9

Spacetime transformations

An important class of symmetries is that which refers to the geometrical disposition of a system. This includes translational invariance, rotational invariance and boosts. Historically, covariant methods were inspired by the fact that the speed of light in a vacuum is constant for all inertial observers. This follows from Maxwell's equations, and it led Einstein to the special theory of relativity and covariance. The importance of covariance has since been applied to many different areas in theoretical physics.

To discuss coordinate transformations we shall refer to figure 9.1, which shows two coordinate systems moving with a relative velocity $\mathbf{v} = \beta c$. The constancy of the speed of light in any inertial frame tells us that the line element (and the corresponding proper time) must be invariant for all inertial observers. For a real constant Ω, this implies that

$$\mathrm{d}s^2 = \Omega^2 \mathrm{d}s'^2 = \Omega^2(-c^2\mathrm{d}t^2 + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}). \tag{9.1}$$

This should not be confused with the non-constancy of the effective speed of light in a material medium; our argument here concerns the vacuum only. This property expresses the constancy, or x-independence, of c. The factor Ω^2 is of little interest here as long as it is constant: one may always re-scale the coordinates to absorb it. Normally one is not interested in re-scaling measuring rods when comparing coordinate systems, since it only make systems harder to compare. However, we shall return to this point in section 9.7.

For particles which travel at the speed of light (massless particles), one has $\mathrm{d}s^2 = 0$ always, or

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = c. \tag{9.2}$$

Now, since $\mathrm{d}s^2 = 0$, it is clearly true that $\Omega^2(x)\,\mathrm{d}s^2 = 0$, for any non-singular, non-zero function $\Omega(x)$. Thus the value of c is preserved by a group of

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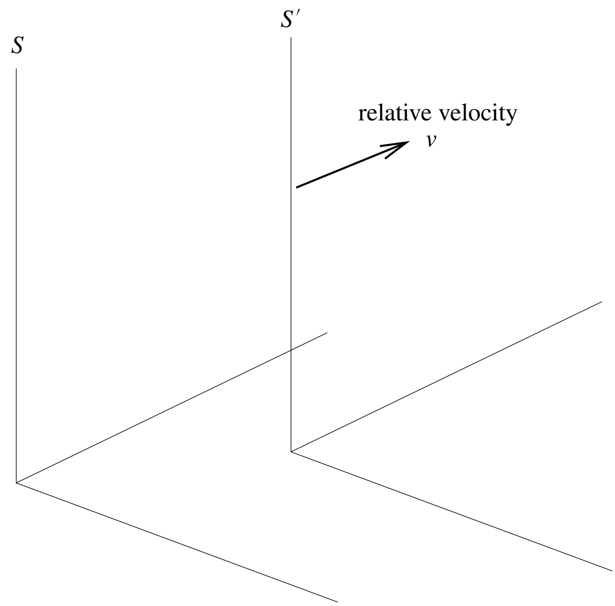

Fig. 9.1. The schematic arrangement for discussing coordinate transformations. Coordinate systems $S(x)$ and $S'(x')$ are in relative motion, with speed $\mathbf{v} = \beta c$.

transformations which obey

$$\mathrm{d}s'^2 = \Omega^2(x)\mathrm{d}s^2. \tag{9.3}$$

This set of transformations forms a group called the *conformal* group.

If all particles moved at the speed of light, we would identify this group as being the fundamental symmetry group for spacetime. However, for particles not moving at c, the line element is non-zero and may be characterized by

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \beta c, \tag{9.4}$$

for some constant $\beta = v/c$. Since we know that, in any frame, a free particle moves in a straight line at constant velocity, we know that β must be a constant and thus

$$\mathrm{d}s'^2 = \mathrm{d}s^2 \neq 0. \tag{9.5}$$

If it were possible for an x-dependence to creep in, then one could transform an inertial frame into a non-inertial frame. The group of transformations which preserve the line element in this way is called the inhomogeneous Lorentz group, or Poincaré group.

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In the non-relativistic limit, coordinate invariances are described by the so-called Galilean group. This group is no smaller than the Lorentz group, but space and time are decoupled, and the speed of light does not play a role at all. The non-relativistic limit assumes that $c \to \infty$. Galilean transformations lie closer to our intuition, but they are often more cumbersome since space and time must often be handled separately.

9.1 Parity and time reversal

In an odd number of spatial dimensions ($n = 2l+1$), a parity, or space-reflection transformation $\mathcal{P}$ has the following non-zero tensor components:

$$
\begin{aligned}
\mathcal{P}^0{}_0 &= 1 \\
\mathcal{P}^i{}_i &= -1,
\end{aligned}
\tag{9.6}
$$

where i is not summed in the last line. When this transformation acts on another tensor object, it effects a change of sign on all space components. In other words, each spatial coordinate undergoes $x^i \to -x^i$. The transformation $A \to -A$ is the discrete group $Z_2 = \{1, -1\}$.

In an even number of spatial dimensions ($n = 2l$), this construction does not act as a reflection, since the combination of an even number of reflections is not a reflection at all. In group language, $(Z_2)^{2n} = \{1\}$. It is easy to check that, in two spatial dimensions, reflection in the x_1 axis followed by reflection in the x_2 axis is equivalent to a continuous rotation. To make a true reflection operator in an even number of space dimensions, one of the spatial indices must be left out. For example,

$$
\begin{aligned}
\mathcal{P}^0{}_0 &= 1 \\
\mathcal{P}^i{}_i &= -1 \quad (i = 1, \ldots, n-1) \\
\mathcal{P}^i{}_i &= +1 \quad (i = n).
\end{aligned}
\tag{9.7}
$$

The time reversal transformation in any number of dimensions performs the analogous function for time coordinates:

$$
\begin{aligned}
\mathcal{T}^0{}_0 &= -1 \\
\mathcal{T}^i{}_i &= 1.
\end{aligned}
\tag{9.8}
$$

These transformations belong to the Lorentz group (and others), and are sometimes referred to as large Lorentz transformations since they cannot be formed by integration or repeated combination of infinitesimal transformations.

9.2 Translational invariance

A general translation in space, or in time, is a coordinate shift. A scalar field transforms simply:

$$\phi(x) \to \phi(x + \Delta x). \tag{9.9}$$

The direction of the shift may be specified explicitly, by

$$\begin{aligned} \phi(t, x^i) &\to \phi(t, x^i + \Delta x^i) \\ \phi(t, x^i) &\to \phi(t + \Delta t, x^i). \end{aligned} \tag{9.10}$$

Invariance under such a *constant* shift of a coordinate is almost always a prerequisite in physical problems found in textbooks. Translational invariance is easily characterized by the coordinate dependence of Green functions. Since the Green function is a two-point function, one can write it as a function of x and x' or in terms of variables rotated by 45 degrees, $\frac{1}{\sqrt{2}}(x - x')$ and $\frac{1}{\sqrt{2}}(x + x')$. These are more conveniently defined in terms of a difference and an average (mid-point) position:

$$\begin{aligned} \tilde{x} &= (x - x') \\ \overline{x} &= \frac{1}{2}(x + x'). \end{aligned} \tag{9.11}$$

The first of these is invariant under coordinate translations, since

$$x - x' = (x + a) - (x' + a). \tag{9.12}$$

The second equation is not, however. Thus, in a theory exhibiting translational invariance, the two-point function must depend only on $\tilde{x} = x - x'$.

9.2.1 Group representations on coordinate space

Translations are usually written in an additive way,

$$x^\mu \to x^\mu + a^\mu, \tag{9.13}$$

but, by embedding spacetime in one extra dimension, $d_R = (n+1) + 1$, one can produce a group vector formulation of the translation group:

$$\begin{pmatrix} x^\mu \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 & a^\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{\mu\prime} \\ 1 \end{pmatrix}. \tag{9.14}$$

This has the form of a group vector multiplication. The final 1 in the column vector is conserved and plays only a formal role. This form is common in computer representations of translation, such as in computer graphics.

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A representation of translations which is particularly important in quantum mechanics is the differential coordinate representation. Consider an infinitesimal translation $a_\mu = \epsilon_\mu$. This transformation can be obtained from an exponentiated group element of the form

$$U(\epsilon) = \exp\left(\mathrm{i}\theta^A T^A\right) \tag{9.15}$$

by writing

$$U(\epsilon) = \exp\left(\mathrm{i}\epsilon_\rho k^\rho\right) \exp\left(\mathrm{i}\epsilon_\rho\, p^\rho / \chi_h\right) = (1 + \mathrm{i}\epsilon_\rho\, p^\rho / \chi_h), \tag{9.16}$$

where

$$p_\mu = \chi_h\, k_\mu = -\mathrm{i}\,\chi_h\,\partial_\mu. \tag{9.17}$$

The action of the infinitesimal group element is thus

$$x^\mu \to U(\epsilon)\, x^\mu = (1 + \chi_h\, \epsilon^\rho \partial_\rho x^\mu) = x^\mu + \epsilon^\rho\, \eta_\rho^{\ \mu} = x^\mu + \epsilon^\mu. \tag{9.18}$$

The reason for writing the generator,

$$T^A \to p_\mu / \chi_h\,, \tag{9.19}$$

in this form, is that p_μ is clearly identifiable as a momentum operator which satisfies

$$[x, p] = \mathrm{i}\,\chi_h\,. \tag{9.20}$$

Thus, it is the momentum divided by a dimensionful scale (i.e. the wavenumber k_μ) which is the generator of translations. In fact, we already know this from Fourier analysis.

The momentum operator closely resembles that from quantum mechanics. The only difference is that the scale χ_h (with dimensions of action), which is required to give p_μ the dimensions of momentum, is not necessarily $\hbar$. It is arbitrary. The fact that $\hbar$ is small is the physical content of quantum mechanics; the remainder is group theory. What makes quantum mechanics special and noticeable is the non-single-valued nature of the exponentiated group element. The physical consequence of a small χ_h is that even a small translation will cause the argument of the exponential to go through many revolutions of 2π. If χ_h is large, then this will not happen. Physically this means that the oscillatory nature of the group elements will be very visible in quantum mechanics, but essentially invisible in classical mechanics. This is why a wavelike nature is important in quantum mechanics.

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9.2.2 Bloch's theorem: group representations on field space

Bloch's theorem, well known in solid state physics, is used to make predictions about the form of wavefunctions in systems which have periodic potentials. In metals, for instance, crystal lattices look like periodic arrays of potential wells, in which electrons move. The presence of potentials means that the eigenfunctions are not plane waves of the form

$$e^{ik(x-x')}, \tag{9.21}$$

for any x, x'. Nevertheless, translational invariance by discrete vector jumps a_i is a property which must be satisfied by the eigenfunctions

$$\phi_k(t, \mathbf{x} + \mathbf{a}) = U(\mathbf{a})\ \phi_k(t, \mathbf{x}) = e^{i\,\mathbf{k}\cdot\mathbf{a}}\ \phi_k(t, \mathbf{x}). \tag{9.22}$$

9.2.3 Spatial topology and boundary conditions

Fields which live on spacetimes with non-trivial topologies require boundary conditions which reflect the spacetime topology. The simplest example of this is the case of periodic boundary conditions:

$$\phi(x) = \alpha\, \phi(x + L), \tag{9.23}$$

for some number α. Periodic boundary conditions are used as a model for homogeneous crystal lattices, where the periodicity is interpreted as translation by a lattice cell; they are also used to simulate infinite systems with finite ones, allowing the limit $L \to \infty$ to be taken in a controlled manner. Periodic boundary conditions are often the simplest to deal with.

The value of the constant α can be specified in a number of ways. Setting it to unity implies a strict periodicity, which is usually over-restrictive. Although it is pragmatic to specify a boundary condition on the field, it should be noted that the field itself is not an observable. Only the probability $P = (\phi, \phi)$ and its associated operator $\hat{P}$ are observables. In Schrödinger theory, for example, $\hat{P} = \psi^*(x)\psi(x)$, and one may have $\psi(x + L) = e^{i\theta(x)}\psi(x)$ and still preserve the periodicity of the probability.

In general, if the field $\phi(x)$ is a complex field or has some multiplet symmetry, then it need only return to its original value up to a gauge transformation; thus $\alpha = U(x)$. For a multiplet, one may write

$$\Phi_A(x + L) = U_A^{\ B}(x)\ \Phi_B(x). \tag{9.24}$$

The transformation U is the exponentiated phase factor belonging to the group of symmetry transformations which leaves the action invariant. This is sometimes referred to as a *non-integrable phase*. Note that, for a local gauge transformation, one also has a change in the vector field:

$$A_\mu(x + L) = \beta A_\mu(x). \tag{9.25}$$

This kind of transformation is required in order to obtain a consistent energy–momentum tensor for gauge symmetric theories (see section 11.5). The value of β depends now on the type of couplings present. From the spacetime symmetry, a real field, A_μ, has only a Z_2 reflection symmetry, i.e. $\beta = \pm 1$, which corresponds heuristically to ferromagnetic and anti-ferromagnetic boundary conditions. Usually $\beta = 1$ to avoid multiple-valuedness.

In condensed matter physics, conduction electrons move in a periodic potential of crystallized valence ions. The potential they experience is thus periodic:

$$V(\mathbf{x}) = V(\mathbf{x} + \mathbf{L}), \tag{9.26}$$

and it follows that, for plane wave eigenfunctions,

$$\phi_k(t, \mathbf{x} + \mathbf{L}) = U(\mathbf{L})\, \phi_k(t, \mathbf{x}) = \mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{L}}\, \phi_k(t, \mathbf{x}). \tag{9.27}$$

This is a straightforward application of the scalar translation operator; the result is known as Bloch's theorem.

On toroidal spacetimes, i.e. those which have periodicities in several directions, the symmetries of the boundary conditions are linked in several directions. This leads to boundary conditions called co-cycle conditions [126]. Such conditions are responsible for flux quantization of magnetic fields in the Hall effect [65, 85].

In order to define a self-consistent set of boundary conditions, it is convenient to look at the so-called Wilson loops in the two directions of the torus, since they may be constructed independently of the eigenfunctions of the Hamiltonian. Normally this is presented in such a way that any constant part of the vector potential would cancel out, giving no information about it. This is the co-cycle condition, mentioned below. The Wilson line is defined by

$$W_j(x) = P \exp\left\{ \mathrm{i}g \int_{\vec{x}_0}^{\vec{x}} A_j\, \mathrm{d}x'_j \right\}, \tag{9.28}$$

j not summed, for some fixed point $\vec{x}_0$. It has an associated Wilson loop $W_j(L'_j)$ around a cycle of length L'_j in the x_j direction by

$$W_j(x_j + L'_j) = W_j(L'_j)\, W_j(x_j). \tag{9.29}$$

The notation here means that the path-dependent Wilson line $W_j(\vec{x})$ returns to the same value multiplied by a phase $W_j(L'_j, \vec{x})$ on translation around a closed curve from x_j to $x_j + L'_j$. The coordinate dependence of the phase usually arises in the context of a uniform magnetic field passing through the torus. In the presence of a constant magnetic field strength, the two directions of the torus are closely linked, and thus one has

$$W_1(u_1 + L_1, u_2) = \exp\left\{ \mathrm{i}L_1 u_2 + \mathrm{i}c_1 L_1 \right\} W_1(u_1, u_2) \tag{9.30}$$

$$W_2(u_1, u_2 + L_2) = \exp\left\{ \mathrm{i}c_2 L_2 \right\} W_2(u_1, u_2). \tag{9.31}$$

At this stage, it is normal to demonstrate the quantization of flux by opening out the torus into a rectangle and integrating around its edges:

$$W_1(u_2 + L_2)W_2(u_1)W_1^{-1}(u_2)W_2^{-1}(u_1 + L_1) = 1. \tag{9.32}$$

This is known as the co-cycle condition, and has the effect of cancelling the contributions to the c's and thus flux quantization is found independently of the values of c_i due to the nature of the path. The most general consistency requirement for the gauge field (Abelian or non-Abelian), which takes into account the phases c_i, has been constructed in ref. [18].

The results above imply that one is not free to choose, say, periodic boundary conditions for bosons and anti-periodic boundary conditions for fermions in the presence of a uniform field strength. All fields must satisfy the same consistency requirements. Moreover, the spectrum may not depend on the constants, c_i, which have no invariant values. One may understand this physically by noting that a magnetic field causes particle excitations to move in circular Landau orbits, around which the line integral of the constant vector potential is null. The constant part of the vector potential has no invariant meaning in the presence of a magnetic field.

In more complex spacetimes, such as spheres and other curved surfaces, boundary conditions are often more restricted. The study of eigenfunctions (spherical harmonics) on spheres shows that general phases are not possible at identified points. Only the eigenvalues ± 1 are consistent with a spherical topology [17].

9.3 Rotational invariance: $SO(n)$

Rotations are clearly of special importance in physics. In n spatial dimensions, the group of rotations is the group which preserves the Riemannian, positive definite, inner product between vectors. In Cartesian coordinates this has the well known form

$$\mathbf{x} \cdot \mathbf{y} = x^i \, y_i. \tag{9.33}$$

The rotation group is the group of orthogonal matrices with unit determinant $SO(n)$. Rotational invariance implies that the Green function only depends on squared combinations of this type:

$$G(x, x') = G\left((x_1 - x_1')^2 + (x_2 - x_2)^2 + \cdots + (x_n - x_n')^2\right). \tag{9.34}$$

The exception here is the Dirac Green function.

9.3.1 Group representations on coordinate space

Three-dimensional rotations are generated by infinitesimal matrices:

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}$$

$$T^2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}$$

$$T^3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9.35}$$

which satisfy a Lie algebra

$$[T_i, T_j] = \mathrm{i}\epsilon_{ijk} T_k. \tag{9.36}$$

These exponentiate into the matrices for a three-dimensional rotation, parametrized by three Euler angles,

$$R_x \equiv U_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \tag{9.37}$$

$$R_y \equiv U_y = \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \tag{9.38}$$

$$R_z \equiv U_z = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \sin\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9.39}$$

The rotation group is most often studied in $n = 3$ dimensions, for obvious reasons, though it is worth bearing in mind that its properties differ quite markedly with n. For instance, in two dimensions it is only possible to have rotation about a point. With only one angle of rotation, the resulting rotation group, $SO(2)$, is Abelian and is generated by the matrix

$$T_1 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}. \tag{9.40}$$

This exponentiates into the group element

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{9.41}$$

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A two-dimensional world can also be represented conveniently by adopting complex coordinates on the Argand plane. In this representation, a vector is simply a complex number z, and a rotation about the origin by an angle θ is accomplished by multiplying:

$$z \to e^{i\theta}\, z. \tag{9.42}$$

9.3.2 Eigenfunctions: circular and spherical harmonics

The eigenfunctions of the rotation operators form a set of basis functions which span representation space. The rotational degrees of freedom in quantum fields can be expanded in terms of these eigenfunctions.

Eigenfunctions in $n = 2$ In two dimensions, there is only a single axis of rotation to consider. Then the action of the rotation operator T_1 has the form

$$-i\partial_\phi\, |\phi\rangle = \Lambda\, |\phi\rangle. \tag{9.43}$$

This equation is trivially solved to give

$$|\phi\rangle = e^{i\Lambda\phi}. \tag{9.44}$$

In two spatial dimensions, there are no special restrictions on the value of Λ. Notice that this means that the eigenfunctions are not necessarily single-valued functions: under a complete rotation, they do not have to return to their original value. They may differ by a phase:

$$|\phi + 2\pi\rangle = e^{i\Lambda(\phi+2\pi)} = e^{i\delta}\, e^{i\Lambda\phi}, \tag{9.45}$$

where $\delta = 2\Lambda\pi$. In higher dimensions δ must be unity because of extra topological restrictions (see below).

Eigenfunctions in $n = 3$ The theory of matrix representations finds all of the irreducible representations of the rotation algebra in $n = 3$ dimensions. These are characterized by their highest weight, or *spin*, with integral and half-integral values. There is another approach, however, which is to use a differential representation of the operators. The advantage of this is that it is then straightforward to find orthonormal basis functions which span the rotational space.

A set of differential operators which satisfies the Lie algebra is easily constructed, and has the form

$$\mathbf{T} = \mathbf{r} \times i\nabla, \tag{9.46}$$

or

$$T_i = i\epsilon_{ijk}\, x_j\, \partial_k. \tag{9.47}$$

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This has the form of an orbital angular momentum operator $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, and it is no coincidence that it re-surfaces also in chapter 11 in that context with only a factor of $\hbar$ to make the dimensions right. It is conventional to look for the simultaneous eigenfunctions of the operators L_1 and L^2 by writing these operators in spherical polar coordinates (with constant radius):

$$\begin{aligned} L_1 &= \mathrm{i}\left(\sin\phi\,\partial_\theta + \cot\theta\ \cos\phi\,\partial_\phi\right) \\ L_2 &= \mathrm{i}\left(-\cos\phi\,\partial_\theta + \cot\theta\ \sin\phi\,\partial_\phi\right) \\ L_3 &= -\mathrm{i}\,\partial_\phi, \end{aligned} \tag{9.48}$$

and

$$L^2 = \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\right) + \frac{1}{\sin^2\theta}\ \partial_\phi^2. \tag{9.49}$$

The eigenvectors and eigenvalues involve two angles, and may be defined by

$$\begin{aligned} L^2\,|\phi,\theta\rangle &= T^2|\phi,\theta\rangle \\ L_3\,|\phi,\theta\rangle &= \Lambda_\mathrm{c}|\phi,\theta\rangle. \end{aligned} \tag{9.50}$$

The solution to the second equation proceeds as in the two-dimensional case, with only minor modifications due to the presence of the other coordinates. The eigenfunctions are written as a direct product,

$$|\phi,\theta\rangle = \Theta(\theta)\Phi(\phi), \tag{9.51}$$

so that one may identify $\Phi(\phi)$ with the solution to the two-dimensional problem, giving

$$|\phi,\theta\rangle = \Theta(\theta)\ \mathrm{e}^{\mathrm{i}\Lambda_\mathrm{c}\phi}. \tag{9.52}$$

The values of Λ_c are not arbitrary in this case: the solution of the constraints for the θ coordinate imposes extra restrictions, because of the topology of a three-dimensional space. Suppose we consider a rotation through an angle of 2π in the ϕ direction in the positive and negative directions:

$$\begin{aligned} |\phi + 2\pi\rangle &= \mathrm{e}^{\mathrm{i}\Lambda_\mathrm{c}(\phi+2\pi)} = \mathrm{e}^{\mathrm{i}\delta}\,\mathrm{e}^{\mathrm{i}\Lambda_\mathrm{c}\phi}, \\ |\phi - 2\pi\rangle &= \mathrm{e}^{\mathrm{i}\Lambda_\mathrm{c}(\phi-2\pi)} = \mathrm{e}^{-\mathrm{i}\delta}\,\mathrm{e}^{\mathrm{i}\Lambda_\mathrm{c}\phi}. \end{aligned} \tag{9.53}$$

In two spatial dimensions, these two rotations are distinct, but in higher dimensions they are not. This is easily seen by drawing the rotation as a circle with an arrow on it (see figure 9.2). By flipping the circle about an axis in its plane we can continuously deform the positive rotation into the negative one, and vice versa. This is not possible in $n = 2$ dimensions. This means that they are, in fact, different expressions of the same rotation. Thus,

$$\mathrm{e}^{\mathrm{i}\delta} = \mathrm{e}^{-\mathrm{i}\delta} = \pm 1. \tag{9.54}$$

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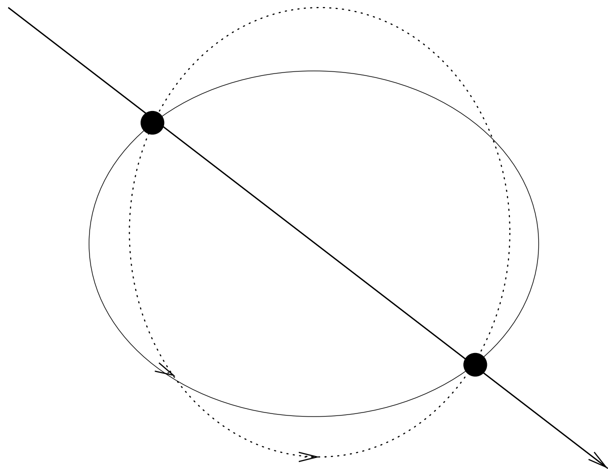

Fig. 9.2. Exchange of particles in two and three spatial dimensions. In the plane, there is only one rotation about the centre of mass which exchanges identical particles. Clockwise and anti-clockwise are inequivalent. In three dimensions or greater, one may rotate this plane around another axis and deform clockwise into anti-clockwise.

These two values are connected with the existence of two types of particle: bosons and fermions, or

$$\Lambda_c = 0, \pm\frac{1}{2}, \pm 1, \ldots, \tag{9.55}$$

for integer m. Note that, in older texts, it was normal to demand the single-valuedness of the wavefunction, rather than using the topological argument leading to eqn. (9.54). If one does this, then only integer values of Λ_c are found, and there is an inconsistency with the solution of the group algebra. This illustrates a danger in interpreting results based on coordinate systems indiscriminately. The result here tells us that the eigenfunctions may be either single-valued for integer Λ_c, or double-valued for half-integral Λ_c. In quantum mechanics, it is normal to use the notation

$$T^2 = l(l+1) \tag{9.56}$$

$$\Lambda_c = m. \tag{9.57}$$

If we now use this result in the eigenvalue equation for L^2, we obtain

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0. \tag{9.58}$$

Putting $z = \cos\theta$ in this equation turns it into the associated Legendre equation,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z^2)\frac{\mathrm{d}P}{\mathrm{d}z}\right] + \left[l(l+1) - \frac{m^2}{1-z^2}\right]P = 0, \tag{9.59}$$

where $P = \Theta(\cos\theta)$. The solutions of the associated Legendre equation may be found for integral and half-integral values of Λ_c, though most books ignore the half-integral solutions. They are rather complicated, and their form is not specifically of interest here. They are detailed, for instance, in Gradshteyn and Ryzhik [63]. Since the magnitude of L_3 cannot exceed that of L^2, by virtue of the triangle (Schwartz) inequality,

$$m^2 \leq l(l+1), \tag{9.60}$$

or

$$-l \leq m \leq l. \tag{9.61}$$

The rotational eigenfunctions are

$$|l, m\rangle = N_{lm}\, P_l^m(\cos\theta)\, \mathrm{e}^{\mathrm{i}m\phi}, \tag{9.62}$$

with normalization factor

$$N_{lm} = (-1)^m \sqrt{\left[\frac{2l+1}{4\pi}\;\frac{(l-m)!}{(l+m)!}\right]}. \tag{9.63}$$

These harmonic eigenfunctions reflect the allowed boundary conditions for systems on spherical spacetimes. They also reflect particle statistics under the interchange of identical particles. The eigenvalues of the spherical harmonics are ± 1 in $3+1$ dimensions, corresponding to (symmetrical) bosons and (anti-symmetrical) fermions; in $2+1$ dimensions, the Abelian rotation group has arbitrary boundary conditions corresponding to the possibility of anyons, or particles with 'any' statistics [83, 89].

9.4 Lorentz invariance

9.4.1 Physical basis

The Lorentz group is a non-compact Lie group which lies at the heart of Einsteinian relativistic invariance. Lorentz transformations are coordinate transformations which preserve the relativistic scalar product

$$x^\mu y_\mu = -x^0\, y^0 + x^i\, y^i, \tag{9.64}$$

and therefore also the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{9.65}$$

Lorentz transformations include, like the Galilean group, translations, rotations and boosts, or changes of relative speed. Under a linear transformation of x^μ, we may write generally

$$x^\mu \to x'^\mu = U^\mu_{\ \nu} x^\nu + a^\mu, \tag{9.66}$$

where a^μ is a constant translation and

$$U^\mu_{\ \nu} = \frac{\partial x'^\mu}{\partial x^\nu} \tag{9.67}$$

is constant.

9.4.2 Lorentz boosts and rotations

A boost is a change of perspective from one observer to another in relative motion to the first. The finite speed of light makes boosts special in Einsteinian relativity. If we refer to figure 9.1 and consider the case of relative motion along the x^1 axis, such that the two frames S and S' coincide at $x^0 = 0$, the Lorentz transformation relating the primed and unprimed coordinates may be written

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) = x^0 \cosh\alpha - x^1 \sinh\alpha \\ x'^1 &= \gamma(x^1 - \beta x^0) = x^1 \cosh\alpha - x^0 \sinh\alpha \\ x'^2 &= x^2 \\ x'^3 &= x^3, \end{aligned} \tag{9.68}$$

where

$$\begin{aligned} \gamma &= 1/\sqrt{1-\beta^2} \\ \beta^i &= v^i/c \\ \beta &= \sqrt{\beta^i \beta_i} \\ \alpha &= \tanh^{-1}\beta. \end{aligned} \tag{9.69}$$

The appearance of hyperbolic functions here, rather than, say, sines and cosines means that there is no limit to the numerical values of the group elements. The group is said to be *non-compact*. In matrix form, in (3 + 1) dimensional spacetime we may write this:

$$L(B) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9.70}$$

where the 'rapidity' $\alpha = \tanh^{-1}\beta$. This may be compared with the explicit form of a rotation about the x^1 axis:

$$L(R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{9.71}$$

Notice that the non-trivial parts of these matrices do not overlap. This leads to an important result, which we shall derive below, namely that rotations and boosts are independent transformations which can be used to parametrize general transformations.

The form of these matrix representations makes it clear that the n-dimensional group of rotations, $SO(n)$, is a sub-group with irreducible representations

$$L_{\mu\nu}(R) = \begin{pmatrix} 1 & 0 \\ 0 & R_{ij} \end{pmatrix}, \tag{9.72}$$

and similarly that boosts in a single direction also form a sub-group. General boosts in multiple directions do not form a group, however.

The form of a general boost can be derived as a generalization of the formulae in eqns. (9.68) on the basis of general covariance. We can write a general form based on figure 9.1 and eqns. (9.68)

$$\begin{aligned} \mathrm{d}x^{0'} &= \gamma(\mathrm{d}x^0 - \beta^i\, \mathrm{d}x_i) \\ \mathrm{d}x^{i'} &= \gamma\left(c_1\, \delta^i{}_j + c_2\, \frac{\beta^i \beta_j}{\beta^2}\right) \mathrm{d}x^j - \gamma\, \beta^i\, \mathrm{d}x^0. \end{aligned} \tag{9.73}$$

The unknown coefficients label projection operators for longitudinal and transverse parts with respect to the n-component velocity vector β^i. By squaring the above expressions and using the invariance of the line element

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^i)^2 = -(\mathrm{d}x^{0'})^2 + (\mathrm{d}x^{i'})^2, \tag{9.74}$$

giving

$$-(\mathrm{d}x^{0'})^2 = -\gamma^2\left((\mathrm{d}x^0)^2 - 2(\beta^i \mathrm{d}x_i)\mathrm{d}x^0 + (\beta^i \mathrm{d}x_i)^2\right), \tag{9.75}$$

and

$$\begin{aligned} (\mathrm{d}x^{0'})^2 = &\left(c_1^2\, \delta_{jk} + (2c_1c_2 + c_2^2)\, \frac{\beta_j \beta_k}{\beta^2}\right) \mathrm{d}x^j \mathrm{d}x^k \\ &+ \gamma^2\beta^2(\mathrm{d}x^0)^2 - 2\gamma(c_1 + c_2)(\beta^i \mathrm{d}x_i)\mathrm{d}x^0, \end{aligned} \tag{9.76}$$

one compares the coefficients of similar terms with the untransformed $\mathrm{d}s^2$ to obtain

$$\begin{aligned} c_1 &= 1 \\ c_2 &= \gamma - 1. \end{aligned} \tag{9.77}$$

Thus, in $1 + n$ block form, a general boost may be written as

$$L_{\mu\nu}(B) = \begin{pmatrix} \gamma & -\gamma\beta^i \\ -\gamma\beta^i & \delta_{ij} + (\gamma - 1)\frac{\beta_i \beta_j}{\beta^2} \end{pmatrix}. \tag{9.78}$$

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9.4.3 The homogeneous Lorentz group: $SO(1, n)$

It is convenient to divide the formal discussion of the Lorentz group into two parts. In the first instance, we shall set the inhomogeneous term, a_μ, to zero. A homogeneous coordinate transformation takes the form

$$x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu, \tag{9.79}$$

where $L_{\mu\nu}$ is a constant matrix. It does not include translations. After a transformation of the line element, one has

$$\begin{aligned} ds'^2 &= g'_{\mu\nu}(x')dx^{\mu\prime}dx^{\nu\prime} \\ &= g_{\mu\nu}(x)L^\mu{}_\rho L^\nu{}_\lambda dx^\rho dx^\lambda. \end{aligned} \tag{9.80}$$

The metric must compensate for this change by transforming like this:

$$g_{\mu\nu}(x) = L_\mu^{\ \rho} L_\nu^{\ \lambda}\, g'_{\rho\lambda}(x'). \tag{9.81}$$

This follows from the above transformation property. We can see this in matrix notation by considering the constant metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1, \ldots)$, which must be invariant if the scalar product is to be preserved. In a Cartesian basis, we have

$$\begin{aligned} x^\mu y_\mu &= \eta_{\mu\nu}\, x^\mu y^\nu = \eta_{\mu\nu}\, (Lx)^\mu (Ly)^\nu \\ x^{\mathrm{T}}\, \eta\, y &= (Lx)^{\mathrm{T}}\, \eta\, (Ly) \\ &= x^{\mathrm{T}} L^{\mathrm{T}}\, \eta\, L y. \end{aligned} \tag{9.82}$$

Comparing the left and right hand sides, we have the matrix form of eqn. (9.81) in a Cartesian basis:

$$\eta = L^{\mathrm{T}}\, \eta\, L. \tag{9.83}$$

The matrices L form a group called the homogeneous Lorentz group. We can now check the group properties of the transformation matrices L. The existence of an associative combination rule is automatically satisfied since matrix multiplication has these properties (any representation in terms of matrices automatically belongs to the general linear group $G(n, R)$). Thus we must show the existence of an inverse and thus an identity element. Acting on the left of eqn. (9.83) with the metric

$$\eta\, L^{\mathrm{T}}\, \eta\, L = \eta^2 = I = L^{-1}\, L, \tag{9.84}$$

where I is the identity matrix belonging to $GL(n, R)$. Thus, the inverse of L is

$$L^{-1} = \eta\, L^{\mathrm{T}}\, \eta. \tag{9.85}$$

In components we have

$$(L^{-1})^\mu{}_\nu = \eta^{\mu\lambda} L^\rho{}_\lambda \eta_{\rho\nu} = L_\nu{}^\mu. \tag{9.86}$$

Since the transpose matrix is the inverse, we can write the Lorentz group as $SO(1, 3)$.

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Dimension and structure of the group The symmetry in $(n+1)^2$ components of $L_{\mu\nu}$ implies that not all of the components may be chosen independently. The fact that only half of the off-diagonal components are independent means that there are

$$d_G = \frac{(n+1)^2 - (n+1)}{2} \tag{9.87}$$

independent components in $n+1$ dimensions, given by the independent elements of $\tilde{\omega}_{\mu\nu}$ to be defined below. Another way of looking at this is that there are $(n+1)^2$ components in the matrix $L_\mu{}^\nu$, but the number of constraints in eqn. (9.83) limits this number. Eqn. (9.83) tells us that the transpose of the equation is the same, thus the independent components of this equation are the diagonal pieces plus half the off-diagonal pieces. This is turn means that the other half of the off-diagonal equations represent the remaining freedom, or dimensionality of the group. d_G is the dimension of the inhomogeneous Lorentz group. The components of

$$g_{\mu\nu} L^\mu{}_\alpha L^\nu{}_\beta = g_{\alpha\beta}$$

may be written out in $1+n$ form, $\mu = (0, i)$ form as follows:

$$\begin{aligned} L^0{}_0 L^0{}_0 \; g_{00} + L^i{}_0 L^j{}_0 \; g_{ij} &= g_{00} \\ L^0{}_i L^0{}_0 \; g_{00} + L^k{}_i L^l{}_0 \; g_{kl} &= g_{i0} = 0 \\ L^0{}_i L^0{}_j \; g_{00} + L^k{}_i L^l{}_j \; g_{ij} &= g_{ij}. \end{aligned} \tag{9.88}$$

This leads to the extraction of the following equations:

$$\begin{aligned} (L^0{}_0)^2 &= 1 + L^i{}_0 L^i{}_0 \\ L^0{}_0 L^0{}_i &= L^k{}_i L_{k0} \\ L^0{}_i L^{j0} + L_{ki} L^{kj} &= \delta_{ij}. \end{aligned} \tag{9.89}$$

These may also be presented in a schematic form in terms of a scalar S, a vector $\mathbf{V}$ and an $n \times n$ matrix M:

$$L^\mu{}_\nu = \begin{pmatrix} S & \mathbf{V}_\mathrm{i}^\mathrm{T} \\ \mathbf{V}_j & M_{ij} \end{pmatrix}, \tag{9.90}$$

giving

$$\begin{aligned} S^2 &= 1 + \mathbf{V}^i \mathbf{V}_i \\ S\mathbf{V}^\mathrm{T} &= \mathbf{V}^\mathrm{T} M \\ I &= M^\mathrm{T} M + \mathbf{V}\mathbf{V}^\mathrm{T}. \end{aligned} \tag{9.91}$$

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It is clear from eqn. (9.90) how the n-dimensional group of rotations, $SO(n)$, is a sub-group of the homogeneous Lorentz group acting on only the spatial components of spacetime vectors:

$$L^{\mu}{}_{\nu}(R) = \begin{pmatrix} 1 & 0 \\ 0 & R_{ij} \end{pmatrix}. \tag{9.92}$$

Notice that it is sufficient to know that $L_0^{\ 0} = 1$ to be able to say that a Lorentz transformation is a rotation, since the remaining equations then imply that

$$M^{\mathrm{T}}M = R^{\mathrm{T}}R = I, \tag{9.93}$$

i.e. that the n-dimensional sub-matrix is orthogonal. The discussion of the Lorentz group can, to a large extent, be simplified by breaking it down into the product of a continuous, connected sub-group together with a few discrete transformations. The elements of the group for which $\det L = +1$ form a sub-group which is known as the proper or restricted Lorentz group. From the first line of eqn. (9.89) or (9.91), we have that $L^{0}{}_{0} \geq 1$ or $L^{0}{}_{0} \leq -1$. The group elements with $L^{0}{}_{0} \geq 1$ and $\det L = +1$ form a sub-group called the *proper orthochronous* Lorentz group, or the restricted Lorentz group. This group is continuously connected, but, since there is no continuous change of any parameter that will deform an object with $\det L = +1$ into an object with $\det L = -1$ (since this would involve passing through $\det L = 0$), this sub-group is not connected to group elements with negative determinants. We can map these disconnected sub-groups into one another, however, with the help of the discrete or *large* Lorentz transformations of *parity* (space reflection) and *time reversal*.

Group parametrization and generators The connected part of the homogeneous Lorentz group may be investigated most easily by considering an infinitesimal transformation in a representation which acts directly on spacetime tensors, i.e. a transformation which lies very close to the identity and whose representation indices A, B are spacetime indices μ, ν. This is the form which is usually required, and the only form we have discussed so far, but it is not the only representation of the group, as the discussion in the previous chapter should convince us. We can write such an *infinitesimal* transformation, $L(\epsilon)$, in terms of a symmetric part and an anti-symmetric part, without loss of generality:

$$L(\epsilon) = I + \epsilon(\tilde{\omega} + \overline{\omega}), \tag{9.94}$$

where $\tilde{\omega}$ is an anti-symmetric matrix, and I and $\overline{\omega}$ together form the symmetric part. ϵ is a vanishingly small (infinitesimal) number. Thus we write, with indices,

$$L_{\mu}^{\ \rho}(\epsilon) = \delta_{\mu}^{\ \rho} + \epsilon(\tilde{\omega}_{\mu}^{\ \rho} + \overline{\omega}_{\mu}^{\ \rho}). \tag{9.95}$$

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Note that, for general utility, the notation commonly appearing in the literature is used here, but beware that the notation is used somewhat confusingly. Some words of explanation are provided below. Substituting this form into eqn. (9.81) gives, to first order in ϵ,

$$g_{\mu\nu}(x)L^{\mu}{}_{\rho}L^{\nu}{}_{\lambda} = g_{\rho\lambda} + \epsilon(\tilde{\omega}_{\rho\lambda} + \overline{\omega}_{\rho\lambda} + \tilde{\omega}_{\lambda\rho} + \overline{\omega}_{\lambda\rho}) + \cdots + \mathrm{O}(\epsilon^2). \tag{9.96}$$

Comparing the left and right hand sides of this equation, we find that

$$\begin{aligned} \tilde{\omega}_{\mu\nu} &= -\tilde{\omega}_{\nu\mu} \\ \overline{\omega}_{\mu\nu} &= -\overline{\omega}_{\nu\mu} = 0. \end{aligned} \tag{9.97}$$

Thus, the off-diagonal terms in $L(\epsilon)$ are anti-symmetric. This property survives exponentiation and persists in finite group elements with one subtlety, which is associated with the indefinite metric. We may therefore identify the structure of a finite Lorentz transformation, L, in spacetime block form. Note that a Lorentz transformation has one index up and one down, since it must map vectors to vectors of the same type:

$$L_{\mu}{}^{\nu} = \begin{pmatrix} L_0^{\ 0} & L_0^{\ i} \\ L_j^{\ 0} & L_i^{\ j} \end{pmatrix}. \tag{9.98}$$

There are two independent (reducible) parts to this matrix representing boosts $\mu, \nu = 0, i$ and rotations $\mu, \nu = \mathrm{i}, j$. Although the generator $\tilde{\omega}_{\mu\nu}$ is purely anti-symmetric, the $0, i$ components form a symmetric matrix under transpose since the act of transposition involves use of the metric:

$$\left(L_0^{\ i}\right)^{\mathrm{T}} = -L^{\mathrm{i}}{}_0 = L_i^{\ 0}. \tag{9.99}$$

The second, with purely spatial components, is anti-symmetric since the generator is anti-symmetric, and the metric leaves the signs of spatial indices unchanged:

$$\left(L_i^{\ j}\right)^{\mathrm{T}} = -L_j^{\ i}. \tag{9.100}$$

Thus, the summary of these two may be written (with both indices down)

$$L_{\mu\nu} = -L_{\nu\mu}. \tag{9.101}$$

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The matrix generators in a (3 + 1) dimensional representation for the Lorentz group in (3 + 1) spacetime dimensions, $T^{AB} = T^{\mu\nu}$, are given explicitly by

$$T_{3+1}{}^{01} = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_{3+1}{}^{02} = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_{3+1}{}^{03} = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix}$$

$$T_{3+1}{}^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_{3+1}{}^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}$$

$$T_{3+1}{}^{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}. \tag{9.102}$$

Note that, because of the indefinite metric, only the spatial components of these generators are Hermitian. This will lead us to reparametrize the components in terms of positive definite group indices below. It is now conventional, if not a little confusing, to write a general infinitesimal Lorentz transformation in the form

$$U_R = L_R(\omega) = I_R + \frac{1}{2}\mathrm{i}\omega_{\mu\nu}T_R^{\mu\nu}, \tag{9.103}$$

where I_R and T_R are the identity and generator matrices of a given representation G_R. In terms of their components A, B,

$$U^A_{\ B} = L^A_{\ B}(\omega) = \delta^A_{\ B} + \frac{\mathrm{i}}{2}\omega_{\rho\sigma}[T_R^{\rho\sigma}]^A_{\ B}. \tag{9.104}$$

The second term here corresponds to the second term in eqn. (9.95), but the spacetime-specific indices μ in eqn. (9.95) have now been replaced by representation indices A, B, anticipating a generalization to other representations. A general finite element of the group in a representation G_R is obtained by exponentiation,

$$L^A_{\ B} = \exp\left(\frac{\mathrm{i}}{2}\omega_{\rho\sigma}\left[T_R{}^{\rho\sigma}\right]^A_{\ B}\right) \tag{9.105}$$

Let us take a moment to understand this form, since it appears repeatedly in the literature without satisfactory explanation. The $\omega^{\mu\nu}$ which appears here is not the same as $\epsilon\tilde{\omega}^{\mu\nu}$ *a priori* (but see the next point). In fact, it plays the role of the group parameters θ^a in the previous chapter. Thus, in the language of the previous chapter, one would write

$$\begin{aligned} U^A_{\ B} = L^A_{\ B}(\epsilon) = \delta^A_{\ B} + \frac{\mathrm{i}}{2}\theta^a[T^a_R]^A_{\ B} \\ L^A_{\ B} = \exp\left(\frac{\mathrm{i}}{2}\theta^a\left[T_R{}^a\right]^A_{\ B}\right). \end{aligned} \tag{9.106}$$

It is easy to see that the use of two indices is redundant notation, since most of the elements of the generators are zeros. It is simply a convenient way to count to the number of non-zero group dimensions d_G in terms of spacetime indices $\mu, \nu = 0, \ldots, n{+}1$ rather than positive definite $a, b = 1, \ldots, d_G$ indices of the group space. The factor of $\frac{1}{2}$ in eqn. (9.105) accounts for the double counting due to the anti-symmetry in the summation over all μ, ν indices. The fact that two indices are used in this summation, rather than the usual one index in T^a, should not lead to confusion. To make contact with the usual notation for generators, we may take the $(3+1)$ dimensional case as an example. In $(3+1)$ dimensions, the homogeneous Lorentz group has $d_G = 6$, and its complement of generators may be written:

$$T^a = \left\{T_{3+1}{}^{10}, T_{3+1}{}^{20}, T_{3+1}{}^{30}, T_{3+1}{}^{12}, T_{3+1}{}^{23}, T_{3+1}{}^{31}\right\}, \tag{9.107}$$

where $a = 1, \ldots, 6$ and the group elements in eqn. (9.105) have the form

$$\exp\left(\mathrm{i}\theta^a T^a\right). \tag{9.108}$$

The first three T^a are the generators of boosts (spacetime rotations), while the latter three are the generators of spatial rotations. The anti-symmetric matrix of parameters $\omega_{\mu\nu}$ contains the components of the rapidity α^i from eqn. (9.68) as well as the angles θ^i which characterize rotations. Eqn. (9.105) is general for any representation of the Lorentz group in $n+1$ dimensions with an appropriate set of matrix generators $T_{\mu\nu}$.

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Lie algebra in 3 + 1 dimensions The generators above satisfy a Lie algebra relation which can be written in several equivalent forms. In terms of the two-index parametrization, one has

$$[T_R{}^{\mu\nu}, T_R{}^{\rho\sigma}] = \mathrm{i}\,(\eta^{\nu\sigma} T_R{}^{\mu\rho} + \eta^{\mu\sigma} T_R{}^{\nu\rho} - \eta^{\mu\rho} T_R{}^{\nu\sigma} - \eta^{\rho\nu} T_R{}^{\mu\sigma})\,. \tag{9.109}$$

This result applies in any number of dimensions. To see this, it is necessary to tie up a loose end from the discussion of the parameters $\omega_{\mu\nu}$ and $\epsilon\tilde{\omega}_{\mu\nu}$ above. While these two quantities play formally different roles, in the way they are introduced above they are in fact equivalent to one another and can even be defined to be equal. This is not in contradiction with what is stated above, where pains were made to distinguish these two quantities formally. The resolution of this point comes about by distinguishing carefully between which properties are special for a specific representation and which properties are general for all representations. Let us try to unravel this point.

The Lorentz transformation is defined in physics by the effect it has on spacetime reference frames (see figure 9.1). If we take this as a starting point, then we must begin by dealing with a representation in which the transformations act on spacetime vectors and tensors. This is the representation in which $A, B \to \mu\nu$, and we can write an infinitesimal transformation as in eqn. (9.95). The alternative form in eqn. (9.104) applies for any representation. If we compare the two infinitesimal forms, it seems clear that $\tilde{\omega}_{\mu\nu}$ plays the role of a generator T_{AB}, and in fact we can make this identification complete by defining

$$\epsilon\tilde{\omega}^{\mu}{}_{\nu} = \frac{\mathrm{i}}{2}\left[\omega_{\rho\lambda} T^{\rho\lambda}_{3+1}\right]^{\mu}{}_{\nu}\,. \tag{9.110}$$

This is made clearer if we make the identification again, showing clearly the representation specific indices:

$$\epsilon\tilde{\omega}^{A}{}_{B} = \frac{\mathrm{i}}{2}\left[\omega_{\rho\lambda} T^{\rho\lambda}_{3+1}\right]^{A}{}_{B}\,. \tag{9.111}$$

This equation is easily satisfied by choosing

$$\left[T^{\rho\sigma}_{3+1}\right] \sim \eta^{\rho A}\eta^{\sigma}{}_{B}. \tag{9.112}$$

However, we must be careful about preserving the anti-symmetry of T_{3+1}, so we have

$$\left[T^{\rho\sigma}_{3+1}\right]^{A}{}_{B} = \frac{2}{\mathrm{i}} \times \frac{1}{2}\left(\eta^{\rho A}\eta^{\sigma}{}_{B} - \eta^{\rho}{}_{B}\eta^{\sigma A}\right). \tag{9.113}$$

Clearly, this equation can only be true when A, B representation indices belong to the set of (3 + 1) spacetime indices, so this equation is only true in one

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representation. Nevertheless, we can use this representation-specific result to determine the algebra relation which is independent of representation as follows. By writing

$$\left[T^{\mu\nu}_{3+1}\right]^A{}_B = \mathrm{i}\left(\eta^{\mu A}\eta^{\nu}{}_B - \eta^{\mu}{}_B\eta^{\nu A}\right)$$
$$\left[T^{\rho\sigma}_{3+1}\right]^B{}_C = \mathrm{i}\left(\eta^{\rho B}\eta^{\sigma}{}_C - \eta^{\rho}{}_C\eta^{\sigma B}\right), \tag{9.114}$$

it is straightforward to compute the commutator,

$$[T^{\mu\nu}_R, T^{\rho\sigma}_R]^A{}_C, \tag{9.115}$$

in terms of η tensors. Each contraction over B leaves a new η with only spacetime indices. The remaining η's have mixed A, μ indices and occur in pairs, which can be identified as generators by reversing eqn. (9.113). The result with A, C indices suppressed is given by eqn. (9.109). In fact, the expression is uniform in indices A, C and thus these 'cancel' out of the result; more correctly they may be generalized to any representation.

The representations of the restricted homogeneous Lorentz group are the solutions to eqn. (9.109). The finite-dimensional, irreducible representations can be labelled by two discrete indices which can take values in the positive integers, positive half-integers and zero. This may be seen by writing the generators in a vector form, analogous to the electric and magnetic components of the field strength $F^{\mu\nu}$ in $(3+1)$ dimensions:

$$J^i \equiv T^i_B = \frac{1}{2}\epsilon_{ijk}T^{jk} = (T^{32}, T^{13}, T^{21})$$
$$K^i \equiv T^i_E/c = \qquad T^{0i} = (T^{01}, T^{02}, T^{03}). \tag{9.116}$$

These satisfy the Lie algebra commutation rules

$$[T^i_B, T^j_B] = \mathrm{i}\epsilon^{ijk}T^k_B$$
$$[T^i_E, T_E{}^j] = -\mathrm{i}\epsilon^{ijk}T^k_E/c^2$$
$$[T^i_E, T^j_B] = \mathrm{i}\epsilon^{ijk}T^k_E. \tag{9.117}$$

Also, as with electromagnetism, one can construct the invariants

$$T^a T^a = \frac{1}{2}T_{R\mu\nu}T^{\mu\nu}_R = T^2_B - T^2_E/c^2$$
$$\frac{1}{8}\epsilon^{\mu\nu\rho\sigma}T^{\mu\nu}_R T^{\rho\sigma}_R = -T^i_E\, T_{Bi}/c. \tag{9.118}$$

These quantities are Casimir invariants. They are proportional to the identity element in any representation, and thus their values can be used to label the representations. From this form of the generators we obtain an interesting

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perspective on electromagnetism: its form is an inevitable expression of the properties of the Lorentz group for vector fields. In other words, the constraints of relativity balanced with the freedom in a vector field determine the form of the action in terms of representations of the restricted group.

The structure of the group can be further unravelled and related to earlier discussions of the Cartan–Weyl basis by forming the new Hermitian operators

$$E_i = \frac{1}{2}\chi_h\,(T_B + \mathrm{i}T_E/c)$$
$$F_i = \frac{1}{2}\chi_h\,(T_B - \mathrm{i}T_E/c) \tag{9.119}$$

which satisfy the commutation rules

$$[E_i, E_j] = \mathrm{i}\,\chi_h\,\epsilon_{ijk}E_k$$
$$[F_i, F_j] = \mathrm{i}\,\chi_h\,\epsilon_{ijk}F_k$$
$$[E_i, F_j] = 0. \tag{9.120}$$

The scale factor, χ_h, is included for generality. It is conventional to discuss angular momentum directly in quantum mechanics texts, for which $\chi_h \to \hbar$. For pure rotation, we can take $\chi_h = 1$. As a matter of principle, we choose to write χ_h rather than $\hbar$, since there is no reason to choose a special value for this scale on the basis of group theory alone. The special value $\chi_h = \hbar$ is the value which is measured for quantum mechanical systems. The restricted Lorentz group algebra now has the form of two copies of the rotation algebra $su(2)$ in three spatial dimensions, and the highest weights of the representations of these algebras will be the two labels which characterize the full representation of the Lorentz group representations.

From the commutation rules (and referring to section 8.5.10), we see that the algebra space may be spanned by a set of basis vectors ($(2\Lambda_{\max}+1)(2\Lambda'_{\max}+1)$ of them). It is usual to use the notation

$$\Lambda_{\mathrm{c}} = \chi_h\;(m_e, m_f)$$
$$\Lambda_{\max} = \chi_h\;(e, f) \tag{9.121}$$

in physics texts, where they are referred to as quantum numbers rather than algebra eigenvalues. Also, the labels j_1, j_2 are often used for e, f, but, in the interest of a consistent and unique notation, it is best not to confuse these with the eigenvalues of the total angular momentum J_i which is slightly different. In terms of these labels, the Lorentz group basis vectors may be written as $|e, m_e;\, f, m_f\rangle$, where $-e \le m_e \le e$, $-f \le m_f \le f$, and e, m_e, f, m_f take on integer or half-integer values. The Cartan–Weyl stepping operators are then,

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by direct transcription from section 8.5.10,

$$\begin{aligned} E_{\pm}|e, m_e;\ f, m_f\rangle &= (E_1 \pm \mathrm{i}E_2)|e, m_e;\ f, m_f\rangle \\ &= \chi_h\ \sqrt{(e \mp m_e)(e \pm m_e + 1)}\ |e, m_e \pm 1;\ f, m_f\rangle \\ E_3|e, m_e;\ f, m_f\rangle &= \chi_h\ m_e|e, m_e;\ f, m_f\rangle \end{aligned} \tag{9.122}$$

and

$$\begin{aligned} F_{\pm}|e, m_e;\ f, m_f\rangle &= (F_1 \pm \mathrm{i}F_2)|e, m_e;\ f, m_f\rangle \\ &= \chi_h\ \sqrt{(f \mp m_f)(f \pm m_f + 1)}\ |e, m_e;\ f, m_f \pm 1\rangle \\ F_3|e, m_j;\ e, m_e\rangle &= \chi_h\ m_f|e, m_e;\ f, m_f\rangle. \end{aligned} \tag{9.123}$$

The algebra has factorized into two $su(2)$ sub-algebras. Each irreducible representation of this algebra may be labelled by a pair $(e,\ f)$, which corresponds to boosts and rotations, from the factorization of the algebra into E and F parts. The number of independent components in such an irreducible representation is $(2e + 1)(2f + 1)$ since, for every e, f can run over all of its values, and vice versa. The physical significance of these numbers lies in the extent to which they may be used to construct field theories which describe a real physical situations. Let us round off the discussion of representations by indicating how these irreducible labels apply to physical fields.

9.4.4 *Different representations of the Lorentz group in* $3+1$ *dimensions*

The explicit form of the Lorentz group generators given in eqns. (9.102) is called the defining representation. It is also the form which applies to the transformation of a spacetime vector. Using this explicit form, we can compute the Casimir invariants for E_i and F_i to determine the values of e and f which characterize that representation. It is a straightforward exercise to perform the matrix multiplication and show that

$$E^2 = E^i E_i = \frac{1}{4}\,\chi_h{}^2(T_B^2 - T_E^2/c^2) = \frac{3}{4}\,\chi_h{}^2\ I_{3+1}, \tag{9.124}$$

where I_{3+1} is the identity matrix for the defining representation. Now, this value can be likened to the general form to determine the highest weight of the representation e:

$$E^2 = \frac{3}{4}\,\chi_h{}^2\ I_{3+1} = e(e+1)\,\chi_h{}^2\ I_{3+1}, \tag{9.125}$$

whence we deduce that $e = \frac{1}{2}$. The same argument may be applied to F^2, with the same result. Thus, the defining representation is characterized by the pair of numbers $(e,\ f) = (\frac{1}{2}, \frac{1}{2})$.

The Lorentz transformations have been discussed so far in terms of tensors, but the independent components of a tensor are not always in an obvious form. A vector, for instance, transforms as

$$A^\mu \to L^\mu_{\ \nu}\, A^\nu, \tag{9.126}$$

but a rank 2-tensor transforms with two such Lorentz transformation matrices

$$A^{\mu\nu} \to L^\mu_{\ \rho} L^\nu_{\ \sigma}\, A^{\rho\sigma}. \tag{9.127}$$

The independent components of a rank 2-tensor might be either diagonal or off-diagonal, and there might be redundant zeros or terms which are identical by symmetry or anti-symmetry, but one could think of re-writing eqn. (9.127) in terms of a single larger matrix acting on a new vector where only the independent components were present, rather than two smaller matrices acting on a tensor. Again, this has to do with a choice of representations. We just pick out the components and re-write the transformations in a way which preserves their content, but changes their form.

Suppose then we do this: we collect all of the independent components of any tensor field into a column vector,

$$A^{\mu\nu\lambda\ldots}_{\ \ \ \ \rho\sigma\ldots} \to \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}, \tag{9.128}$$

where N is the total number of independent components in the object being acted upon, and is therefore the dimension of this representation. The array of matrices $L_\mu^{\ \nu}$ (one for each index) can now be replaced by a single matrix $L_\oplus$ which will have as many independent components as the product of the L's. Often such a single matrix will be reducible into block-diagonal form, i.e. a direct sum of irreducible representations.

The irreducible blocks of any (3+1) spacetime-dimensional Lorentz transformation of arbitrary representation d_R are denoted $D^{(e,f)}(G_R)$. A tensor transformation of rank N might therefore decompose into a number of irreducible blocks in equivalent-vector form:

$$L_\oplus{}^A{}_B = D^{(e_1,f_1)} \oplus D^{(e_2,f_2)} \ldots \oplus D^{(e_N,f_N)}. \tag{9.129}$$

The decomposition of a product of transformations as a series of irreducible representations

$$D^{(A)} \otimes D^{(B)} = \sum_\oplus c_M\, D^M \tag{9.130}$$

is called the Clebsch–Gordon series. The indices A, B run over $1, \ldots, (2e + 1)(2f + 1)$ for each irreducible block. For each value of e, we may take all the

Table 9.1. Spin/helicity properties of some representations of the Lorentz group in $(3+1)$ dimensions.

The number of degrees of freedom (D.F.) $\phi = (2e+1)(2f+1)$. Note that the electromagnetic field $F_{\mu\nu}$ lacks the longitudinal mode $m_s = 0$ of the massive vector field A_μ.

Representation (e, f)	'Spin' $m_s = e + f$	D.F. ϕ	Description
$(\frac{1}{2}, 0)$	$\frac{1}{2}$	2	Weyl 2-spinor
$(0, \frac{1}{2})$	$\frac{1}{2}$	2	Weyl 2-spinor
$(0, 0)$	0	1	trivial scalar
$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$	$\pm\frac{1}{2}$	4	Dirac 4-spinor
$(\frac{1}{2}, \frac{1}{2})$	$0, \pm 1$	4	4-vector A_μ
$(1, 0) \oplus (0, 1)$	± 1	6	anti-symm. $F_{\mu\nu}$
$(1, 1) \oplus (1, 0) \oplus (0, 1) \oplus (0, 0)$	$0, \pm 1, \pm 2$	16	rank 2-tensor

values of f in turn, and vice versa. So which representation applies to which field? We can look at this in two ways.

- We see that e, f are allowed by the general solution of the Lorentz symmetry. The values are $0, \frac{1}{2}, 1, \ldots$. We then simply construct fields which transform according to these representations and match them with physical phenomena.
- We look at fields which we know about $(\phi, A_\mu, g_{\mu\nu}, \ldots)$ and determine what e, f these correspond to.

Some common values of 'spin' are listed in table 9.1. Counting the highest weights of the blocks is not difficult, but to understand the difference between a massless vector field and a massive vector field, for example (both with highest spin weight 1), we must appreciate that these fields have quite different spacetime transformation properties. This is explained by the fact that there are two ways in which a spin 1 field can be constructed from irreducible representations of the Lorentz group, and they form inequivalent representations. Since we are dealing with the homogeneous Lorentz group in a given frame, the spin is the same as the total intrinsic angular momentum of the frame, and is defined by a sum of the two vectors

$$S_i \equiv J_i = E_i + F_i, \tag{9.131}$$

with maximum helicity s given by $e + f$; the range of allowed values follows in integer steps from the rules of vector addition (see section 11.7.4). The maximum value is when the vectors are parallel and the minimum value is when they are anti-parallel. Thus

$$s = \pm(e + f), \pm(e + f - 1), \ldots, \pm|e - f|. \tag{9.132}$$

The spin s is just the highest weight of the Lorentz representation. Of all the representations which one might construct for physical models, we can narrow down the possibilities by considering further symmetry properties. Most physical fields do not change their properties under parity transformations or spatial reflection. Under a spatial reflection, the generators E_i, F_i are exchanged:

$$\begin{aligned} \mathcal{P} E_i \mathcal{P}^{-1} &= F_i \\ \mathcal{P} F_i \mathcal{P}^{-1} &= E_i. \end{aligned} \tag{9.133}$$

In order to be consistent with spatial reflections, the representations of parity-invariant fields must be symmetrical in (e, f). This means we can either make irreducible representations of the form

$$(e, e) \tag{9.134}$$

or symmetrized composite representations of the form

$$(e, f) \oplus (f, e), \tag{9.135}$$

such that exchanging $e \leftrightarrow f$ leaves them invariant.

Helicity values for spin 1 For example, a spin 1 field can be made in two ways which correspond to the massless and massive representations of the Poincaré algebra. In the first case, a spin 1 field can be constructed with the irreducible transformational properties of a vector field,

$$\left(\frac{1}{2}, \frac{1}{2}\right). \tag{9.136}$$

A field of this type would exist in nature with spin/helicities $s = 0, \pm 1$. These correspond to: (i) $2s + 1 = 1$, i.e. one longitudinal scalar component A_0, and (ii) $2s + 1 = 3$, a left or right circularly polarized vector field. This characterizes the massive Proca field, A_μ, which describes W and Z vector bosons in the electro-weak theory. However, it is also possible to construct a field which transforms as

$$(1, 0) \oplus (0, 1). \tag{9.137}$$

The weight strings from this representation have only the values $m_s = \pm 1$, the left and right circular polarizations. There is no longitudinal zero component. The values here apply to the photon field, $F_{\mu\nu}$. The symmetrization corresponds to the anti-symmetry of the electromagnetic field strength tensor. The anti-symmetry is also the key to understanding the difference between these two representations.

One reason for looking at this example is that, at first glance, it seems confusing. After all, the photon is also usually represented by a vector potential A_μ, but here we are claiming that a vector formulation is quite different from an anti-symmetric tensor formulation. There is a crucial difference between the massive vector field and the massless vector field, however. The difference can be expressed in several equivalent ways which all knit together to illuminate the theme of representations nicely.

The physical photon field, $F_{\mu\nu}$, transforms like a tensor of rank 2. Because of its anti-symmetry, it can also be written in terms of a *massless* 4-vector potential, which transforms like a *gauge-invariant* vector field. Thus, the massless vector field is associated with the anti-symmetric tensor form. The massive Proca field only transforms like a vector field with no gauge invariance. The gauge invariance is actually a direct manifestation of the difference in transformation properties through a larger invariance group with a deep connection to the Lorentz group. The true equation satisfied by the photon field is

$$\partial_\mu F^{\mu\nu} = (\Box\, \delta^\mu_\nu - \partial^\mu \partial_\nu) A_\mu = 0, \tag{9.138}$$

while the Proca field satisfies

$$(-\Box\ + m^2) A_\mu = 0. \tag{9.139}$$

This displays the difference between the fields. The photon field has a degree of freedom which the Proca field does not; namely, its vector formulation is invariant under

$$A_\mu \rightarrow A_\mu + (\partial_\mu s), \tag{9.140}$$

for any scalar function $s(x)$. The Proca field is not. Because of the gauge symmetry, for the photon, no coordinate transformation is complete without an associated, arbitrary gauge transformation. A general coordinate variation of these fields illustrates this (see section 4.5.2).

$$\begin{array}{ll} \text{Photon field} & \delta_x A^\mu = \epsilon_\nu F^{\nu\mu} \\ \text{Proca field} & \delta_x A^\mu = \epsilon_\nu (\partial^\nu A^\mu). \end{array}$$

The difference between these two results is a gauge term. This has the consequence that the photon's gauge field formulation behaves like an element of the *conformal group*, owing to the spacetime-dependent function $s(x)$. This

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is very clearly illustrated in section 11.5. The gauge field A_μ must transform like this if the tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which derives from it is to transform like an element of the Lorentz group. The same is not true of the Proca field, A_μ, which is simply a vector field without complication.

Appearances can therefore be deceptive. The spin 1 vector fields might look the same, but the gauge invariance of the gauge field associates it with an anti-symmetric second-rank tensor. The anti-symmetric property of the photon tensor endows it with a property called *transversality*, which means that the physical excitations of the field E_i, B_i are transverse to the direction of propagation (i.e. to the direction of its momentum or wavenumber) k^i. This is not the case for the Proca field. It has components of its field in the direction of motion, i.e. *longitudinal* components. The extra $s = 0$ mode in the helicity values for the Proca field corresponds to a longitudinal mode.

For a massless field travelling in the x^3 direction, $k_\mu = (k, 0, 0, k)$. Transversality means that

$$k^i F_{i\mu} = \partial^i F_{i\mu} = 0, \tag{9.141}$$

which is guaranteed by Maxwell's equations away from sources. In gauge form,

$$k^i A_i = 0, \tag{9.142}$$

which can always be secured by a gauge transformation. For the massive vector field, the lack of gauge invariance means that this condition cannot be secured.

9.4.5 *Other spacetime dimensions*

In a different number of spacetime dimensions $n + 1$, the whole of the above $(3 + 1)$ dimensional procedure for finding the irreducible representations must be repeated, and the spin labels must be re-evaluated in the framework of a new set of representations for the Lorentz group. This will not be pursued here.

9.4.6 *Factorization of proper Lorentz transformations*

From the discussion of the Lie algebra above, one sees that an arbitrary element of the proper or restricted Lorentz group can be expressed as a product of a rotation and a boost. This only applies to the restricted transformations, and is only one possible way of parametrizing such a transformation. The result follows from the fact that a general boost may be written as

$$L(B) = \begin{pmatrix} \gamma & -\gamma\beta^i \\ -\gamma\beta^i & \delta_{ij} + (\gamma - 1)\frac{\beta_i\beta_j}{\beta^2} \end{pmatrix}, \tag{9.143}$$

and a rotation may be written

$$L(R) = \begin{pmatrix} 1 & 0 \\ 0 & R_{ij} \end{pmatrix}. \tag{9.144}$$

The result can be shown starting from a general Lorentz transformation as in eqn. (9.98). Suppose we operate on this group element with an inverse boost (a boost with $\beta^i \to -\beta^i$:

$$L^{-1}(B)L = \begin{pmatrix} \gamma & -\gamma\beta^i \\ -\gamma\beta^i & \delta_{ij} + (\gamma - 1)\frac{\beta_i\beta_j}{\beta^2} \end{pmatrix} \begin{pmatrix} L_0^{\ 0} & L_0^{\ i} \\ L_j^{\ 0} & L_i^{\ j} \end{pmatrix}, \tag{9.145}$$

where we define the velocity to be

$$\beta^i = -\left(\frac{L^i_{\ 0}}{L^0_{\ 0}}\right). \tag{9.146}$$

This makes

$$\gamma = L^0_{\ 0}, \tag{9.147}$$

and it then follows from eqns. (9.89) that this product has the form

$$L^{-1}(B)L = \begin{pmatrix} 1 & 0 \\ 0 & M_i^{\ j} \end{pmatrix} = L(R). \tag{9.148}$$

This result is clearly a pure rotation, meaning that we can rearrange the formula to express the original arbitrary proper Lorentz transformation as a product of a boost and a rotation,

$$L = L(B)L(R). \tag{9.149}$$

9.4.7 The inhomogeneous Lorentz group or Poincaré group in 3 + 1 dimensions

If the inhomogeneous translation term, a_μ, is not set to zero in eqn. (9.66), one is led to a richer and more complex group structure [137]. This is described by the so-called inhomogeneous Lorentz group, or Poincaré group. It is a synthesis of the physics of translations, from earlier in this chapter, and the fixed origin behaviour of the homogeneous Lorentz group. The most general transformation of this group can be written

$$x'^\mu = L^\mu_{\ \nu} x^\nu + a^\mu, \tag{9.150}$$

where a^μ is an x^μ-independent constant translation vector. These transformations cannot be represented by a $d_R = 4$ representation by direct matrix

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multiplication, but a $d_R = 5$ representation is possible, by analogy with eqn. (9.14), by embedding in one extra dimension:

$$U_{3+1+1}x^\mu = \begin{pmatrix} L^\mu{}_\nu & a_\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^\mu \\ 1 \end{pmatrix} = x^\mu + a^\mu. \tag{9.151}$$

The generic infinitesimal Poincaré transformation may be written

$$U = 1 + \frac{\mathrm{i}}{2}\omega_{\mu\nu}T_R^{\mu\nu} + \mathrm{i}\epsilon_\rho\, k_R^\rho, \tag{9.152}$$

for some scale χ_h with dimensions of action. Inspired by the differential representation for the translation group, we find a differential form for the homogeneous Lorentz group, which might be combined with the translation group in a straightforward way. These are:

$$\begin{aligned} T_{\mathrm{diff}}^{\mu\nu} &= -\mathrm{i}(x^\mu\partial^\nu - x^\nu\partial^\mu) \\ J_i &= \frac{1}{2}\epsilon_{ijk}T^{jk} = -\frac{\mathrm{i}}{2}\,\chi_h\,\epsilon_{ijk}(x_j\partial_k - x_k\partial_j) \\ K_i &= T_{0i} \\ p_\mu &= \chi_h\; k_\mu = -\mathrm{i}\,\chi_h\,\partial_\mu. \end{aligned} \tag{9.153}$$

An important difference between the inhomogeneous Lorentz group and the homogeneous Lorentz group is that the total angular momentum generator, J_i, is no longer just the intrinsic angular momentum of a field, but it can include orbital angular momentum about a point displaced from the origin. This means that we have to be more careful than before in distinguishing spin s from $j = e + k$ by defining it only in an inertial rest frame with zero momentum. It is easily verified that these representations satisfy the algebra relations. Using these forms, it is a matter of simple algebra to evaluate the full algebraic content of the Poincaré group:

$$[k_\mu, T_{\rho\sigma}] = -\mathrm{i}(\eta_{\mu\rho}k_\sigma - \eta_{\mu\sigma}k_\rho), \tag{9.154}$$

or equivalently

$$\begin{aligned} [k_0, J_i] &= 0 \\ [k_i, J_l] &= -\mathrm{i}\,\chi_h\,\epsilon_{ilm}k_m. \end{aligned} \tag{9.155}$$

These relations are trivial statements about the transformation properties of k_0 (scalar) and k_i (vector) under rotations. Using the definitions above, we also find that

$$\begin{aligned} [k_0, K_i] &= \mathrm{i}k_i \\ [k_i, K_j] &= -\mathrm{i}\,\chi_h\,\eta_{ij}k_0. \end{aligned} \tag{9.156}$$

These relations show that a boost K_i affects k_0, k_i, but not k_j for $j \neq \mathrm{i}$.

Massive fields It is a curious feature of the Poincaré group, which comes about because it arises in connection with the finite speed of light, that the mass of fields plays a role in their symmetry properties. Physically, massless fields are bound to move at the speed of light so they have no rest frame about which to define intrinsic properties, like spin, which depend on coordinate concepts. It is therefore necessary to find another way to characterize intrinsic rotation. We can expect mass to play a role since it is linked to the momentum, which is the generator of translations.

The Poincaré group leaves invariant the relation

$$\mathbf{p}^2 c^2 + m^2 c^4 = \text{const}, \tag{9.157}$$

where $p_\mu = (mc, p_i)$. This is, in fact, a Casimir invariant, $p^\mu p_\mu$, up to dimensional factors. Recall from the discussion of translations that the momentum may be written

$$p_\mu = \chi_h k_\mu, \tag{9.158}$$

where k_μ is the wavenumber or reciprocal lattice vector. As in the case of the other groups, we can label the field by invariant quantities. Here we have the quadratic Casimir invariants

$$\begin{aligned} J^2 &= j(j+1)\,\chi_h{}^2 \\ p^2 &= \mathbf{p}^2 c^2 + m^2 c^4, \end{aligned} \tag{9.159}$$

which commute with the group generators and are thus independent of symmetry basis:

$$\begin{aligned} [p^2, p_\mu] &= 0 \\ [p^2, J_i] &= 0 \\ [p^2, K_i] &= 0. \end{aligned} \tag{9.160}$$

A covariant rotation operator can be identified which will be useful for discussing intrinsic in chapter 11. It is called the Pauli–Lubanski vector, and it is defined by

$$W_\mu = \frac{1}{2}\,\chi_h\,\epsilon_{\mu\nu\lambda\rho} T^{\nu\lambda} p^\rho. \tag{9.161}$$

The quadratic form, W^2, is Lorentz- and translation-invariant:

$$\begin{aligned} [W^2, p_\mu] &= 0 \\ [W^2, T_{\mu\nu}] &= 0. \end{aligned} \tag{9.162}$$

W satisfies

$$W^\mu p_\mu = 0 \tag{9.163}$$

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and

$$[W_\mu, W_\nu] = \mathrm{i}\epsilon_{\mu\nu\rho\sigma} W^\rho p^\sigma \tag{9.164}$$

$$W^2 = -\frac{1}{2}\,\chi_h{}^2\, T^{\mu\nu} T_{\mu\nu}\, p^2 + \chi_h{}^2\, T^{\mu\nu} T_\nu{}^\lambda\, p_\nu p_\lambda. \tag{9.165}$$

If we consider W_μ in a rest frame where $p_i = 0$, we have

$$W^\mu_{\text{rest}} = -mc(0, J_1, J_2, J_3)_{\text{rest}} = -\frac{1}{2} mc(0, S_1, S_2, S_3), \tag{9.166}$$

where S_i may be thought of as the intrinsic (non-orbital) rotation of the field (called *spin* of the representation), which is defined by

$$S_i = J_i\Big|_{\text{rest}}. \tag{9.167}$$

Thus, W^μ is clearly a 4-vector with the properties of intrinsic rotations in a rest frame. Indeed, evaluating eqn. (9.164) in a rest frame, we find that

$$[W_i, W_j] = -\mathrm{i}mc\, \epsilon_{ijk} W^k. \tag{9.168}$$

Or setting $W_i = -mc\, J_i$, we recover the rotational algebra

$$[J_i, J_j] = \mathrm{i}\, \chi_h\, \epsilon_{ijk} J^k. \tag{9.169}$$

Thus the Poincaré-invariant quadratic form is

$$W^2_{\text{rest}} = m^2c^2\, J^2 = m^2c^2\, j(j+1)\, \chi_h{}^2\, I_R. \tag{9.170}$$

For classifying fields, we are interested in knowing which of the properties of the field can be determined independently (or which simultaneous eigenvalues of the symmetry operators exist). Since the rest mass m is fixed by observation, we need only specify the 3-momentum, p_i, to characterize linear motion. From eqns. (9.155), we find that J_i and p_j do not commute so they are not (non-linearly) independent, but there is a rotation (or angular momentum) which does commute with p_j. It is called the *helicity* and is defined by

$$\lambda \equiv J_i \hat{p}^i, \tag{9.171}$$

where $\hat{p}^i$ is a unit vector in the direction of the spatial 3-momentum. The commutator then becomes

$$[p_i, J_j] p^j = \mathrm{i}\, \chi_h\, \epsilon_{ijk} p^k p^j = 0. \tag{9.172}$$

Thus, λ can be used to label the state of a field. A state vector is therefore characterized by the labels (‘quantum numbers’ in quantum mechanics)

$$|\Theta\rangle \equiv |m, j, p^i, \lambda\rangle, \tag{9.173}$$

i.e. the mass, the linear momentum, the highest weight of the rotational symmetry and the helicity. In a rest frame, the helicity becomes ill defined, so one must choose an arbitrary component of the spin, usually m_j as the limiting value.

We would like to know how these states transform under a given Poincaré transformation. Since the states, as constructed, are manifestly eigenstates of the momentum, a translation simply incurs a phase

$$|\Theta\rangle \to \exp\left(\mathrm{i}p^{\mu}a_{\mu}\right)|\Theta\rangle. \tag{9.174}$$

Homogeneous Lorentz transformations can be used to halt a moving state. The state $|m, j, p^i, \lambda\rangle$ can be obtained from $|m, j, 0, s_i\rangle$ by a rotation into the direction of p_i followed by a boost $\exp(\mathrm{i}\theta^i K_i)$ to set the frame in motion. Thus

$$|m, j, p^i, \lambda\rangle = L\,|m, j, 0, s_i\rangle. \tag{9.175}$$

The sub-group which leaves the momentum p_μ invariant is called the *little group* and can be used to classify the intrinsic rotational properties of a field. For massive fields in $3+1$ dimensions, the little group is covered by $SU(2)$, but this is not the case for massless fields.

Massless fields For massless fields, something special happens as a result of motion at the speed of light in a special direction. It is as though a field is squashed into a plane, and the rotational behaviour becomes two-dimensional and Abelian. The direction of motion decouples from the two orthogonal directions. Consider a state of the field

$$\Theta_\pi\rangle = |m, s, \pi, \lambda\rangle, \tag{9.176}$$

where the momentum $\pi_\mu = \pi(1, 0, 0, 1)$ is in the x^3 direction, and the Lorentz energy condition becomes $p^2c^2 = 0$ or $p_0 = |p_i|$. This represents a 'particle' travelling in the x^3 direction at the speed of light. The little group, which leaves p_μ invariant, may be found and is generated by

$$\begin{aligned}\Lambda_1 &= J_1 + K_1\\ \Lambda_2 &= J_1 - K_1\\ \Lambda_3 &= J_3.\end{aligned} \tag{9.177}$$

Clearly, the x^3 direction is privileged. These are the generators of the two-dimensional Euclidean group of translations and rotations called $ISO(2)$ or E_2. It is easily verified from the Poincaré group generators that the little group generators commute with the momentum operator

$$[\Lambda_i, p_\mu]\,|\Theta_\pi\rangle = 0. \tag{9.178}$$

The commutation relations for Λ_i are

$$\begin{aligned}[\Lambda_3, \Lambda_1] &= \mathrm{i}\Lambda_2 \\ [\Lambda_3, \Lambda_2] &= -\mathrm{i}\Lambda_1 \\ [\Lambda_1, \Lambda_2] &= 0.\end{aligned} \tag{9.179}$$

The last line signals the existence of an invariant sub-group. Indeed, one can define a Cartan–Weyl form and identify an invariant sub-algebra H,

$$\begin{aligned}E_\pm &= \Lambda_1 \pm \mathrm{i}\Lambda_2 \\ H &= \Lambda_3,\end{aligned} \tag{9.180}$$

with Casimir invariant

$$\begin{aligned}C_2 &= \Lambda_1^2 + \Lambda_2^2 \\ 0 &= [C_2, \Lambda_i].\end{aligned} \tag{9.181}$$

The stepping operators satisfy

$$[H, E_\pm] = \pm E_\pm, \tag{9.182}$$

i.e. $\Lambda_c = \pm 1$. This looks almost like the algebra for $su(2)$, but there is an important difference, namely the Casimir invariant. Λ_3 does not occur in the Casimir invariant since it would spoil its commutation properties (it has decoupled). This means that the value of $\Lambda_c = m_j$ is not restricted by the Schwarz inequality, as in section 8.5.10, to less than $\pm\Lambda_{\max} = \pm j$. The stepping operators still require the solutions for $\Lambda_c = m_j$ to be spaced by integers, but there is no upper or lower limit on the allowed values of the spin eigenvalues. In order to make this agree, at least in notation, with the massive case, we label physical states by Λ_3 only, taking

$$\Lambda_1|\Theta_\pi\rangle = \Lambda_2|\Theta_\pi\rangle = 0. \tag{9.183}$$

Thus, we may take the single value $H = \Lambda_3 = \Lambda_c = m_j = \lambda$ to be the angular momentum in the direction x^3, which is the helicity, since we have taken the momentum to point in this direction. See section 11.7.5 for further discussion on this point.

9.4.8 Curved spacetime: Killing's equation

In a curved spacetime, the result of an infinitesimal translation from a point can depend on the local curvature there, i.e. the translation is position-dependent. Consider an infinitesimal inhomogeneous translation $\epsilon^\mu(x)$, such that

$$x^\mu \to L^\mu{}_\nu x^\nu + \epsilon^\mu(x). \tag{9.184}$$

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Then we have

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = L^{\mu}_{\ \nu} + (\partial_{\nu}\epsilon^{\mu}), \tag{9.185}$$

and

$$\begin{aligned} ds'^2 &= g_{\mu\nu}\left(L^{\mu}_{\ \rho} + (\partial_{\rho}\epsilon^{\mu})\right)\left(L^{\nu}_{\ \sigma} + (\partial_{\sigma}\epsilon^{\nu})\right)dx^{\rho}dx^{\lambda} \\ &= g_{\mu\nu}\left[L^{\mu}_{\ \rho}L^{\nu}_{\ \lambda} + L^{\mu}_{\ \rho}(\partial_{\sigma}\epsilon^{\nu}) + (\partial_{\rho}\epsilon^{\mu})L^{\nu}_{\ \sigma} + \cdots + O(\epsilon^2)\right]dx^{\rho}dx^{\lambda}. \end{aligned} \tag{9.186}$$

The first term here vanishes, as above, owing to the anti-symmetry of $\omega_{\mu}^{\ \rho}$. Expanding the second term using eqn. (9.95), and remembering that both $\omega_{\mu\nu}$ and $\epsilon_{\mu}(x)$ are infinitesimal so that $\epsilon^{\mu}\omega_{\rho\sigma}$ is second-order and therefore negligible, we have an additional term, which must vanish if we are to have invariance of the line element:

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = 0. \tag{9.187}$$

The covariant generalization of this is clearly

$$\nabla_{\mu}\epsilon_{\nu} + \nabla_{\nu}\epsilon_{\mu} = 0. \tag{9.188}$$

This equation is known as Killing's equation, and it is a constraint on the allowed transformations, $\epsilon^{\mu}(x)$, which preserve the line element, in a spacetime which is curved. A vector, $\xi^{\mu}(x)$, which satisfies Killing's equation is called a Killing vector of the metric $g_{\mu\nu}$. Since this equation is symmetrical, it has $\frac{1}{2}(n+1)^2+(n+1)$ independent components. Since ξ^{μ} has only $n+1$ components, the solution is over-determined. However, there are $\frac{1}{2}(n+1)^2 - (n+1)$ anti-symmetric components in Killing's equation which are unaffected; thus there must be

$$m = (n+1) + \frac{1}{2}(n+1)^2 - (n+1) \tag{9.189}$$

free parameters in the Killing vector, in the form:

$$\begin{aligned} \nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} &= 0 \\ \xi_{\mu}(x) &= a_{\mu} + \omega_{\mu\nu}x^{\nu}, \end{aligned} \tag{9.190}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$. A manifold is said to be 'maximally symmetric' if it has the maximum number of Killing vectors, i.e. if the line element is invariant under the maximal number of transformations.

9.5 Galilean invariance

The relativity group which describes non-Einsteinian physics is the Galilean group. Like the Poincaré group, it contains translations, rotations and boosts.

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As a group, it is no smaller, and certainly no less complicated, than the Lorentz group. In fact, it may be derived as the $c \to \infty$ limit of the Poincaré group. But there is one conceptual simplification which makes Galilean transformations closer to our everyday experience: the absence of a cosmic speed limit means that arbitrary boosts of the Galilean transformations commute with one another. This alters the algebra of the generators.

9.5.1 Physical basis

The Galilean group applies physically to objects moving at speeds much less than the speed of light. For this reason, it cannot describe massless fields at all. The care required in distinguishing massless from massive concepts in the Poincaré algebra does not arise here for that simple reason. An infinitesimal Galilean transformation involves spatial and temporal translations, now written separately as

$$\begin{aligned} x^{i'} &= x^i + \delta x^i \\ t' &= t + \delta t, \end{aligned} \tag{9.191}$$

rotations by $\theta^i = \frac{1}{2}\epsilon^{ijk}\omega_{jk}$ and boosts by incremental velocity δv^i

$$x^{i'} = x^i - \delta v^i\, t. \tag{9.192}$$

This may be summarized by the standard infinitesimal transformation form

$$\begin{aligned} x^{i'} &= \left(1 + \frac{\mathrm{i}}{2}\omega_{lm} T^{lm}\right)^i{}_j\, x^j \\ x^{i'} &= (1 + \mathrm{i}\Theta)^i{}_j\, x^j, \end{aligned} \tag{9.193}$$

where the matrix

$$\Theta \equiv k_i \delta x^i - \tilde{\omega}\delta t + \theta_i T_B^i + \delta v_i\, T_E^i. \tag{9.194}$$

The exponentiated translational part of this is clearly a plane wave:

$$U \sim \exp \mathrm{i}(\mathbf{k}\cdot\delta\mathbf{x} - \tilde{\omega}\delta t). \tag{9.195}$$

Galilean transformations preserve the Euclidean scalar product

$$\mathbf{x}\cdot\mathbf{y} = x^i\, y_i. \tag{9.196}$$

9.5.2 Retardation and boosts

Retardation is the name given to the delay experienced in observing the effect of a phenomenon which happened at a finite distance from the source. The delay is caused by the finite speed of the disturbance. For example, the radiation at great distances from an antenna is retarded by the finite speed of light. A disturbance in a fluid caused at a distant point is only felt later because the disturbance travels at the finite speed of sound in the fluid. The change in momentum felt by a ballistic impulse in a solid or fluid travels at the speed of transport, i.e. the rate of flow of the fluid or the speed of projectiles emanating from the source.

Retardation expresses causality, and it is important in many physical problems. In Galilean physics, it is less important than in Einsteinian physics because cause and effect in a Galilean world (where $v \ll c$) are often assumed to be linked instantaneously. This is the Galilean approximation, which treats the speed of light as effectively infinite. However, retardation transformations become a useful tool in systems where the action is not invariant under boosts. This is because they allow us to derive a covariant form by transforming a non-covariant action. For example, the action for the Navier–Stokes equation can be viewed as a retarded snapshot of a particle field in motion. It is a snapshot because the action is not covariant with respect to boosts. We also derived a retarded view of the electromagnetic field arising from a particle in motion in section 7.3.4.

Retardation can be thought of as the opposite of a boost transformation. A boost transformation is characterized by a change in position due to a finite speed difference between two frames. In a frame x' moving with respect to a frame x we have

$$x^i(t)' = x^i(t) + v^i\, t. \tag{9.197}$$

Rather than changing the position variable, we can change the way we choose to measure time taken for the moving frame to run into an event which happened some distance from it:

$$t_{\text{ret}} = t - \frac{(x' - x)^i}{v^i}. \tag{9.198}$$

Whereas the idea of simultaneity makes this idea more complicated in the Einsteinian theory, here the retarded time is quite straightforward for constant velocity, v^i, between the frames. If we transform a system into a new frame, it is sometimes convenient to parametrize it in terms of a retarded time. To do this, we need to express both coordinates and derivatives in terms of the new quantity. Considering an infinitesimal retardation

$$t_{\text{ret}} = t - \frac{\mathrm{d}x^i}{v^i}, \tag{9.199}$$

it is possible to find the transformation rule for the time derivative, using the requirement that

$$\frac{\mathrm{d}t_{\text{ret}}}{\mathrm{d}t_{\text{ret}}} = 1. \tag{9.200}$$

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It may be verified that

$$\left[\partial_t + v^i\, \partial_i\right]\left[t - \frac{\mathrm{d}x^j}{v^j}\right] = 1. \tag{9.201}$$

Thus, one identifies

$$\frac{\mathrm{d}}{\mathrm{d}t_{\mathrm{ret}}} = \partial_t + v^i\, \partial_j. \tag{9.202}$$

This retarded time derivative is sometimes called the *substantive derivative*. In fluid dynamics books it is written

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t_{\mathrm{ret}}}. \tag{9.203}$$

It is simply the retarded-time total derivative. Compare this procedure with the form of the Navier–Stokes equation in section 7.5.1 and the field of a moving charge in section 7.3.4.

9.5.3 Generator algebra

The generators T_B and T_E are essentially the same generators as those which arise in the context of the Lorentz group in eqn. (9.116). The simplest way to derive the Galilean group algebra at this stage is to consider the $c \to \infty$ properties of the Poincaré group. The symbols T_B and T_E help to identify the origins and the role of the generators within the framework of Lorentzian symmetry, but they are cumbersome for more pedestrian work. Symbols for the generators, which are in common usage are

$$\begin{aligned} J^i &= T_B^i \\ N^i &= T_E^i. \end{aligned} \tag{9.204}$$

These are subtly different from, but clearly related to, the symbols used for rotations and boosts in the Poincaré algebra. The infinitesimal parameters, θ^a, of the group are

$$\theta^a = \left\{\delta t, \delta x^i, \theta^i, \delta v^i\right\}. \tag{9.205}$$

In $3 + 1$ dimensions, there are ten such parameters, as there are in the Poincaré group. These are related to the symbols of the Lorentz group by

$$\begin{aligned} \delta v_i &= \frac{1}{2}\omega_{0i} \\ \delta x^i &= \epsilon^i \\ \delta t &= \epsilon^0/c, \end{aligned} \tag{9.206}$$

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and

$$H + mc^2 = cp_0 = \chi_h \, c \, k_0$$
$$H = \chi_h \, \tilde{\omega} \, (= \hbar\tilde{\omega}). \tag{9.207}$$

Note that the zero point is shifted so that the energy H does not include the rest energy mc^2 of the field in the Galilean theory. This is a definition which only changes group elements by a phase and the action by an irrelevant constant. The algebraic properties of the generators are the $c \to \infty$ limit of the Poincaré algebra. They are summarized by the following commutators:

$$
\begin{aligned}
[k_i, k_j] &= 0 \\
[N_i, N_j] &= 0 \\
[H, k_i] &= 0 \\
[H, J_i] &= 0 \\
[H, N_i] &= \mathrm{i} \, \chi_h \, k_i \\
[k_i, J_l] &= -\mathrm{i} \, \chi_h \, \epsilon_{ilm} k_m \\
[k_i, N_j] &= \mathrm{i}m \, \chi_h \, \delta_{ij} \\
[J_i, N_l] &= \mathrm{i}\epsilon_{ilm} N_m \\
[J_i, J_j] &= \mathrm{i}\epsilon_{ijk} J_k,
\end{aligned} \tag{9.208}
$$

where $p_0/c \to m$ is the mass, having neglected $H/c = \chi_h \, \tilde{\omega}/c$. The Casimir invariants of the Galilean group are

$$J^i J_i, k^i K_i, N^i N_i. \tag{9.209}$$

The energy condition is now the limit of the Poincaré Casimir invariant, which is singular and asymmetrical:

$$\frac{p^i p_i}{2m} = E \tag{9.210}$$

(see section 13.5).

9.6 Conformal invariance

If we relax the condition that the line element $\mathrm{d}s^2$ must be preserved, and require it only to transform isotropically (which preserves $\mathrm{d}s^2 = 0$), then we can allow transformations of the form

$$
\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\
&\to \Omega^2(x) \left(-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right),
\end{aligned} \tag{9.211}
$$

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where $\Omega(x)$ is a non-singular, non-vanishing function of x^μ. In the action, we combine this with a similar transformation of the fields, e.g. in $n+1$ dimensions,

$$\phi(x) \to \Omega^{(1-n)/2}\phi(x). \tag{9.212}$$

This transformation stretches spacetime into a new shape by deforming it with the function $\Omega(x)$ equally in all directions. For this reason, the conformal transformation preserves the angle between any two lines which meet at a vertex, even though it might bend straight lines into curves or vice versa.

Conformal transformations are important in physics for several reasons. They represent a deviation from systems of purely Markov processes. If a translation in spacetime is accompanied by a change in the environment, then the state of the system must depend on the history of changes which occurred in the environment. This occurs, for instance, in the curvature of spacetime, where parallel transport is sensitive to local curvature; it also occurs in gauge theories, where a change in a field's internal variables (gauge transformation) accompanies translations in spacetime, and in non-equilibrium statistical physics where the environment changes alongside dynamical processes, leading to conformal distortions of the phase space. Conformal symmetry has many applications.

Because the conformal transformation is a scaling of the metric tensor, its effect is different for different kinds of fields and their interactions. The number of powers of the metric which occurs in the action (or, loosely speaking, the number of spacetime indices on the fields) makes the invariance properties of the action and the field equations quite different. Amongst all the fields, Maxwell's free equations (a massless vector field in) in $3 + 1$ dimensions stand out for their general conformal invariance. This leads to several useful properties of Maxwell's equations, which many authors unknowingly take for granted. Scalar fields are somewhat different, and are conformally invariant in $1+1$ dimensions, in the massless case, in the absence of self-interactions. We shall consider these two cases below.

Consider now an infinitesimal change of coordinates, as we did in the case of Lorentz transformations:

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + \epsilon^\mu(x). \tag{9.213}$$

The line element need not be invariant any longer; it may change by

$$\mathrm{d}s^{2'} = \Omega^2(x)\,\mathrm{d}s^2. \tag{9.214}$$

Following the same procedure as in eqn. (9.186), we obtain now a condition for eqn. (9.214) to be true. To first order, we have:

$$\Omega^2(x) g_{\mu\nu} = g_{\mu\nu} + \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu. \tag{9.215}$$

Clearly, ϵ^μ and $\Omega(x)$ must be related in order to satisfy this condition. The relationship is easily obtained by taking the trace of this equation, multiplying

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through by $g^{\mu\nu}$. This gives, in $n+1$ dimensions,

$$(\Omega^2 - 1)(n+1) = 2(\partial_\lambda \epsilon^\lambda). \tag{9.216}$$

Using this to replace $\Omega(x)$ in eqn. (9.215) gives us the equation analogous to eqn. (9.187), but now for the full conformal symmetry:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{(n+1)}(\partial_\lambda \epsilon^\lambda) g_{\mu\nu}. \tag{9.217}$$

This is the Killing equation for the conformal symmetry. Its general solution in $n+1$ dimensions, for $n > 1$, is

$$\epsilon^\mu(x) = a^\mu + b x^\mu + \omega^{\mu\nu} x_\nu + 2x^\mu c^\nu x_\nu - c^\mu x^2, \tag{9.218}$$

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$. In $(1+1)$ dimensional Minkowski space, eqn. (9.217) reduces to two equations

$$\begin{aligned} \partial_0 \epsilon_0 &= -\partial_1 \epsilon_1 \\ \partial_0 \epsilon_1 &= -\partial_1 \epsilon_0. \end{aligned} \tag{9.219}$$

In two-dimensional Euclidean space, i.e. $n = 1$, followed by a Wick rotation to a positive definite metric, this equation reduces simply to the Cauchy–Riemann relations for $\epsilon^\mu(x)$, which is solved by any analytic function in the complex plane. After a Wick rotation, one has

$$\begin{aligned} \partial_0 \epsilon_0 &= \partial_1 \epsilon_1 \\ \partial_0 \epsilon_1 &= -\partial_1 \epsilon_0. \end{aligned} \tag{9.220}$$

To see that this is simply the Cauchy–Riemann relations,

$$\frac{\mathrm{d}}{\mathrm{d}z^*} f(z) = 0, \tag{9.221}$$

we make the identification

$$\begin{aligned} z &= x^0 + \mathrm{i}x^1 \\ f(z) &= \epsilon_0 + \mathrm{i}\epsilon_1 \end{aligned} \tag{9.222}$$

and note that

$$\frac{\mathrm{d}}{\mathrm{d}z^*} = \frac{1}{2}\left(\partial_0 + \mathrm{i}\partial_1\right). \tag{9.223}$$

This property of two-dimensional Euclidean space reflects the well known property of analytic functions in the complex plane, namely that they all are conformally invariant and solve Laplace's equation:

$$\nabla^2 f(x^i) = 4\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}z^*} f(z) = 0. \tag{9.224}$$

It makes two-dimensional, conformal field theory very interesting. In particular it is important for string theory and condensed matter physics of critical phenomena, since the special analyticity allows one to obtain Green functions and conservation laws in the vicinity of so-called fixed points.

9.6.1 Scalar fields in $n+1$ dimensions

We begin by writing down the action, making the appearance of the metric explicit:

$$S = \int \mathrm{d}^{n+1}x \sqrt{g} \frac{1}{c} \left\{ \frac{1}{2} (\partial_\mu \phi) g^{\mu\nu} (\partial_\nu \phi) + V(\phi) - J\phi \right\}. \tag{9.225}$$

Note the factor of the determinant of the metric in the volume measure: this will also scale in the conformal transformation. We now let

$$\begin{aligned} g_{\mu\nu} &\to \Omega^2(x) \overline{g}_{\mu\nu} \\ g &\to \Omega^{2(n+1)}(x)\, \overline{g} \\ \phi(x) &\to \Omega^{(1-n)/2}(x) \overline{\phi}(x) \\ J &\to \Omega^{\alpha}(x) \overline{J}, \end{aligned} \tag{9.226}$$

where α is presently unknown. It is also useful to define the 'connection' $\Gamma_\mu = \Omega^{-1}\partial_\mu \Omega$. We now examine the variation of the action under this transformation:

$$\begin{aligned} \delta S = \int \mathrm{d}^{n+1}x \sqrt{\overline{g}} \Omega^{n+1} \frac{1}{c} \Big\{ & (\partial_\mu \Omega^{(1-n)/2} \delta\overline{\phi}) \frac{\overline{g}^{\mu\nu}}{\Omega^2} (\partial_\nu \Omega^{(1-n)/2} \overline{\phi}) \\ & + \delta V - \Omega^{(1-n)/2+\alpha} \overline{J}\, \delta\overline{\phi} \Big\}. \end{aligned} \tag{9.227}$$

Integrating by parts to separate $\delta\overline{\phi}$ gives

$$\begin{aligned} \delta S = \int \mathrm{d}^{n+1}x\, \sqrt{\overline{g}}\, \Omega^{n+1} \frac{1}{c} & \\ \Big\{ -(1+n-2)\Gamma_\mu\, & \Omega^{(1-n)/2-2} \delta\overline{\phi}\overline{g}^{\mu\nu} (\partial_\nu \Omega^{(1-n)/2} \overline{\phi}) \\ - \Omega^{(1-n)/2-2} & \delta\overline{\phi}\overline{g}^{\mu\nu} (\partial_\mu \partial_\nu \Omega^{(1-n)/2} \overline{\phi}) + \delta V \Big\}. \end{aligned} \tag{9.228}$$

Notice how the extra terms involving Γ_μ, which arise from derivatives acting on Ω, are proportional to $(1+n-2) = n-1$. These will clearly vanish in $n = 1$ dimensions, and thus we see how $n = 1$ is special for the scalar field. To fully express the action in terms of barred quantities, we now need to commute the factors of Ω through the remaining derivatives and cancel them against the factors in the integration measure. Each time $\Omega^{(1-n)/2}$ passes through

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a derivative, we pick up a term containing $\frac{1}{2}(1-n)\Gamma_\mu$, thus, provided we have $\alpha = -(n+3)/2$ and $\delta V = 0$, we may write

$$\delta S = \int \mathrm{d}^{n+1}x \,\sqrt{\overline{g}}\, \frac{1}{c} \left\{ -\overline{\Box}\, \overline{\phi} - \overline{J} \right\} \delta\overline{\phi} + \text{terms} \times (n-1). \tag{9.229}$$

Clearly, in 1 + 1 dimensions, this equation is conformally invariant, provided the source J transforms in the correct way, and the potential V vanishes. The invariant equation of motion is

$$-\overline{\Box}\, \overline{\phi}(x) = \overline{J}. \tag{9.230}$$

9.6.2 *The Maxwell field in $n+1$ dimensions*

The conformal properties of the Maxwell action are quite different to those of the scalar field, since the Maxwell action contains two powers of the inverse metric, rather than one. Moreover, the vector source coupling $J^\mu A_\mu$ contains a power of the inverse metric because of the indices on A_μ. Writing the action with metric explicit, we have

$$S = \int \mathrm{d}^{n+1}x \sqrt{g} \frac{1}{c} \left\{ \frac{1}{4} F_{\mu\nu} g^{\mu\rho} g^{\nu\lambda} F_{\rho\lambda} + J_\mu g^{\mu\nu} A_\nu \right\}. \tag{9.231}$$

We now re-scale, as before, but with slightly different dimensional factors

$$\begin{aligned} g_{\mu\nu} &\to \Omega^2(x) \overline{g}_{\mu\nu} \\ g &\to \Omega^{2(n+1)}(x)\, \overline{g} \\ A_\mu(x) &\to \Omega^{(3-n)/2}(x) \overline{A}_\mu(x) \\ J_\mu &\to \Omega^\alpha \overline{J}_\mu, \end{aligned} \tag{9.232}$$

and vary the action to find the field equations:

$$\begin{aligned} \delta S = \int \mathrm{d}^{n+1}x \sqrt{\overline{g}} \Omega^{n+1} \frac{1}{c} \Big\{ &\partial_\mu (\delta \overline{A}_\nu \Omega^{(3-n)/2}) \frac{\overline{g}^{\mu\rho} \overline{g}^{\nu\lambda}}{\Omega^4} F_{\rho\lambda} \\ &+ \overline{J}_\mu \overline{g}^{\mu\nu} \Omega^{(3-n)/2-2+\alpha} \delta \overline{A}_\nu \Big\}. \end{aligned} \tag{9.233}$$

Integrating by parts, we obtain

$$\begin{aligned} \delta S = \int \mathrm{d}^{n+1}x \, \frac{1}{c} \sqrt{\overline{g}}\, \Omega^{(n-3)/2}\, \delta \overline{A}_\nu \Big\{ &(n-3) \Gamma_\mu \overline{g}^{\mu\rho} \overline{g}^{\nu\lambda} F_{\rho\lambda} \\ &- \partial_\mu F_{\rho\lambda} \overline{g}^{\mu\rho} \overline{g}^{\nu\lambda} + \overline{J}_\mu \overline{g}^{\mu\nu} \Omega^{\alpha-2} \Big\}. \end{aligned} \tag{9.234}$$

On commuting the scale factor through the derivatives using

$$\partial_\mu F_{\rho\lambda} = \frac{1}{2}(3-n) \partial_\mu \left[\Gamma_\rho \overline{A}_\lambda - \Gamma_\lambda \overline{A}_\rho \right] + \partial_\mu \overline{F}_{\rho\lambda}, \tag{9.235}$$

we see that we acquire further terms proportional to $n - 3$. Three dimensions clearly has a special significance for Maxwell's equations, so let us choose $n = 3$ now and use the notation $\overline{\partial}_\mu$ to denote the fact that the derivative is contracted using the transformed metric $\overline{g}_{\mu\nu}$. This gives

$$\delta S = \int \mathrm{d}^{n+1}x \frac{1}{c}\sqrt{g}\left\{-\overline{\partial}_\mu \overline{F}^{\mu\nu} + \overline{J}^\nu \Omega^{\alpha-2}\right\}\delta\overline{A}_\nu = 0. \tag{9.236}$$

Notice that the invariance of the equations of motion, in the presence of a current, depends on how the current itself scales. Suppose we couple to the current arising from a scalar field which has the general form $J_\mu \sim \phi^*\partial_\mu\phi$, then, from the previous section, this would scale by Ω^{n-1}. For $n = 1$, this gives precisely $\alpha = n - 1 = 2$. Note, however, that the matter field itself is not conformally invariant in $n = 3$. As far as the electromagnetic sector is concerned, however, $n = 3$ gives us the conformally invariant equation of motion

$$\overline{\partial}_\mu \overline{F}^{\mu\nu} = \overline{J}^\nu. \tag{9.237}$$

The above treatment covers only two of the four Maxwell's equations. The others arise from the Bianchi identity,

$$\epsilon^{\mu\nu\lambda\rho}\partial_\mu F_{\lambda\rho} = 0. \tag{9.238}$$

The important thing to notice about this equation is that it is independent of the metric. All contractions are with the metric-independent, anti-symmetric tensor; the other point is precisely that it is anti-symmetric. Moreover, the field scale factor $\Omega^{3-n}/2$ is simply unity in $n = 3$, thus the remaining Maxwell equations are trivially invariant.

In non-conformal dimensions, the boundary terms are also affected by the scale factor, Ω. The conformal distortion changes the shape of a boundary, which must be compensated for by the other terms. Since the dimension in which gauge fields are invariant is different to the dimension in which matter fields are invariant, no gauge theory can be conformally invariant in flat spacetime. Conformally improved matter theories can be formulated in curved spacetime, however, in any number of dimensions (see section 11.6.3).

9.7 Scale invariance

Conformal invariance is an exacting symmetry. If we relax the x-dependence of $\Omega(x)$ and treat it as a constant, then there are further possibilities for invariance of the action. Consider

$$S = \int (\mathrm{d}x)\left\{\frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) + \sum_l \frac{1}{l!}\, a_l\, \phi^l\right\}. \tag{9.239}$$

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Table 9.2. Scale-invariant potentials.

$n = 1$	$n = 2$	$n = 3$
All	$\frac{1}{6!} g\phi^6$	$\frac{1}{4!}\lambda\phi^4$

Let us scale

$$\begin{aligned} g_{\mu\nu} &\to \overline{g}_{\mu\nu}\ \Omega^2 \\ \phi(x) &\to \overline{\phi}(x)\ \Omega^{-\alpha}, \end{aligned} \tag{9.240}$$

where α is to be determined. Since the scale factors now commute with the derivatives, we can secure the invariance of the action for certain l which satisfy,

$$\Omega^{n+1}\,\Omega^{-2-2\alpha} = 1 = \Omega^{-l\alpha}, \tag{9.241}$$

which solves to give $\alpha = \frac{n+1}{2} - 1$, and hence,

$$l = \frac{n+1}{(n+1)/2 - 1}. \tag{9.242}$$

For $n = 3$, $l = 4$ solves this; for $n = 2$, $l = 6$ solves this; and for $n = 1$, it is not solved for any l since the field is dimensionless. We therefore have the globally scale-invariant potentials in table 9.2.

9.8 Breaking spacetime symmetry

The breakdown of a symmetry means that a constraint on the uniformity of a system is lost. This sometimes happens if systems develop structure. For example, if a uniformly homogeneous system suddenly becomes lumpy, perhaps because of a phase transition, then translational symmetry will be lost. If a uniform external magnetic field is applied to a system, rotational invariance is lost. When effects like these occur, one or more symmetry generators are effectively lost, together with the effect on any associated eigenvalues of the symmetry group. In a sense, the loss of a constraint opens up the possibility of more freedom or more variety in the system. In the opposite sense, it restricts the type of transformations which leave the system unchanged. Symmetry breakdown is often associated with the lifting of *degeneracy* of group eigenvalues, or quantum numbers.

There is another sense in which symmetry is said to be broken. Some calculational procedures break symmetries in the sense that they invalidate the

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assumptions of the original symmetry. For example, the imposition of periodic boundary conditions on a field in a crystal lattice is sometimes said to break Lorentz invariance,

$$\psi(x + L) = \psi(x). \tag{9.243}$$

The existence of a topological property such as periodicity does not itself break the Lorentz symmetry. If there is a loss of homogeneity, then translational invariance would be lost, but eqn. (9.243) does not imply this in any way: it is purely an identification of points in the system at which the wavefunction should have a given value. The field still transforms faithfully as a spacetime scalar. However, the condition in eqn. (9.243) does invalidate the assumptions of Lorentz invariance because the periodicity length L is a constant and we know that a boost in the direction of that periodicity would cause a length contraction. In other words, the fact that the boundary conditions themselves are stated in a way which is not covariant invalidates the underlying symmetry.

Another example is the imposition of a finite temperature scale $\beta = 1/kT$. This is related to the last example because, in the Euclidean representation, a finite temperature system is represented as being periodic in imaginary time (see section 6.1.5). But whether we use imaginary time or not, the idea of a constant temperature is also a non-covariant concept. If we start in a heat bath and perform a boost, the temperature will appear to change because of the Doppler shift. Radiation will be red- and blue-shifted in the direction of travel, and thus it is only meaningful to measure a temperature at right angles to the direction of travel. Again, the assumption of constant temperature does not break any symmetry of spacetime, but the ignorance of the fact that temperature is a function of the motion leads to a contradiction.

These last examples cannot be regarded as a breakdown of symmetry, because they are not properties of the system which are lost, they are only a violation of symmetry by the assumptions of a calculational procedure.

9.9 Example: Navier–Stokes equations

Consider the action for the velocity field:

$$S = \tau \int (\mathrm{d}x) \left\{ \frac{1}{2}\rho v_i (D_t v^i) + \rho v^i v^j (D^k_{ij} v_k) + \frac{\mu}{2}(\partial_i v^i)^2 + J_i v^i \right\}, \tag{9.244}$$

where

$$J_i \equiv F_i + \partial_i P, \tag{9.245}$$

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and

$$D_t = \partial_t + \Gamma = \partial_t + \frac{1}{2}\left(\frac{\partial_t \rho}{\rho}\right)$$

$$D^k_{ij} = \delta^l{}_l \delta^m_j \, \partial^k + \Gamma^k_{ij} = \delta^l{}_l \delta^m_j \, \partial^k + \frac{v_i v_j}{v^4} \, \partial_m (v^m v^k), \tag{9.246}$$

$$\rho \frac{\mathrm{D} v^i}{\mathrm{D} t} + (\partial_i P) - \mu \nabla^2 v^i = F^i, \tag{9.247}$$

where P is the pressure and F is a generalized force. This might be the effect of gravity or an electric field in the case of a charged fluid.

These connections result from the spacetime dependence of the coordinate transformation. They imply that our transformation belongs to the conformal group rather than the Galilean group, and thus we end up with connection terms

$$\frac{\mathrm{D} u^i}{\mathrm{D} t} = (\partial_t + v^j \partial_j) v^i, \tag{9.248}$$

where

$$\partial_\mu N^\mu = 0 \tag{9.249}$$

and $N^\mu = (N, N v^i)$.

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