Bull. Aust. Math. Soc. **93** (2016), 19–30
doi:10.1017/S0004972715000623

A NOTE ON SIMPLE ZEROS OF PRIMITIVE DIRICHLET *L*-FUNCTIONS

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(Received 24 March 2015; accepted 8 April 2015; first published online 8 July 2015)

Abstract

In this paper, by using the theory of reproducing kernel Hilbert spaces and the pair correlation formula constructed by Chandee *et al.* ['Simple zeros of primitive Dirichlet L-functions and the asymptotic large sieve', *Q. J. Math.* **65**(1) (2014), 63–87], we prove that at least 93.22% of low-lying zeros of primitive Dirichlet L-functions are simple in a proper sense, under the assumption of the generalised Riemann hypothesis.

2010 *Mathematics subject classification*: primary 11M26.

Keywords and phrases: primitive Dirichlet L-functions, simple zeros, pair correlation formula, reproducing kernel.

1. Introduction

It is generally conjectured that all nontrivial zeros of the Riemann zeta function (or other arithmetical L-functions) lie on the central line of the functional equation and that these zeros are all simple. The latter statement is called the simple zero conjecture. One of the standard approaches to this conjecture is to make use of the pair correlation function under the assumption of the Riemann hypothesis (RH) for the relevant L-function. In the celebrated paper [6], assuming the RH, Montgomery obtained a fundamental formula on pair correlation of zeros of the Riemann zeta function $\zeta(s)$ on the critical line $\mathrm{Re}(s) = 1/2$. Using this formula, he proved that for any $\epsilon > 0$, at least $2/3 - \epsilon$ of nontrivial zeros of $\zeta(s)$ are simple, assuming RH. Slightly later, Montgomery and Taylor [7] improved the proportion of simple zeros to $3/2 - 2^{-1/2}\cot(2^{-1/2}) - \epsilon = 0.67250\ldots$ by using a variational argument, and Gallagher [5] has treated a slightly more general problem. There are several results on the proportion of simple zeros of other L-functions. For example, Özlük [8] considered the q-analogue of the pair correlation conjecture. Assuming the generalised Riemann hypothesis (GRH), he constructed certain formulas for the weighted pair correlation functions of zeros of Dirichlet L-functions. The weight emphasises the nontrivial zeros near the real axis. Using his formula, he proved that at least 11/12 of the zeros of

Dirichlet L-functions, averaged over characters (both primitive and nonprimitive ones) and conductors, are simple in a proper sense.

Recently Conrey *et al.* [4] developed a new method named the asymptotic large sieve. Roughly speaking, their method is an asymptotic version of the large sieve inequality for linear forms of Dirichlet characters, restricted to only primitive ones. Using their techniques, several remarkable results on primitive L-functions have been obtained. One such example is the main theorem of Chandee *et al.* [2]. Throughout their paper (and this paper), the GRH is assumed. Let Φ be a real-valued smooth function which has compact support in (a, b) with $0 < a < b < \infty$, and let

$$\widehat{\Phi}(s) = \int_0^\infty \Phi(x)x^{s-1}\,dx$$

be its Mellin transform. We put

$$N_\Phi(Q) = \sum_q \frac{W(q/Q)}{\phi(q)} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} \sum_{\gamma_\chi} |\widehat{\Phi}(i\gamma_\chi)|^2,$$

where W is a smooth function compactly supported in $(1, 2)$, the second sum is over all primitive characters χ modulo q and the last sum is over all nontrivial zeros $1/2 + i\gamma_\chi$ of $L(s,\chi)$. It is proved in [2, Lemma 1] that

$$N_\Phi(Q) \sim \frac{A}{2\pi} Q \log Q \int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2\,dx,$$

where

$$A = \widehat{W}(1) \prod_p \Big(1 - \frac{1}{p^2} - \frac{1}{p^3}\Big).$$

For $Q > 1$ and $\alpha \in \mathbf{R}$, we define the pair correlation function F_Φ by

$$F_\Phi(Q^\alpha; W) = \frac{1}{N_\Phi(Q)} \sum_q \frac{W(q/Q)}{\phi(q)} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} \Big|\sum_{\gamma_\chi} \widehat{\Phi}(i\gamma_\chi) Q^{i\gamma_\chi\alpha}\Big|^2. \tag{1.1}$$

By using the asymptotic large sieve, they proved that for any $\epsilon > 0$, the asymptotic formula

$$\begin{aligned} F_\Phi(Q^\alpha; W) = (1 + o(1))\Big(f(\alpha) + \Phi(Q^{-|\alpha|})^2 \log Q\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2\,dx\Big)^{-1}\Big) \\ + O(\Phi(Q^{-|\alpha|})\sqrt{f(\alpha)\log Q}) \end{aligned} \tag{1.2}$$

holds uniformly for $|\alpha| \le 2 - \epsilon$ as $Q \to \infty$, where

$$f(\alpha) := \begin{cases} |\alpha| & \text{if } |\alpha| \le 1, \\ 1 & \text{if } |\alpha| > 1. \end{cases}$$

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As a corollary, they proved that assuming the GRH, the proportion of simple zeros of primitive Dirichlet L-functions is at least $11/12 = 0.917\ldots$ in the sense that the inequality

$$\sum_q \frac{W(q/Q)}{\phi(q)} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} \sum_{\substack{\gamma_\chi \\ \text{simple}}} |\widehat{\Phi}(i\gamma_\chi)|^2 \geq \Big(\frac{11}{12} + o(1)\Big) N_\Phi(Q) \tag{1.3}$$

holds as $Q \to \infty$, where the function Φ is chosen such that

$$\widehat{\Phi}(ix) = (\sin x/x)^2.$$

They note that this Φ does not satisfy the condition for smoothness, but their asymptotic formula (1.2) remains valid for this Φ, since the condition $\widehat{\Phi}(ix) \ll |x|^{-2}$ is good enough for their proof. We also use this function throughout this paper.

The aim of this paper is to improve the arguments in [2] and obtain a better lower bound for the left-hand side of (1.3). With the notation above, we now describe the main theorem of this paper.

Theorem 1.1. *Assume the GRH. Then*

$$\sum_q \frac{W(q/Q)}{\phi(q)} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} \sum_{\substack{\gamma_\chi \\ \text{simple}}} \widehat{\Phi}(i\gamma_\chi)^2 \geq (M + o(1)) N_\Phi(Q)$$

as $Q \to \infty$. Here,

$$\begin{aligned} M &:= 1 - \int_{-\infty}^{\infty} g(\beta)\Big(1 - \Big(\frac{\sin \pi\beta}{\pi\beta}\Big)^2\Big)\, d\beta \\ &= 0.93228262\ldots, \end{aligned} \tag{1.4}$$

where the function g is given by

$$\begin{aligned} g(x) = \frac{\sin^2 1}{16c_1^2(1-\cos 1)^2}\Bigg(&\frac{-\frac{2(1-\cos 1)}{\sin 1}(\cos 2\pi x + 1) + 4\pi x \sin 2\pi x}{4\pi^2x^2 - 1} c_1 \\ &- \frac{\frac{2\sqrt{3}(1+\cos\sqrt{3})}{\sin\sqrt{3}}(\cos 2\pi x - 1) + 4\pi x \sin 2\pi x}{4\pi^2 x^2 - 3} c_2\Bigg)^2. \end{aligned} \tag{1.5}$$

The constants c_1, c_2 are given by

$$\begin{aligned} c_1 &= \Big(\cos 1 - \frac{(1-\cos 1)(2+\cos\sqrt{3})\sin\sqrt{3}}{3\sqrt{3}\sin 1(1+\cos\sqrt{3})} \\ &\qquad - \frac{(1-\cos 1)(4\sqrt{3} - 3\sqrt{3}\sin 1 - \sin\sqrt{3})}{3\sqrt{3}\sin 1}\Big)^{-1} \\ &= 6.757821\ldots, \\ c_2 &= \frac{(1-\cos 1)\sin\sqrt{3}}{\sqrt{3}\sin 1(1+\cos\sqrt{3})} c_1 = 2.506205\ldots. \end{aligned}$$

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In [2], the function $r(x) = (\sin 2\pi x/2\pi x)^2$ is used instead of $g(x)$. This $r(x)$ is rather useful, since its Fourier transform is quite simple and it gives a result close to the best for our aim of evaluating the proportion of simple zeros. On the other hand, the choice of the function $g(x)$ is the best for our purposes because it maximises the value of M defined by (1.4) under the conditions on $g(x)$ given in the next section. To obtain this function, we use the theory of reproducing kernel Hilbert spaces. We solve a differential equation characterising the reproducing kernel for the relevant Hilbert space, and by normalising its solution at the origin, we obtain (1.5).

Remark 1.2.

(1) We note that the singularities of the function $g(x)$ defined by (1.5) at $1/(2\pi)$ and $\sqrt{3}/(2\pi)$ are removable. The function $g(x)$ is an L^1-function on the real line (since $g(x) = O(x^{-2})$ as $x \to \pm\infty$) and is entire if we extend it to the whole complex plane.

(2) The value of M defined by (1.4) is explicitly given by

$$M = 1 - \frac{\sin 1}{4c_1(1 - \cos 1)}.$$

See the argument in Remark 4.1 for details.

2. Preparations for the proof

The basic idea for evaluating the proportion of simple zeros is the same as in [2]. We will explain briefly how the asymptotic formula (1.2) works to give a certain lower bound for the proportion of simple zeros. Let $g(x) \in L^1(\mathbf{R})$ be a positive-valued even function normalised so that $g(0) = 1$. We also assume that its Fourier transform $\tilde{g}(u)$ has a compact support in $[-2, 2]$. First, by the definition of the pair correlation function (1.1),

$$\frac{1}{N_\Phi(Q)} \sum_q \frac{W(q/Q)}{\phi(q)} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} \sum_{\gamma_\chi, \gamma'_\chi} \widehat{\Phi}(i\gamma_\chi)\widehat{\Phi}(i\gamma'_\chi) g\Big(\frac{(\gamma_\chi - \gamma'_\chi)\log Q}{2\pi}\Big)$$
$$= \int_{-\infty}^{\infty} F_\Phi(Q^\beta; W)\tilde{g}(\beta)\, d\beta. \tag{2.1}$$

As can be seen in many papers dealing with pair correlation functions, it causes no difference to our lower bound if we apply the asymptotic formula (1.2) for $|\alpha| \le 2$. By (1.2) with our assumption, the right-hand side of (2.1) is

$$\int_{-\infty}^{\infty} F_\Phi(Q^\beta; W)\tilde{g}(\beta)\, d\beta \sim \int_{-2}^{2} f(\beta)\tilde{g}(\beta)\, d\beta$$
$$+ (\log Q)\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{\Phi}(ix)|^2\, dx\Big)^{-1} \int_{-2}^{2} \Phi(Q^{-|\beta|})^2 \tilde{g}(\beta)\, d\beta. \tag{2.2}$$

The second term of the right-hand side of (2.2) is

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$$\sim \tilde{g}(0)(\log Q)\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty}|\widehat{\Phi}(ix)|^2\,dx\Big)^{-1}\int_{-\infty}^{\infty}\Phi(Q^{-|\beta|})^2\,d\beta$$
$$= \tilde{g}(0)\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty}|\widehat{\Phi}(ix)|^2\,dx\Big)^{-1}\int_{-\infty}^{\infty}\Phi(e^{-|\beta|})^2\,d\beta = \tilde{g}(0).$$

Here, we use Plancherel's formula for the Mellin transform

$$\int_{-\infty}^{\infty}\Phi(e^{-|\beta|})^2\,d\beta = \int_{-\infty}^{\infty}\Phi(e^{-\beta})^2\,d\beta = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\widehat{\Phi}(ix)|^2\,dx,$$

and the fact that the function Φ satisfies $\Phi(x) = \Phi(x^{-1})$ (see [2]). Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty}F_{\Phi}(Q^{\beta};W)\tilde{g}(\beta)\,d\beta &\sim \tilde{g}(0) + \int_{-2}^{2}f(\beta)\tilde{g}(\beta)\,d\beta\\ &= \int_{-\infty}^{\infty}(\delta(\beta)+f(\beta))\tilde{g}(\beta)\,d\beta\\ &= \int_{-\infty}^{\infty}\Big(1+\Big(\delta(\beta)-\Big(\frac{\sin\pi\beta}{\pi\beta}\Big)^2\Big)\Big)g(\beta)\,d\beta\\ &= g(0) + \int_{-\infty}^{\infty}g(\beta)\Big(1-\Big(\frac{\sin\pi\beta}{\pi\beta}\Big)^2\Big)\,d\beta, \qquad (2.3)\end{aligned}$$

where $\delta(\beta)$ is the Dirac delta function. Here, we use Parseval's theorem and the fact that the Fourier transform of f is given by

$$\tilde{f}(\beta) = \delta(\beta) - \Big(\frac{\sin\pi\beta}{\pi\beta}\Big)^2.$$

Combining (2.1) and (2.3), we obtain the following formula.

Lemma 2.1. *Assume the GRH. For any function g satisfying the conditions above,*

$$\frac{1}{N_{\Phi}(Q)}\sum_{q}\frac{W(q/Q)}{\phi(q)}\sideset{}{^*}\sum_{\chi(\mathrm{mod}\,q)}\sum_{\gamma_\chi,\gamma'_\chi}\widehat{\Phi}(i\gamma_\chi)\widehat{\Phi}(i\gamma'_\chi)g\Big(\frac{(\gamma_\chi-\gamma'_\chi)\log Q}{2\pi}\Big)$$
$$= 1 + \int_{-\infty}^{\infty}g(\beta)\Big(1-\Big(\frac{\sin\pi\beta}{\pi\beta}\Big)^2\Big)\,d\beta + o(1). \qquad (2.4)$$

We denote by m_{ρ_χ} the multiplicity of the zero $\rho_\chi = 1/2 + i\gamma_\chi$ of $L(s,\chi)$. Since $g(0) = 1$,

$$\begin{aligned}\sum_{\gamma_\chi\ \text{simple}}\widehat{\Phi}(i\gamma_\chi)^2 &\geq \sum_{\gamma_\chi}(2-m_{\rho_\chi})\widehat{\Phi}(i\gamma_\chi)^2\\ &= 2\sum_{\gamma_\chi}\widehat{\Phi}(i\gamma_\chi)^2 - \sum_{\gamma_\chi,\gamma'_\chi}g\Big(\frac{(\gamma_\chi-\gamma'_\chi)\log Q}{2\pi}\Big)\widehat{\Phi}(i\gamma_\chi)\widehat{\Phi}(i\gamma'_\chi)\\ &\quad + \sum_{\substack{\gamma_\chi,\gamma'_\chi\\ \gamma_\chi\neq\gamma'_\chi}}g\Big(\frac{(\gamma_\chi-\gamma'_\chi)\log Q}{2\pi}\Big)\widehat{\Phi}(i\gamma_\chi)\widehat{\Phi}(i\gamma'_\chi).\end{aligned}$$

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Therefore, by the definition of $N_\Phi(Q)$ and (2.4),

$$\begin{aligned}
&\frac{1}{N_\Phi(Q)}\sum_q \frac{W(q/Q)}{\phi(q)} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} \sum_{\substack{\gamma_\chi \\ \text{simple}}} \widehat{\Phi}(i\gamma_\chi)^2 \\
&\quad \geq \frac{2}{N_\Phi(Q)}\sum_q \frac{W(q/Q)}{\phi(q)} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} \sum_{\gamma_\chi} \widehat{\Phi}(i\gamma_\chi)^2 \\
&\qquad - \frac{1}{N_\Phi(Q)}\sum_q \frac{W(q/Q)}{\phi(q)} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} \sum_{\gamma_\chi, \gamma'_\chi} g\Big(\frac{(\gamma_\chi - \gamma'_\chi)\log Q}{2\pi}\Big)\widehat{\Phi}(i\gamma_\chi)\widehat{\Phi}(i\gamma'_\chi) \\
&\qquad + \frac{1}{N_\Phi(Q)}\sum_q \frac{W(q/Q)}{\phi(q)} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} \sum_{\substack{\gamma_\chi, \gamma'_\chi \\ \gamma_\chi \neq \gamma'_\chi}} g\Big(\frac{(\gamma_\chi - \gamma'_\chi)\log Q}{2\pi}\Big)\widehat{\Phi}(i\gamma_\chi)\widehat{\Phi}(i\gamma'_\chi) \\
&\quad \geq 1 - \int_{-\infty}^{\infty} g(\beta)\Big(1 - \Big(\frac{\sin \pi\beta}{\pi\beta}\Big)^2\Big)\, d\beta. \qquad (2.5)
\end{aligned}$$

We would like to minimise the second term on the right-hand side of (2.5) under the conditions proposed at the beginning of this section. The best choice of g is the function given by (1.5), and by using this function, the consequence of Theorem 1.1 is obtained. We will see how this function is obtained in the next section.

3. Finding the best possible function

In this section, we explain how we found the function $g(x)$ defined by (1.5). Basically, we use the idea (and notation) introduced in [1]. We recall that an entire function $g : \mathbf{C} \to \mathbf{C}$ is of exponential type $C\pi$ if for any $\epsilon > 0$ there exists a positive constant C_ϵ such that $|g(z)| \leq C_\epsilon \exp((C\pi + \epsilon)|z|)$ holds for all $z \in \mathbf{C}$. In Section 2, we imposed on g the condition that the Fourier transform

$$\tilde{g}(\beta) = \int_{-\infty}^{\infty} g(x)e^{-2\pi i\beta x}\, dx$$

is supported in $[-2, 2]$. By the Paley–Wiener theorem, this class of functions is exactly the class of entire functions of exponential type at most 4π whose restriction to the real axis is integrable. We define a measure $d\mu(x)$ by

$$d\mu(x) = \Big\{1 - \Big(\frac{\sin \pi x}{\pi x}\Big)^2\Big\}\, dx$$

and denote by $\mathcal{H} = \mathcal{B}_2(2\pi, \mu)$ the class of entire functions f of exponential type at most 2π for which

$$\int_{-\infty}^{\infty} |f(x)|^2\, d\mu(x) < \infty.$$

This space is a Hilbert space with the norm given by

$$\|f\|_{\mathcal{H}}^2 = \int_{-\infty}^{\infty} |f(x)|^2\, d\mu(x).$$

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There exists a function $K(w,\cdot) \in \mathcal{H}$ such that

$$f(w) = \langle f, K(w,\cdot)\rangle_{\mathcal{H}} = \int_{-\infty}^{\infty} f(x)\overline{K(w,x)}\,dx$$

holds for all $f \in \mathcal{H}$. This is the so-called reproducing kernel for the Hilbert space $\mathcal{H}$.

Let g be an entire function satisfying $g(0) = 1$, and assume that its restriction to the real axis is nonnegative and that the Fourier transform $\tilde{g}$ has support in $[-2,2]$. Since g is of exponential type at most 4π, by Krein's decomposition,

$$g(z) = S(z)\overline{S(\overline{z})},$$

where S is an entire function of exponential type at most 2π. We see that S belongs to $\mathcal{H}$. Then, since $g(0) = 1$, by the Cauchy–Schwarz inequality,

$$1 = |S(0)|^2 = |\langle S, K(0,\cdot)\rangle_{\mathcal{H}}|^2 \le \|S\|_{\mathcal{H}}^2 \|K(0,\cdot)\|_{\mathcal{H}}^2 = \|S\|_{\mathcal{H}}^2 K(0,0).$$

Therefore,

$$\int_{-\infty}^{\infty} g(x)\Big\{1 - \Big(\frac{\sin \pi x}{\pi x}\Big)^2\Big\}\,dx = \int_{-\infty}^{\infty} |S(x)|^2\,d\mu(x) = \|S\|_{\mathcal{H}}^2 \ge K(0,0)^{-1}, \tag{3.1}$$

and the equality holds if and only if $S(z) = cK(0,z)$, where c is a complex constant whose absolute value is equal to $K(0,0)^{-1}$. Hence, for our purpose, it suffices to find the function $K(0,x)$, whereas in [1], the reproducing kernel of $\mathcal{B}_2(\pi,\mu)$ is completely obtained. We put $\kappa_w(x) = \overline{K(w,x)}$. This function satisfies

$$\int_{-\infty}^{\infty} f(x)\kappa_w(x)\Big\{1 - \Big(\frac{\sin \pi x}{\pi x}\Big)^2\Big\}\,dx = f(w)$$

for any $f \in \mathcal{B}_2(2\pi,\mu)$. By considering the Fourier transform, we obtain the functional equation

$$e^{-2\pi i t w} = \tilde{\kappa}_w(t) - \int_{-1}^{1} (1 - |u|)\tilde{\kappa}_w(t-u)\,du \tag{3.2}$$

for $-1 < t < 1$. By (formally) differentiating both sides twice, we obtain the differential equation

$$-4\pi^2 w^2 e^{-2\pi i t w} = \tilde{\kappa}_w''(t) - (\tilde{\kappa}_w(t+1) + \tilde{\kappa}_w(t-1) - 2\tilde{\kappa}_w(t)) \quad (-1 < t < 1).$$

Since κ_w is of exponential type 2π, its Fourier transform $\tilde{\kappa}_w$ has a support in $[-1,1]$. By putting $w = 0$,

$$\tilde{\kappa}_0''(t) - \tilde{\kappa}_0(t+1) - \tilde{\kappa}_0(t-1) + 2\tilde{\kappa}_0(t) = 0 \quad (-1 < t < 1).$$

Since $\tilde{\kappa}_0(t+1) = 0$ for $0 < t < 1$,

$$\tilde{\kappa}_0''(t) - \tilde{\kappa}_0(t-1) + 2\tilde{\kappa}_0(t) = 0 \quad (0 < t < 1). \tag{3.3}$$

On the other hand, for $-1 < t < 0$, $\tilde{\kappa}_0(t-1) = 0$ and so $\tilde{\kappa}_0''(t) - \tilde{\kappa}_0(t+1) + 2\tilde{\kappa}_0(t) = 0$. Replacing t with $t-1$ $(0 < t < 1)$ yields

$$\tilde{\kappa}_0''(t-1) - \tilde{\kappa}_0(t) + 2\tilde{\kappa}_0(t-1) = 0 \quad (0 < t < 1). \tag{3.4}$$

By (3.3),

$$\tilde{\kappa}_0(t-1) = \tilde{\kappa}_0''(t) + 2\tilde{\kappa}_0(t) \quad (0 < t < 1), \tag{3.5}$$

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and hence

$$\tilde{\kappa}_0''(t-1) = \tilde{\kappa}_0''''(t) + 2\tilde{\kappa}_0''(t) \quad (0 < t < 1). \tag{3.6}$$

By inserting (3.5) and (3.6) into (3.4),

$$\tilde{\kappa}_0''''(t) + 4\tilde{\kappa}_0''(t) + 3\tilde{\kappa}_0(t) = 0 \quad (0 < t < 1). \tag{3.7}$$

The general solution for (3.7) is

$$\tilde{\kappa}_0(t) = c_1 \cos t + c_2 \cos\sqrt{3}t + c_3 \sin t + c_4 \sin\sqrt{3}t \quad (0 < t < 1). \tag{3.8}$$

On the other hand, by (3.3), since

$$\tilde{\kappa}_0(t) = \tilde{\kappa}_0''(t+1) + 2\tilde{\kappa}_0(t+1)$$

for $-1 < t < 0$, using the solution above,

$$\tilde{\kappa}_0(t) = c_1 \cos(t+1) - c_2 \cos\sqrt{3}(t+1) + c_3 \sin(t+1) - c_4 \sin\sqrt{3}(t+1) \quad (-1 < t < 0). \tag{3.9}$$

4. The determination of the coefficients

To determine the constants $c_1, \ldots, c_4$, it suffices to insert (3.8) and (3.9) into (3.2). First we assume $0 < t < 1$. We decompose

$$\begin{aligned}
&\int_{-1}^{1} (1-|u|)\tilde{\kappa}_0(t-u)\,du \\
&\quad = \int_{t-1}^{0} (1+u)\tilde{\kappa}_0(t-u)\,du \\
&\qquad + \int_{0}^{t} (1-u)\tilde{\kappa}_0(t-u)\,du + \int_{t}^{1} (1-u)\tilde{\kappa}_0(t-u)\,du.
\end{aligned} \tag{4.1}$$

By applying (3.8) to the first and second terms and (3.9) to the third term on the right-hand side of (4.1),

$$\begin{aligned}
&\int_{t-1}^{0} (1+u)\tilde{\kappa}_0(t-u)\,du \\
&\quad = (-c_1 + c_3)\sin t + \Big(-\frac{c_2}{\sqrt{3}} + \frac{c_4}{3}\Big)\sin\sqrt{3}t + (c_1 + c_3)\cos t + \Big(\frac{c_2}{3} + \frac{c_4}{\sqrt{3}}\Big)\cos\sqrt{3}t \\
&\qquad + t\Big(c_1 \sin 1 + \frac{c_2}{\sqrt{3}}\sin\sqrt{3} - c_3 \cos 1 - \frac{c_4}{\sqrt{3}}\cos\sqrt{3}\Big) \\
&\qquad - c_1 \cos 1 - \frac{c_2}{3}\cos\sqrt{3} - c_3 \sin 1 - \frac{c_4}{3}\sin\sqrt{3},
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\int_{0}^{t} (1-u)\tilde{\kappa}_0(t-u)\,du \\
&\quad = (c_1 + c_3)\sin t + \Big(\frac{c_2}{\sqrt{3}} + \frac{c_4}{3}\Big)\sin\sqrt{3}t + (c_1 - c_3)\cos t + \Big(\frac{c_2}{3} - \frac{c_4}{\sqrt{3}}\Big)\cos\sqrt{3}t \\
&\qquad - t\Big(c_3 + \frac{c_4}{\sqrt{3}}\Big) - c_1 - \frac{c_2}{3} + c_3 + \frac{c_4}{\sqrt{3}},
\end{aligned} \tag{4.3}$$

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$$\begin{aligned}\int_t^1 &(1-u)\tilde{\kappa}_0(t-u)\,du\\ &= -c_3\sin t + \frac{c_4}{3}\sin\sqrt{3}t - c_1\cos t + \frac{c_2}{3}\cos\sqrt{3}t\\ &\quad + t\Big(-c_1\sin 1 + \frac{c_2}{\sqrt{3}}\sin\sqrt{3} + c_3\cos 1 - \frac{c_4}{\sqrt{3}}\cos\sqrt{3}\Big) + (\sin 1+\cos 1)c_1\\ &\quad - \Big(\frac{\sin\sqrt{3}}{\sqrt{3}} + \frac{\cos\sqrt{3}}{3}\Big)c_2 + (-\cos 1+\sin 1)c_3 + \Big(\frac{\cos\sqrt{3}}{\sqrt{3}} - \frac{\sin\sqrt{3}}{3}\Big)c_4, \qquad (4.4)\end{aligned}$$

respectively. Combining (4.2)–(4.4),

$$\begin{aligned}\int_{-1}^1 &(1-|u|)\tilde{\kappa}_0(t-u)\,du\\ &= c_1\cos t + c_2\cos\sqrt{3}t + c_3\sin t + c_4\sin\sqrt{3}t\\ &\quad + t\Big\{\frac{2}{\sqrt{3}}(\sin\sqrt{3})c_2 - c_3 - \Big(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\cos\sqrt{3}\Big)c_4\Big\} + (\sin 1 - 1)c_1\\ &\quad - \Big(\frac{\sin\sqrt{3}}{\sqrt{3}} + \frac{2\cos\sqrt{3}}{3} + \frac{1}{3}\Big)c_2\\ &\quad + (1-\cos 1)c_3 + \Big(\frac{1}{\sqrt{3}} + \frac{\cos\sqrt{3}}{\sqrt{3}} - \frac{2\sin\sqrt{3}}{3}\Big)c_4 \qquad (4.5)\end{aligned}$$

for $0 < t < 1$. By inserting (3.8) and (4.5) into the identity

$$\tilde{\kappa}_0(t) - \int_{-1}^1 (1-|u|)\tilde{\kappa}_0(t-u)\,du = 1, \qquad (4.6)$$

we find that the constants $c_1, \ldots, c_4$ must satisfy

$$\frac{2}{\sqrt{3}}(\sin\sqrt{3})c_2 - c_3 - \Big(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\cos\sqrt{3}\Big)c_4 = 0, \qquad (4.7)$$

$$\begin{aligned}(\sin 1 - 1)c_1 &- \Big(\frac{\sin\sqrt{3}}{\sqrt{3}} + \frac{2\cos\sqrt{3}}{3} + \frac{1}{3}\Big)c_2\\ &+ (1-\cos 1)c_3 + \Big(\frac{1}{\sqrt{3}} + \frac{\cos\sqrt{3}}{\sqrt{3}} - \frac{2\sin\sqrt{3}}{3}\Big)c_4 = -1. \qquad (4.8)\end{aligned}$$

Next we assume $-1 < t < 0$. We decompose

$$\begin{aligned}\int_{-1}^1 &(1-|u|)\tilde{\kappa}_0(t-u)\,du\\ &= \int_{-1}^t (1+u)\tilde{\kappa}_0(t-u)\,du\\ &\quad + \int_t^0 (1+u)\tilde{\kappa}_0(t-u)\,du + \int_0^{t+1} (1-u)\tilde{\kappa}_0(t-u)\,du. \qquad (4.9)\end{aligned}$$

By applying (3.8) to the first term and (3.9) to the second and third terms on the right-hand side of (4.9),

$$\int_{-1}^{t}(1+u)\tilde{\kappa}_0(t-u)\,du$$
$$= -c_1\cos(t+1) - \frac{c_2}{3}\cos\sqrt{3}(t+1) - c_3\sin(t+1) - \frac{c_4}{3}\sin\sqrt{3}(t+1)$$
$$+ t\Big(c_3 + \frac{c_4}{\sqrt{3}}\Big) + c_1 + \frac{c_2}{3} + c_3 + \frac{c_4}{\sqrt{3}}, \tag{4.10}$$

$$\int_{t}^{0}(1+u)\tilde{\kappa}_0(t-u)\,du$$
$$= (c_1+c_3)\cos(t+1) - \Big(\frac{c_2}{3} + \frac{c_4}{\sqrt{3}}\Big)\cos\sqrt{3}(t+1) + (-c_1+c_3)\sin(t+1)$$
$$+ \Big(\frac{c_2}{\sqrt{3}} - \frac{c_4}{3}\Big)\sin\sqrt{3}(t+1)$$
$$+ t\Big(c_1\sin 1 - \frac{c_2}{\sqrt{3}}\sin\sqrt{3} - c_3\cos 1 + \frac{c_4}{\sqrt{3}}\cos\sqrt{3}\Big)$$
$$+ (\sin 1 - \cos 1)c_1 + \Big(\frac{\cos\sqrt{3}}{3} - \frac{\sin\sqrt{3}}{\sqrt{3}}\Big)c_2 - (\sin 1 + \cos 1)c_3$$
$$+ \Big(\frac{\cos\sqrt{3}}{\sqrt{3}} + \frac{\sin\sqrt{3}}{3}\Big)c_4, \tag{4.11}$$

$$\int_{0}^{t+1}(1-u)\tilde{\kappa}_0(t-u)\,du$$
$$= (c_1-c_3)\cos(t+1) + \Big(-\frac{c_2}{3} + \frac{c_4}{\sqrt{3}}\Big)\cos\sqrt{3}(t+1) + (c_1+c_3)\sin(t+1)$$
$$- \Big(\frac{c_2}{\sqrt{3}} + \frac{c_4}{3}\Big)\sin\sqrt{3}(t+1) + t\Big(-c_3 + \frac{c_4}{\sqrt{3}}\Big) - c_1 + \frac{c_2}{3}, \tag{4.12}$$

respectively. Combining (4.10)–(4.12),

$$\int_{-1}^{1}(1-|u|)\tilde{\kappa}_0(t-u)\,du$$
$$= c_1\cos(t+1) - c_2\cos\sqrt{3}(t+1) + c_3\sin(t+1) - c_4\sin\sqrt{3}(t+1)$$
$$+ t\Big(c_1\sin 1 - \frac{\sin\sqrt{3}}{\sqrt{3}}c_2 - c_3\cos 1 + \frac{2+\cos\sqrt{3}}{\sqrt{3}}c_4\Big)$$
$$+ (\sin 1 - \cos 1)c_1 + \Big(\frac{2}{3} + \frac{\cos\sqrt{3}}{3} - \frac{\sin\sqrt{3}}{\sqrt{3}}\Big)c_2 + (1 - \cos 1 - \sin 1)c_3$$
$$+ \Big(\frac{1+\cos\sqrt{3}}{\sqrt{3}} + \frac{\sin\sqrt{3}}{3}\Big)c_4 \tag{4.13}$$

for $-1 < t < 0$. By inserting (3.9) and (4.13) into (4.6), we find that the coefficients

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$c_1, \dots, c_4$ must satisfy

$$c_1 \sin 1 - \frac{\sin\sqrt{3}}{\sqrt{3}} c_2 - c_3 \cos 1 + \frac{2 + \cos\sqrt{3}}{\sqrt{3}} c_4 = 0, \tag{4.14}$$

$$\begin{aligned}(\sin 1 - \cos 1)c_1 &+ \Big(\frac{2}{3} + \frac{\cos\sqrt{3}}{3} - \frac{\sin\sqrt{3}}{\sqrt{3}}\Big)c_2 \\ &+ (1 - \sin 1 - \cos 1)c_3 + \Big(\frac{1 + \cos\sqrt{3}}{\sqrt{3}} + \frac{\sin\sqrt{3}}{3}\Big)c_4 = -1.\end{aligned} \tag{4.15}$$

By solving (4.7), (4.8), (4.14), (4.15),

$$\begin{aligned}c_1 &= \Big(\cos 1 - \frac{(1-\cos 1)(2+\cos\sqrt{3})\sin\sqrt{3}}{3\sqrt{3}\sin 1(1+\cos\sqrt{3})} \\ &\qquad - \frac{(1-\cos 1)(4\sqrt{3} - 3\sqrt{3}\sin 1 - \sin\sqrt{3})}{3\sqrt{3}\sin 1}\Big)^{-1} \\ &= 6.757821\dots,\end{aligned}$$

$$c_2 = \frac{(1-\cos 1)\sin\sqrt{3}}{\sqrt{3}\sin 1(1+\cos\sqrt{3})}c_1 = 2.506205\dots, \quad c_3 = \frac{1-\cos 1}{\sin 1}c_1 = 3.691814\dots,$$

$$c_4 = -\frac{1-\cos 1}{\sqrt{3}\sin 1}c_1\Big(= -\frac{1+\cos\sqrt{3}}{\sin\sqrt{3}}c_2\Big) = -2.131470\dots.$$

By computing the inverse Fourier transform,

$$\begin{aligned}\kappa_0(x) = &\frac{-\frac{2(1-\cos 1)}{\sin 1}(\cos 2\pi x + 1) + 4\pi x \sin 2\pi x}{4\pi^2 x^2 - 1}c_1 \\ &- \frac{\frac{2\sqrt{3}(1+\cos\sqrt{3})}{\sin\sqrt{3}}(\cos 2\pi x - 1) + 4\pi x \sin 2\pi x}{4\pi^2 x^2 - 3}c_2.\end{aligned}$$

The function (1.5) is given by $g(x) = \kappa_0(0)^{-2}\kappa_0(x)^2$.

REMARK 4.1. (1) The value of $K(0,0)$ is given by

$$K(0,0) = \overline{\kappa_0(0)} = \frac{4(1-\cos 1)}{\sin 1}c_1,$$

and we can confirm that for the function $g(x)$ and the value of $K(0,0)$ given above, the left- and right-hand sides of (3.1) are equal. Hence, the value of M, defined by (1.4), is concretely given by

$$M = 1 - \frac{\sin 1}{4c_1(1-\cos 1)}. \tag{4.16}$$

(2) Since the zeros of our $g(x)$ are not periodic, we can apply the method of Cheer and Goldston [3] and improve the proportion of simple zeros very slightly. That is, by

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picking up the contributions of distinct zeros of $L(s,\chi)$ near the real axis, we can prove that

$$\sum_{q} \frac{W(q/Q)}{\phi(q)} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\substack{\gamma_\chi \\ \text{simple}}} \widehat{\Phi}(i\gamma_\chi)^2 \geq (M + O + o(1)) N_\Phi(Q)$$

as $Q \to \infty$, where M is given by (4.16), and $O \geq 7.68 \times 10^{-10}$. This O comes from the third term of the middle of (2.5). The evaluation of this off-diagonal term requires some complicated arguments, although the final lower bound of O is extremely small. Hence, we omit the details.

Acknowledgement

The author would like to express his gratitude to the referee of this paper for giving him many valuable suggestions.

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https://doi.org/10.1017/S0004972715000623 Published online by Cambridge University Press