0}^4\mathbf{1}^1\mathbf{1}^3\mathbf{0}^2}\mathbf{0}^3\mathbf{1}^1\mathbf{0}^2\mathbf{0}^4 = \underbrace{(\overline{\mathbf{1}^1}\mathbf{1}^1)}_{1}\underbrace{(\overline{\mathbf{0}^2}\mathbf{0}^2)}_{1}\underbrace{(\overline{\mathbf{1}^3}\mathbf{0}^3)}_{0}\underbrace{(\overline{\mathbf{0}^4}\mathbf{0}^4)}_{1} = 1\times 1\times 0\times 1 = 0. \tag{5.6}$$

5.4 Maximal Questions

Given an apparatus represented by r detectors at a given stage, the observer has the freedom to ask any one of a number of questions. For instance, the observer may look at a particular detector to ascertain its signal state and *not* look at any

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of the other detectors. Such a question that does not involve all of the detectors will be called a *partial question*. Partial questions are discussed in Chapter 8.

A *maximal question* is one that asks a binary question of every detector in a register. From (5.5), we see that the elements of the dual basis $\overline{B^{[r]}}$ can be interpreted as maximal questions, and there are 2^r of them. It is important to note that in the absence of any theory that quantizes observers and/or their questions, arbitrary vectors in $\overline{\mathcal{Q}^{[r]}}$ are not in general physically meaningful maximal questions; only the elements of the preferred dual basis $\overline{B^{[r]}}$ are maximal questions in the formulation of QDN being discussed at this point.

It is not impossible, however, to imagine a scenario where questions being asked correspond to elements of $\overline{\mathcal{Q}^{[r]}}$ that are nontrivial linear combinations of preferred basis elements. For instance, a given observer may be conducting an experiment where there are two possible quantum outcomes, such that each outcome would trigger a different maximal question of some state at a later stage. That would correspond to quantization of apparatus and/or observers. This sort of scenario is essentially the focus of Chapter 21.

Example 5.5 Given the rank-two quantum register $\mathcal{Q}^{[2]} \equiv \mathcal{Q}^1\mathcal{Q}^2$, the preferred basis $B^{[2]}$ has four elements:

$$B^{[2]} = \{\mathbf{0}^1\mathbf{0}^2, \mathbf{1}^1\mathbf{0}^2, \mathbf{0}^1\mathbf{1}^2, \mathbf{1}^1\mathbf{1}^2\}. \tag{5.7}$$

The maximal questions are all the elements of the dual basis

$$\overline{B^{[r]}} \equiv \{\overline{\mathbf{0}^1\mathbf{0}^2}, \overline{\mathbf{1}^1\mathbf{0}^2}, \overline{\mathbf{0}^1\mathbf{1}^2}, \overline{\mathbf{1}^1\mathbf{1}^2}\}. \tag{5.8}$$

5.5 Signality

The *signality* of a basis state corresponds to the total "particle number" being detected by the apparatus concerned, *if* any positive signal in any of the detectors is interpreted as registering a particle in the conventional sense.

Definition 5.6 The *signality* of an element $\boldsymbol{i}^1\boldsymbol{i}^2\boldsymbol{i}^3\cdots\boldsymbol{i}^a$ of the signal basis $B^{[r]}$ is the sum $\sum_{a=1}^{r} i^a$.

Example 5.7 The signality of the element $\mathbf{0}^6\mathbf{1}^2\mathbf{1}^1\mathbf{0}^5\mathbf{0}^3\mathbf{1}^4$ in $B^{[6]}$ is $1+1+0+1+0+0=3$.

The particle interpretation of signality has to be viewed cautiously for two reasons. First, detectors register irreversible processes that may have nothing to do with any identifiable, persistent "particle." For example, a *phonon* is a collective phenomenon, a quantum of excitation in a crystal that does not "exist"

in a reductionist sense. Second, it is possible in QDN to deal with labstates that are superpositions of preferred basis states of differing signality. Classically, this would have no particle interpretation but is part and parcel of QDN.

Dynamics may in many situations rule out certain superpositions of labstates in a quantum register. For example, charge conservation rules out superpositions of labstates corresponding to different numbers of electrons. Another example is angular momentum: we are not allowed in conventional quantum mechanics to superpose states of particles with different total spin number, conventionally denoted by j. However, the general principle remains: superposition of labstates of different signality is allowed in principle in QDN; only dynamical context rules out certain mathematical possibilities.

These comments apply to the labstates. As far as the dual basis is concerned, we are not at this stage entitled to create linear superpositions of *any* of the basis elements in the dual basis set $\overline{B^{[r]}}$, as these elements represent questions being asked by the observer. We have introduced no concept yet of a "quantized observer," whatever that might mean. Notwithstanding our comments just before Example 5.5, all questions asked at this stage of this book have to be completely classical. The point about QDN is that the labstates do not have this restriction and can be superpositions of the classical-looking elements of the preferred basis. This remark applies *before* the observer looks at a labstate.

5.6 The Economy of Success

A useful fact now emerges, a fact that underpins the success of CM (a moderately simple theory) in describing a hideously complex reality. Of all the 2^r states in a rank-r classical register, only one of them has a truth value of one relative to any given maximal question, and all the other $2^r - 1$ states have truth value of zero, relative to that question. To see this, consider an arbitrary maximal question $\bar{\boldsymbol{i}}^1\bar{\boldsymbol{i}}^2\ldots\bar{\boldsymbol{i}}^r$ asked of an arbitrary classical register state $\boldsymbol{j}^1\boldsymbol{j}^2\ldots\boldsymbol{j}^r$. The answer is

$$\bar{\boldsymbol{i}}^1\bar{\boldsymbol{i}}^2\ldots\bar{\boldsymbol{i}}^r\boldsymbol{j}^1\boldsymbol{j}^2\ldots\boldsymbol{j}^r = \underbrace{(\bar{\boldsymbol{i}}^1\boldsymbol{j}^1)}_{\delta^{i^1j^1}}\underbrace{(\bar{\boldsymbol{i}}^2\boldsymbol{j}^2)}_{\delta^{i^2j^2}}\ldots\underbrace{(\bar{\boldsymbol{i}}^r\boldsymbol{j}^r)}_{\delta^{i^rj^r}} = \delta^{i^1j^1}\delta^{i^2j^2}\ldots\delta^{i^rj^r}, \qquad (5.9)$$

which is zero in general except for the particular case when $i^a = j^a$ for each possible value of a, and then the answer value is one. This result greatly simplifies calculations in QDN.

This result contributes to the inherent economy of the particle description in CM, as the following example illustrates. Suppose we wanted to describe a single point particle in three-dimensional space. The conventional approach would be to define a Cartesian coordinate system with coordinates x, y, and z along three mutually orthogonal axes, relative to some chosen origin of coordinates. Then the position of a particle P would be fully specified by giving only three numbers, (x_P, y_P, z_P), referred to as the *position coordinates of the particle*. The economy

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of this approach is literally infinite: saying where the particle is immediately says where the particle is not, which is every point in space apart from the one with coordinates (x_P, y_P, z_P).

This economy of information comes about because of context: we have been told that there is only one particle.

But suppose now we are presented with a situation where we were not told beforehand how many particles there were in our laboratory. Then we would have to look at every point in space if we wanted to know the total particle number. This is not a trivial point but a hard fact in astrophysics and cosmology: astronomers have to scan space thoroughly in order to make estimates of matter and dark matter densities on cosmological scales.

In the quantum context, we may use signality to tell us how many signals there are in our detector array. A labstate of signality one corresponds closely to what can be thought of as a single particle state. This means that, if we knew beforehand that we were dealing with a signality-one state, then it would suffice to find a positive signal in just one detector in an array to be sure (by context) that there were no positive signals anywhere else in that array.

It is an intriguing thought that when we specify the classical Cartesian coordinates x, y, z of a point particle, we are not only answering the question **where is the particle?** but also saying that it is not at any of the other points in space, of which there is an uncountable number. Clearly, that is an impressive form of economy, but one occurring only because of our contextual information.

5.7 Quantum Registers

We now extend the discussion from quantum labstates of a single detector to quantum labstates of two or more detectors. These higher rank labstates are elements of some quantum register $\mathcal{Q}^{[r]} \equiv Q^1 Q^2 \dots Q^r$, where $r \geqslant 1$ is the rank of the register and Q^i is the ith qubit in the register. We have suppressed the tensor product symbol $\otimes$ here.

Tensor products of vector spaces are discussed in the Appendix. Unlike Cartesian products, tensor product spaces are genuine vector spaces, the quantum register $\mathcal{Q}^{[r]}$ being a complex vector space of dimension 2^r. This is one of the places where the fundamental differences between CM and QM manifest themselves. In the classical case, there is only a finite number of different possible states in a classical register. For a system of r classical bits, each of which has two possible mutually exclusive answer states, then the total number of integers 0 or 1 we would need in order to parametrize any classical state of the system is just r. Any state in such a classical register can therefore be represented by an r-tuple of the form $\{i^1, i^2, \dots, i^r\}$, where $i^a = 0$ or 1, for $a = 1, 2, \dots r$, and there are just 2^r different such r-tuples. In the quantum case, possible states are elements of a finite-dimensional complex vector space of dimension 2^r, which has infinitely many elements.

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Consider an arbitrary state $\mathbf{\Psi}$ in $\mathcal{Q}^{[2]}$. Given the above preferred basis, we can always write

$$\mathbf{\Psi} = z^1\mathbf{0}^1\mathbf{0}^2 + z^2\mathbf{1}^1\mathbf{0}^2 + z^3\mathbf{0}^1\mathbf{1}^2 + z^4\mathbf{1}^1\mathbf{1}^2, \tag{5.10}$$

where the z^a are labeled complex numbers, not powers of z. The dual state $\overline{\mathbf{\Psi}}$ is given by

$$\begin{aligned}\overline{\mathbf{\Psi}} &\equiv \overline{z^1\mathbf{0}^1\mathbf{0}^2 + z^2\mathbf{1}^1\mathbf{0}^2 + z^3\mathbf{0}^1\mathbf{1}^2 + z^4\mathbf{1}^1\mathbf{1}^2} \\ &= z^{1*}\overline{\mathbf{0}^1\mathbf{0}^2} + z^{2*}\overline{\mathbf{1}^1\mathbf{0}^2} + z^{3*}\overline{\mathbf{0}^1\mathbf{1}^2} + z^{4*}\overline{\mathbf{1}^1\mathbf{1}^2},\end{aligned} \tag{5.11}$$

where z^{a*} is the complex conjugate of z^a. If the state is normalized, we have the condition

$$\overline{\mathbf{\Psi}}\mathbf{\Psi} = |z^1|^2 + |z^2|^2 + |z^3|^2 + |z^4|^2 = 1. \tag{5.12}$$

If now we write $z^a = x^a + iy^a$, where x^a and y^a are real, then the normalization condition (5.12) is equivalent to saying that possible states rank-two quantum register can be identified one-to-one with points on S^7, the unit sphere in eight-dimensional Euclidean space $\mathbb{E}^8$.

The above analysis is for just *two* detectors. If we had in mind, say, a screen consisting of a million detectors, or a neural network in a brain, the numbers shoot up beyond imagination.[1] Clearly, qubit register states have much more structure than their classical counterparts. The exploration of this structure is in a real sense in its infancy at this time. For example, there is still a great deal to be understood about the physics of quantum entangled states in low rank quantum registers. We discuss some aspects of this in Chapter 22.

5.8 The Computational Basis Representation

The SBR $\{\boldsymbol{i}^1\boldsymbol{i}^2\boldsymbol{i}^3\ldots\boldsymbol{i}^r : i^a = 0, 1 : a = 1, 2, \ldots, r\}$ for elements of the preferred basis $B^{[r]}$ is useful in some respects but less so in others. A frequently more useful but equivalent notation for the elements of the preferred basis will be called the *computational basis representation* (CBR) and is obtained as follows. For each element $\boldsymbol{i}^1\boldsymbol{i}^2\boldsymbol{i}^3\cdots\boldsymbol{i}^r$ of the SBR, there is precisely one element $\boldsymbol{i}$ of the CBR, where i is an integer in the range $[0, 2^r - 1]$, given by the *computational basis map*

$$i = i^1 + 2i^2 + 4i^3 + \cdots + 2^{r-1}i^r, \tag{5.13}$$

where i^3 means the third element in the SBR and not i to the power of three.

The computational basis map can be inverted. Given an element $\boldsymbol{p}$ of the CBR, where p is an integer in the range $[0, 2^r - 1]$, we can write

$$p = p^{[1]} + 2p^{[2]} + 4p^{[3]} + \cdots + 2^{r-1}p^{[p]} = \frac{1}{2}\sum_{a=1}^{r} 2^a p^{[a]}, \tag{5.14}$$

[1] The average human brain has about a hundred billion neurons.

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where the symbol $p^{[a]}$ denotes the signal status (zero or one) of the ath detector when the apparatus is in the labstate $\boldsymbol{p}$. The coefficients $p^{[a]}$ will be referred to as the *binary components* of the integer p and are discussed in further detail later on in this chapter. They play an essential role in the formalism of QDN and in many calculations.

Example 5.8 For the rank-three quantum register $Q^1Q^2Q^3$, the SBR of the preferred basis $B^{[3]}$ is

$$B^{[3]} = \{\mathbf{0}^1\mathbf{0}^2\mathbf{0}^3, \mathbf{1}^1\mathbf{0}^2\mathbf{0}^3, \mathbf{0}^1\mathbf{1}^2\mathbf{0}^3, \mathbf{1}^1\mathbf{1}^2\mathbf{0}^3, \mathbf{0}^1\mathbf{0}^2\mathbf{1}^3, \mathbf{1}^1\mathbf{0}^2\mathbf{1}^3, \mathbf{0}^1\mathbf{1}^2\mathbf{1}^3, \mathbf{1}^1\mathbf{1}^2\mathbf{1}^3\}. \tag{5.15}$$

The CBR of $B^{[3]}$ is the set

$$B^{[3]} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}\}, \tag{5.16}$$

where $\mathbf{0} = \mathbf{0}^1\mathbf{0}^2\mathbf{0}^3$, $\mathbf{1} = \mathbf{1}^1\mathbf{0}^2\mathbf{0}^3$, and so on.

The CBR generally has the advantage over the SBR of being more compact and on that account is better suited in many but not all calculations. The dual preferred basis $\overline{B^{[r]}}$ can also be expressed in CBR terms, and then the inner product relations (5.5) take the form

$$\bar{\boldsymbol{i}}\boldsymbol{j} = \delta^{ij}, \quad 0 \leqslant i, j < 2^r, \tag{5.17}$$

which is very useful.

A disadvantage of the CBR is that it masks the signal properties of a given state. For example, the CBR element $\mathbf{3}$ could represent the SBR element $\mathbf{1}^1\mathbf{1}^2$ for a rank-two apparatus or the SBR element $\mathbf{1}^1\mathbf{1}^2\mathbf{0}^3\mathbf{0}^4\mathbf{0}^5$ of a rank-five apparatus and so on. However, context will generally make it clear what is meant by a given CBR expression.

The CBR is useful for representing linear operators over the register. These will generally be denoted in blackboard font in QDN. For example, the classical or quantum register identity operator $\mathbb{I}^{[r]}$ is expressed in the CBR by

$$\mathbb{I}^{[r]} = \sum_{k=0}^{2^r-1} \boldsymbol{k}\bar{\boldsymbol{k}}. \tag{5.18}$$

More generally, if we know the action of a linear register operator on each element of the signal basis, then we can represent that operator as a dyadic in the computation representation, as we show in the next section.

5.9 Register Operators

A *register operator* is a function that maps elements of a classical or quantum register into itself or another classical or quantum register. Linear register operators will be denoted in blackboard bold font, such as $\mathbb{U}$, $\mathbb{A}$, and so on. The same

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labeling convention used for states and questions will be used for linear operators: upper indices refer to qubits and, while lower labels indicate stages.

Suppose $\mathbb{O}_{n,m}$ is a linear register operator mapping states from an initial stage Σ_m rank-r_m classical or quantum register into a final stage Σ_n rank-r_n classical or quantum register. Suppose further that we are given its action on each element $\boldsymbol{i}_m$ of the CBR, that is,

$$\mathbb{O}_{n,m}\boldsymbol{i_m} = \sum_{j=0}^{2^{r_n}-1} O_{n,m}^{ji}\boldsymbol{j}_n, \quad i = 0, 1, \ldots, 2^{r_m} - 1, \tag{5.19}$$

where the $O_{n,m}^{ji}$ are complex coefficients.

Now "multiply" each side from the right by $\overline{\boldsymbol{i_m}}$ and sum over i:

$$\sum_{i=0}^{2^{r_m}-1} \mathbb{O}_{n,m}\boldsymbol{i}_m\overline{\boldsymbol{i}_m} = \sum_{i=0}^{2^{r_m}-1}\sum_{j=0}^{2^{r_n}-1} O_{n,m}^{ji}\boldsymbol{j}_n\overline{\boldsymbol{i}_m} \tag{5.20}$$

The left-hand side simplifies because of linearity:

$$\sum_{i=0}^{2^{r_m}-1} \mathbb{O}_{n,m}\boldsymbol{i}_m\overline{\boldsymbol{i}_m} = \mathbb{O}_{n,m}\left\{\sum_{i=0}^{2^{r_m}-1} \boldsymbol{i}_m\overline{\boldsymbol{i}_m}\right\} = \mathbb{O}_{n,m}\mathbb{I}_m = \mathbb{O}_{n,m}, \tag{5.21}$$

using (5.18). Hence finally we arrive at the dyadic representation of a typical register operator:

$$\mathbb{O}_{n,m} = \sum_{i=0}^{2^{r_m}-1}\sum_{j=0}^{2^{r_n}-1} \boldsymbol{j}_n O_{n,m}^{ji}\overline{\boldsymbol{i}_m}. \tag{5.22}$$

We define the *retraction* $\overline{\mathbb{O}}_{n,m}$ of a register operator $\mathbb{O}_{n,m}$ by

$$\overline{\mathbb{O}}_{n,m} \equiv \sum_{i=0}^{2^{r_m}-1}\sum_{j=0}^{2^{r_n}-1} \boldsymbol{i}_m O_{n,m}^{ji*}\overline{\boldsymbol{j}_n}, \tag{5.23}$$

where $O_{n,m}^{ij*}$ is the complex conjugate of $O_{n,m}^{ij}$. A retraction as defined here is not necessarily the equivalent of an *inverse* operator.

5.10 Classical Register Operators

We saw in Chapter 3 that there are only four classical bit operators $\boldsymbol{C}^1 \equiv \boldsymbol{I}$, $\boldsymbol{C}^2 \equiv \boldsymbol{F}$, $\boldsymbol{C}^3 \equiv \boldsymbol{D}$, and $\boldsymbol{C}^4 \equiv \boldsymbol{U}$ that map bit states to bit states. Likewise, a classical register operator maps classical register states to classical register states. We shall discuss a classical register operator $\mathbb{C}_{n,m}$ mapping register states from a rank-r_m classical register $\mathcal{R}_m$ to a rank-r_n classical register $\mathcal{R}_n$.

As before, the most economical way to discuss classical register operators is via the CBR, as follows. For any element $\boldsymbol{i}_m$ in $\mathcal{R}_m$, where $i = 0, 1, 2, \ldots, 2^{r_m} - 1$, the operator $\mathbb{C}_{n,m}$ necessarily maps it into precisely *one* element in $\mathcal{R}_n$, and not into a linear combination of two or more, as in the quantum case. This means that we can always write

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$$\mathbb{C}_{n,m}\boldsymbol{i}_m = \sum_{j=0}^{2^{r_n}-1} C_{n,m}^{ji}\boldsymbol{j}_n, \quad i = 0, 1, 2, \ldots, 2^{r_m} - 1, \tag{5.24}$$

where for a given i, every element $C_{n,m}^{ji}$ is zero for every integer j in the interval $[0, 2^{r_n} - 1]$ except for one, which has value one. Using the completeness property of the elements $\{\boldsymbol{i}_m\}$ we arrive at the dyadic representation

$$\mathbb{C}_{n,m} = \sum_{j=0}^{2^{r_n}-1}\sum_{i=0}^{2^{r_m}-1} \boldsymbol{j}_n C_{n,m}^{ji}\overline{\boldsymbol{i}_m}. \tag{5.25}$$

This is a special case of (5.22) and there is a total of $(2^{r_n})^{2^{r_m}}$ different possible such operators.

In this set of operators, there are two subsets that are of particular importance in both classical register mechanics and quantum register mechanics. These are the *register projection operators* and the *register signal operators*.

Register Projection Operators

Given a rank-r register, the observer may be interested in asking questions of an individual bit or qubit and leaving all the other bits or qubits alone. The register projection operators $\mathbb{P}^i$ and $\widehat{\mathbb{P}}^i$ will be used in later chapters to construct partial questions. These operators are defined by

$$\begin{aligned}\mathbb{P}^i &= \boldsymbol{I}^1\boldsymbol{I}^2\ldots\boldsymbol{I}^{i-1}\boldsymbol{P}^i\boldsymbol{I}^{i+1}\ldots\boldsymbol{I}^r,\\ \widehat{\mathbb{P}}^i &= \boldsymbol{I}^1\boldsymbol{I}^2\ldots\boldsymbol{I}^{i-1}\widehat{\boldsymbol{P}}^i\boldsymbol{I}^{i+1}\ldots\boldsymbol{I}^r,\end{aligned}$$

where $\boldsymbol{I}^a$ is the bit identity operator for the ath bit or qubit and $\boldsymbol{P}^i$, $\widehat{\boldsymbol{P}}^i$ are the bit projection operators for the ith bit or qubit, as discussed in the previous chapter.

Register Signal Operators

Associated with a rank-r quantum register $\mathcal{Q}^{[r]}$ are some important operators connected to the physics of observation, and these will appear frequently throughout the formalism. The most important of these are the r signal annihilation operators $\mathbb{A}^i$, $i = 1, 2, \ldots, r$ and the related signal creation operators $\widehat{\mathbb{A}}^i$. These operators are defined in terms of the signal bit operators discussed in the previous chapter, as follows:

$$\begin{aligned}\mathbb{A}^i &\equiv \boldsymbol{I}^1\boldsymbol{I}^2\ldots\boldsymbol{I}^{i-1}\boldsymbol{A}^i\boldsymbol{I}^{i+1}\ldots\boldsymbol{I}^r,\\ \widehat{\mathbb{A}}^i &\equiv \boldsymbol{I}^1\boldsymbol{I}^2\ldots\boldsymbol{I}^{i-1}\widehat{\boldsymbol{A}}^i\boldsymbol{I}^{i+1}\ldots\boldsymbol{I}^r, \quad i = 1, 2, \ldots, r,\end{aligned} \tag{5.26}$$

where the superscripts on the right-hand side label the individual qubits in a given rank-r register and we suppress the tensor product symbol $\otimes$.

5.11 The Signal Algebra

We define S^a, the ath *signal set*, to be the set of register operators

$$S^a \equiv \{\mathbb{P}^a, \widehat{\mathbb{P}}^a, \mathbb{A}^a, \widehat{\mathbb{A}}^a\}, \quad a = 1, 2, \ldots, r. \tag{5.27}$$

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Table 5.1 *The signal set algebra*

	$\mathbb{P}^a$	$\widehat{\mathbb{P}}^a$	$\mathbb{A}^a$	$\widehat{\mathbb{A}}^a$
$\mathbb{P}^a$	$\mathbb{P}^a$	0	$\mathbb{A}^a$	0
$\widehat{\mathbb{P}}^a$	0	$\widehat{\mathbb{P}}^a$	0	$\widehat{\mathbb{A}}^a$
$\mathbb{A}^a$	0	$\mathbb{A}^a$	0	$\mathbb{P}^a$
$\widehat{\mathbb{A}}^a$	$\widehat{\mathbb{A}}^a$	0	$\widehat{\mathbb{P}}^a$	0

Then an extraordinarily useful fact is that any two elements from different signal sets commute.

For a given signal set S^a, we have the ath *signal set algebra*, as shown in Table 5.1. For $i = 1, 2, \ldots, r$ we have

$$\mathbb{A}^i\mathbb{A}^i = \widehat{\mathbb{A}}^i\widehat{\mathbb{A}}^i = 0, \qquad \left\{\mathbb{A}^i, \widehat{\mathbb{A}}^i\right\} = \mathbb{P}^i + \widehat{\mathbb{P}}^i = \mathbb{I}^{[r]}, \tag{5.28}$$

where curly brackets denote an *anticommutator*, while for $i \neq j$, we have

$$[\mathbb{A}^i, \mathbb{A}^j] = [\widehat{\mathbb{A}}^i, \widehat{\mathbb{A}}^j] = [\mathbb{A}^i, \widehat{\mathbb{A}}^j] = 0, \tag{5.29}$$

where square brackets denote a *commutator*. Note that in the above, the symbol 0 represents the *zero operator* for the register concerned.

The signal set algebra gives QDN a particular flavor; parts of it are reminiscent of a theory with fermions (anticommuting objects), while other parts have a bosonic (commuting) character. At the signal level, however, we are dealing with neither concept specifically; the signal algebra is determined by the physics of observation as it relates to the apparatus and has its own logic that is distinct from that of conventional particle physics.

5.12 Signal Excitations

The signal operators introduced above are used to construct signal states from the signal ground state $\mathbf{0}$ as follows.

1. The *signality one states* are of the form $\widehat{\mathbb{A}}^i\mathbf{0} \equiv \mathbf{2}^{i-1}, 1 \leq i \leq r$.
2. The *signality two states* are of the form $\widehat{\mathbb{A}}^i\widehat{\mathbb{A}}^j\mathbf{0} \equiv \underline{\mathbf{2}^{i-1} + \mathbf{2}^{j-1}}, 1 \leq i < j \leq r$.
3. The *signality k states* are of the form

$$\widehat{\mathbb{A}}^{i_1}\widehat{\mathbb{A}}^{i_2}\ldots\widehat{\mathbb{A}}^{i_k}\mathbf{0} \equiv \underline{\sum_{j=1}^{k} \mathbf{2}^{i_j-1}}, \tag{5.30}$$

for $1 \leq i_1 < i_2 < \cdots < i_k \leq r$.

Remark 5.9 In the above, we underline expressions such as $\underline{\mathbf{2}^{i-1} + \mathbf{2}^{j-1}}$ to indicate that this is *not* the vector sum of $\mathbf{2}^{i-1}$ and $\mathbf{2}^{j-1}$ but the vector in the CBR corresponding to the sum of the integers 2^{i-1} and 2^{j-1}.

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The same approach can be used to discuss maximal questions. Recall that these are represented by the dual preferred basis elements of the form $\overline{\boldsymbol{i}}$, $0 \leq i < 2^r$. Then we can write

$$\overline{\mathbf{0}}\mathbb{A}^{i_1}\mathbb{A}^{i_2}\ldots\mathbb{A}^{i_k} \equiv \overline{\sum_{j=1}^{k} \mathbf{2}^{i_j-1}}. \tag{5.31}$$

Exercise 5.10 Prove that

$$\mathbb{A}^i\mathbf{0} = 0, \quad \overline{\mathbf{0}}\widehat{\mathbb{A}}^i = 0, \quad 1 \leq i \leq r. \tag{5.32}$$

Exercise 5.11 Use the signal algebra to show that

$$\overline{\boldsymbol{i}}\boldsymbol{j} = \delta^{ij}, \quad 0 \leq i, j, < 2^r. \tag{5.33}$$

5.13 Signality Classes

For each rank r quantum register $\mathcal{Q}^{[r]}$, we may write its preferred basis $B^{[r]}$ in the form

$$B^{[r]} = \{\mathbf{0}, \widehat{\mathbb{A}}^1\mathbf{0}, \widehat{\mathbb{A}}^2\mathbf{0}, \ldots, \widehat{\mathbb{A}}^r\mathbf{0}, \widehat{\mathbb{A}}^1\widehat{\mathbb{A}}^2\mathbf{0}, \ldots, \widehat{\mathbb{A}}^1\widehat{\mathbb{A}}^2\ldots\widehat{\mathbb{A}}^r\mathbf{0}\}. \tag{5.34}$$

For each element, the *signality* of that element is the number of signal creation operators used to create it from the signal ground state.

Example 5.12 For a rank-11 quantum register, the preferred basis state given by **851** in the CBR is given by $\mathbf{1}^1\mathbf{1}^2\mathbf{0}^3\mathbf{0}^4\mathbf{1}^5\mathbf{0}^6\mathbf{1}^7\mathbf{0}^8\mathbf{1}^9\mathbf{1}^{10}\mathbf{0}^{11}$ in the SBR, from which we can read off its signality as 6. Equivalently, we can see that this state is given by $\widehat{\mathbb{A}}^1\widehat{\mathbb{A}}^2\widehat{\mathbb{A}}^5\widehat{\mathbb{A}}^7\widehat{\mathbb{A}}^9\widehat{\mathbb{A}}^{10}\mathbf{0}$, which clearly has signality 6.

Such examples soon show the enormous advantage of using the CBR for large rank registers. The CBR is particularly suited for computer algebra calculations.

By inspection, it is easy to see that the preferred basis $B^{[r]}$ separates into $r+1$ disjoint *signality classes*:

$$\begin{aligned}
&\text{signality zero} && B^{[r,0]} \equiv \{\mathbf{0}\},\\
&\text{signality one} && B^{[r,1]} \equiv \{\widehat{\mathbb{A}}^a\mathbf{0} : a = 1, 2, \ldots, r\},\\
&\text{signality two} && B^{[r,2]} \equiv \{\widehat{\mathbb{A}}^a\widehat{\mathbb{A}}^b\mathbf{0} : 1 \leqslant a < b \leqslant r\},\\
&&&\vdots \quad\quad \vdots\\
&\text{signality } r && B^{[r,r]} \equiv \{\widehat{\mathbb{A}}^1\widehat{\mathbb{A}}^2\ldots\widehat{\mathbb{A}}^r\mathbf{0}\}.
\end{aligned} \tag{5.35}$$

Then we may write the preferred basis as the union of all of these signality classes, that is,

$$B^{[r]} = \bigcup_{a=0}^{r} B^{[r,a]}. \tag{5.36}$$

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We define the *signality count* $\sigma^{[r,a]}$ as the number of elements in $B^{[r,a]}$. Then

$$\sigma^{[r,a]} = \frac{r!}{a!(r-a)!} = \binom{r}{a} \tag{5.37}$$

Hence the *total signality* $\sigma^{[r]}$ is given by

$$\sigma^{[r]} \equiv \sum_{i=0}^{r} \sigma^{[r,a]} = \sum_{a=0}^{r} \binom{r}{a} = 2^a, \tag{5.38}$$

as expected.

A significant feature of QDN is that the addition of register states of different signality is permitted, unless the dynamics specifically rules it out. This will happen, for example, whenever the signals are interpreted as electrically charged particles. Under such circumstances, we expect conservation of total signality.

5.14 Binary Decomposition

Given a nonnegative integer k, the *binary decomposition* of k is the expansion

$$k = \sum_{a=1}^{\infty} k^{[a]} 2^{a-1}, \quad k = 0, 1, 2, \ldots, \tag{5.39}$$

where each of the binary coefficients $k^{[a]}$ is either zero or one. Table 5.2 shows the binary decomposition up to $k = 9$.

From the table, we read off for example that

$$6 = 0 \times 2^{1-1} + 1 \times 2^{2-1} + 1 \times 2^{3-1} + 0 \times 2^{4-1} + 0 \times \cdots = 2 + 4 = 6.$$

For each k, the maximum a for which $k^{[a]} = 1$ is the *minimum rank* of k, denoted $\mu(k)$ and indicated by a box in Table 5.2. The minimum rank of k is the rank of the "smallest" quantum register that contains the basis state $\boldsymbol{k}$.

Table 5.2 *The binary decomposition of the integers up to nine*

$a \rightarrow$		1	2	3	4	...
$2^{a-1} \rightarrow$		1	2	4	8	...
$k \downarrow$	0	0	0	0	0	...
	1	$\boxed{1}$	0	0	0	...
	2	0	$\boxed{1}$	0	0	...
	3	1	$\boxed{1}$	0	0	...
	4	0	0	$\boxed{1}$	0	...
	5	1	0	$\boxed{1}$	0	...
	6	0	1	$\boxed{1}$	0	...
	7	1	1	$\boxed{1}$	0	...
	8	0	0	0	$\boxed{1}$	...
	9	1	0	0	$\boxed{1}$	...

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Example 5.13 From Table 5.2 we read off

$$\begin{aligned}\mu(1) &= 1,\\ \mu(2) = \mu(3) &= 2,\\ \mu(4) = \mu(5) = \mu(6) = \mu(7) &= 3.\end{aligned} \tag{5.40}$$

We see $\mu(9) = 4$ and $9^{[1]} = 1$, $9^{[2]} = 0$, $9^{[3]} = 0$, $9^{[4]} = 1$, so

$$9 = 1 \times 2^{1-1} + 0 \times 2^1 + 0 \times 2^{3-1} + 1 \times 2^{4-1} = 1 + 8 = 9. \tag{5.41}$$

5.15 Computational Basis Representation for Signal Operators

For a given rank-r register $\mathcal{Q}^{[r]}$, the preferred basis elements $\{\boldsymbol{k} : k = 0, 1, 2, \ldots, 2^r - 1\}$ are given in the CBR by

$$\boldsymbol{k} = \underline{\sum_{a=1}^{\mu(k)} k^{[a]} \mathbf{2}^{a-1}}, \quad k = 0, 1, 2, \ldots, 2^r - 1, \tag{5.42}$$

where $k^{[a]} = 0$ or 1 is the ath *binary component* of the integer k; i.e., we have

$$k = k^{[1]}2^{1-1} + k^{[2]}2^{2-1} + k^{[3]}2^{3-1} + \cdots + 2^{\mu(k)-1}. \tag{5.43}$$

Hence we may equate the CBR and SBR elements in the basis $B^{[\mu_k]}$

$$\boldsymbol{k} = \boldsymbol{k}^{[1]}\boldsymbol{k}^{[2]} \ldots \mathbf{1}^{\mu_k} = \left(\widehat{\mathbb{A}}^1\right)^{k_{[1]}} \left(\widehat{\mathbb{A}}^2\right)^{k_{[2]}} \ldots \left(\widehat{\mathbb{A}}^{\mu_k}\right) \mathbf{0}. \tag{5.44}$$

Now use the results

$$\begin{aligned}\widehat{\mathbb{A}}^a \boldsymbol{k} &= 0 && \text{if } k^{[a]} = 1,\\ &= \underline{\boldsymbol{k} + \mathbf{2}^{a-1}} && \text{if } k^{[a]} = 0.\end{aligned} \tag{5.45}$$

Defining

$$\widehat{k}^{[a]} \equiv 1 - k^{[a]}, \tag{5.46}$$

then we can readily show

$$\begin{aligned}\widehat{\mathbb{A}}^a \boldsymbol{k} &= \widehat{k}^{[a]} \underline{\boldsymbol{k} + \mathbf{2}^{a-1}}, & \overline{\boldsymbol{k}}\widehat{\mathbb{A}}^a &= k^{[a]} \overline{\boldsymbol{k} - \mathbf{2}^{a-1}}\\ \mathbb{A}^a \boldsymbol{k} &= k^{[a]} \underline{\boldsymbol{k} - \mathbf{2}^{a-1}}, & \overline{\boldsymbol{k}}\mathbb{A}^a &= \widehat{k}^{[a]} \overline{\boldsymbol{k} + \mathbf{2}^{a-1}}.\end{aligned} \tag{5.47}$$

Hence we must have

$$\widehat{\mathbb{A}}^a = \sum_{k=0}^{2^r-1} \widehat{k}^{[a]} \underline{\boldsymbol{k} + \mathbf{2}^{a-1}} \overline{\boldsymbol{k}} = \sum_{k=0}^{2^r-1} k^{[a]} \boldsymbol{k} \overline{\boldsymbol{k} - \mathbf{2}^{a-1}}, \tag{5.48}$$

$$\mathbb{A}^a = \sum_{k=0}^{2^r-1} k^{[a]} \underline{\boldsymbol{k} - \mathbf{2}^{a-1}} \overline{\boldsymbol{k}} = \sum_{k=0}^{2^r-1} \widehat{k}^{[a]} \boldsymbol{k} \overline{\boldsymbol{k} + \mathbf{2}^{a-1}}, \tag{5.49}$$

which leads to

$$\mathbb{P}^a = \sum_{k=0}^{2^r-1} \widehat{k}^{[a]} \boldsymbol{k}\overline{\boldsymbol{k}}, \qquad \widehat{\mathbb{P}}^a = \sum_{k=0}^{2^r-1} k^{[a]} \boldsymbol{k}\overline{\boldsymbol{k}}. \tag{5.50}$$

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These results are consistent with

$$\mathbb{P}^a + \widehat{\mathbb{P}}^a = \sum_{k=0}^{2^r-1} (k_{[a]} + \widehat{k}_{[a]}) \boldsymbol{k}\overline{\boldsymbol{k}} = \sum_{k=0}^{2^r-1} \boldsymbol{k}\overline{\boldsymbol{k}} = \mathbb{I}^{[r]}, \qquad a = 1, 2, \ldots, r. \tag{5.51}$$

Example 5.14 Using the above expansions, we find

Rank one

$$\begin{aligned} \mathbb{P} &= \mathbf{0}\overline{\mathbf{0}}, & \widehat{\mathbb{P}} &= \mathbf{1}\overline{\mathbf{1}} \\ \mathbb{A} &= \mathbf{0}\overline{\mathbf{1}}, & \widehat{\mathbb{A}} &= \mathbf{1}\overline{\mathbf{0}}. \end{aligned} \tag{5.52}$$

Rank two

$$\begin{aligned} \mathbb{P}^1 &= \mathbf{0}\overline{\mathbf{0}} + \mathbf{2}\overline{\mathbf{2}}, & \widehat{\mathbb{P}}^1 &= \mathbf{1}\overline{\mathbf{1}} + \mathbf{3}\overline{\mathbf{3}}, \\ \mathbb{P}^2 &= \mathbf{0}\overline{\mathbf{0}} + \mathbf{1}\overline{\mathbf{1}}, & \widehat{\mathbb{P}}^2 &= \mathbf{2}\overline{\mathbf{2}} + \mathbf{3}\overline{\mathbf{3}}, \\ \mathbb{A}^1 &= \mathbf{0}\overline{\mathbf{1}} + \mathbf{2}\overline{\mathbf{3}}, & \widehat{\mathbb{A}}^1 &= \mathbf{1}\overline{\mathbf{0}} + \mathbf{3}\overline{\mathbf{2}}, \\ \mathbb{A}^2 &= \mathbf{0}\overline{\mathbf{2}} + \mathbf{1}\overline{\mathbf{3}}, & \widehat{\mathbb{A}}^2 &= \mathbf{2}\overline{\mathbf{0}} + \mathbf{3}\overline{\mathbf{1}}. \end{aligned} \tag{5.53}$$

Rank three

$$\begin{aligned} \mathbb{P}^1 &= \mathbf{0}\overline{\mathbf{0}} + \mathbf{2}\overline{\mathbf{2}} + \mathbf{4}\overline{\mathbf{4}} + \mathbf{6}\overline{\mathbf{6}}, & \widehat{\mathbb{P}}^1 &= \mathbf{1}\overline{\mathbf{1}} + \mathbf{3}\overline{\mathbf{3}} + \mathbf{5}\overline{\mathbf{5}} + \mathbf{7}\overline{\mathbf{7}}, \\ \mathbb{P}^2 &= \mathbf{0}\overline{\mathbf{0}} + \mathbf{1}\overline{\mathbf{1}} + \mathbf{4}\overline{\mathbf{4}} + \mathbf{5}\overline{\mathbf{5}}, & \widehat{\mathbb{P}}^2 &= \mathbf{2}\overline{\mathbf{2}} + \mathbf{3}\overline{\mathbf{3}} + \mathbf{6}\overline{\mathbf{6}} + \mathbf{7}\overline{\mathbf{7}}, \\ \mathbb{P}^3 &= \mathbf{0}\overline{\mathbf{0}} + \mathbf{1}\overline{\mathbf{1}} + \mathbf{2}\overline{\mathbf{2}} + \mathbf{3}\overline{\mathbf{3}}, & \widehat{\mathbb{P}}^3 &= \mathbf{4}\overline{\mathbf{4}} + \mathbf{5}\overline{\mathbf{5}} + \mathbf{6}\overline{\mathbf{6}} + \mathbf{7}\overline{\mathbf{7}}, \\ \mathbb{A}^1 &= \mathbf{0}\overline{\mathbf{1}} + \mathbf{2}\overline{\mathbf{3}} + \mathbf{4}\overline{\mathbf{5}} + \mathbf{6}\overline{\mathbf{7}}, & \widehat{\mathbb{A}}^1 &= \mathbf{1}\overline{\mathbf{0}} + \mathbf{3}\overline{\mathbf{2}} + \mathbf{5}\overline{\mathbf{4}} + \mathbf{7}\overline{\mathbf{6}}, \\ \mathbb{A}^2 &= \mathbf{0}\overline{\mathbf{2}} + \mathbf{1}\overline{\mathbf{3}} + \mathbf{4}\overline{\mathbf{6}} + \mathbf{5}\overline{\mathbf{7}}, & \widehat{\mathbb{A}}^2 &= \mathbf{2}\overline{\mathbf{0}} + \mathbf{3}\overline{\mathbf{1}} + \mathbf{6}\overline{\mathbf{4}} + \mathbf{7}\overline{\mathbf{5}}, \\ \mathbb{A}^3 &= \mathbf{0}\overline{\mathbf{4}} + \mathbf{1}\overline{\mathbf{5}} + \mathbf{2}\overline{\mathbf{6}} + \mathbf{3}\overline{\mathbf{7}}, & \widehat{\mathbb{A}}^3 &= \mathbf{4}\overline{\mathbf{0}} + \mathbf{5}\overline{\mathbf{1}} + \mathbf{6}\overline{\mathbf{2}} + \mathbf{7}\overline{\mathbf{3}}. \end{aligned} \tag{5.54}$$

In general, the CBR for any signal operator in a rank-r register consists of a sum of 2^{r-1} transition operators, all of which annihilate each other, including themselves. Likewise, a product $\widehat{\mathbb{A}}^i\widehat{\mathbb{A}}^j$ of two different signal operators can be expressed as a sum of 2^{r-2} transition operators that mutually annihilate, and so on. This process of representation can be continued until we arrive at the *saturation operator* $\widehat{\mathbb{A}}^1\widehat{\mathbb{A}}^2\ldots\widehat{\mathbb{A}}^r$, which when applied to the signal ground state creates the antithesis of the ground state, the *fully saturated signal state* $\underline{\mathbf{2^r - 1}} \equiv \mathbf{1}^1\mathbf{1}^2\ldots\mathbf{1}^r$.

A particularly useful expression for the signal creation operators is obtained by writing any of them in the form

$$\widehat{\mathbb{A}}^i = \mathbf{2}^{i-1}\overline{\mathbf{0}} + \mathbb{X}^i, \tag{5.55}$$

where the operator $\mathbb{X}^i \equiv \sum_{k=1}^{2^r-1} \widehat{k}^{[i]}\underline{\boldsymbol{k} + \mathbf{2}^{i-1}\overline{\boldsymbol{k}}}$ annihilates the signal ground state. This expression can be used to greatly simplify calculations for those experiments involving signality-one labstates, as in single-photon quantum optics experiments.

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