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Higher Connectedness Properties of Support Points and Functionals of Convex Sets

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Abstract. We prove that the set of all support points of a nonempty closed convex bounded set C in a real infinite-dimensional Banach space X is AR(σ-compact) and contractible. Under suitable conditions, similar results are proved also for the set of all support functionals of C and for the domain, the graph, and the range of the subdifferential map of a proper convex lower semicontinuous function on X.

Introduction

Let C be a nonempty closed convex set in a real Banach space X, with continuous dual X^*. If $x \in C$ and $x^* \in X^* \setminus \{0\}$ are such that $x^*(x) = \sup x^*(C)$, we say that x is a *support point* of C and x^* is a *support functional* of C. Let $\operatorname{supp}(C)$ and $\Sigma(C)$ denote respectively the set of all support points of C and the set of all support functionals of C. Moreover we denote by $\operatorname{Supp}(C)$ the set

$$\left\{ (x, x^*) \in \operatorname{supp}(C) \times \Sigma(C); \quad x^*(x) = \sup x^*(C) \right\}.$$

By using a parametric smooth variational principle, L. Veselý [15] proved that if C is nonempty closed convex and does not contain any hyperplane, then the set $\operatorname{supp}(C)$ is pathwise connected (this was proved for bounded sets in [4]). In [10, 11], G. Luna published some results about higher connectedness properties of the set of all support points of a convex closed set in a Banach space. In particular, he stated that if C is a nonempty closed convex boundedly w-compact (that is, the intersection of C with any closed ball is w-compact) set in X, not containing any closed affine subset of finite codimension, then every continuous map defined on the unit sphere of $\mathbb{R}^n$ and with values in $\operatorname{supp}(C)$ can be extended to a continuous map with values in $\operatorname{supp}(C)$ and defined on the unit ball of $\mathbb{R}^n$. However, the proof of this result is not correct; in fact, if we put

$$C = \left\{ (x_n)_n \in l_2 \,;\, 0 \le x_1 \le 1, |x_n| \le 1/n \quad (n > 1) \right\}$$

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and $a = 0$, it is easy to see that [10, Lemma 3] is false. After some preliminaries, contained in Section 1, we prove in Section 2 the result of Luna, cited above, but without the strong assumption that C is boundedly w-compact. In the case C is bounded, $C = -C$ and X is infinite-dimensional, we obtain that $\mathrm{supp}(C)$ is AR (*cf.* Theorem 2.9 and Definition 2.1). Moreover, under the hypothesis that the space X admits an equivalent Fréchet differentiable norm, we prove similar results for the sets $\mathrm{Supp}(C)$ and $\Sigma(C)$. Similar results are also proved for the domain, the graph, and the range of the subdifferential map of a proper convex lower semicontinuous function on X (*cf.* Proposition 2.6).

Suppose that C is a nonempty closed convex bounded subset of X and let Y be a finite-codimensional w^*-closed subspace of X^*. E. Bishop and R. R. Phelps [2] proved that $Y \cap \Sigma(C)$ is dense in Y; this is an easy consequence of the famous Bishop–Phelps theorem. In Section 3, we obtain some results in this direction. In particular, under the hypothesis that X admits an equivalent Fréchet differentiable norm, we study cardinality and connectedness properties of the intersection of the set $\Sigma(C)$ with a continuous finite-codimensional surface of X^* (see Definition 3.4).

The main tools used in this paper are the application of the parametric smooth variational principle to the study of support properties for convex sets and functions in Banach spaces, introduced by L. Veselý in [15], the Leray-Schauder continuation principle, and the Michael selection theorem.

1 Notation and Preliminaries

Throughout the paper, X denotes a real Banach space with the dual X^*. We denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. We denote by $\mathbb{R}^+$ the interval $(0,\infty)$. For $x, y \in X$, $[x, y]$ denotes the closed segment in X with endpoints x and y, and $(x, y) = [x, y] \setminus \{x, y\}$ is the corresponding "open" segment. We denote by w and w^* the weak-topology and the weak*-topology, respectively. Moreover, we denote by $\mathcal{BCC}(X)$ the set of all nonempty bounded closed convex subsets of X. If A is a subset of X, we denote by $\dim(A)$ the dimension of the affine hull of A.

Let C be a nonempty closed convex set in X, suppose that the sets $\mathrm{supp}(C)$, $\Sigma(C)$ and $\mathrm{Supp}(C)$ are defined as in the introduction, and put $\Sigma_1(C) = \Sigma(C) \cap S_{X^*}$.

Let $h\colon X \to (-\infty, \infty]$ be a proper convex lower semicontinuous (l.s.c.) function. We denote by $\mathrm{dom}(h)$ the domain of h, that is, the set $\{x \in X;\, h(x) < \infty\}$. Let us recall that the sets

$$\mathbf{D}(\partial h) = \{x \in \mathrm{dom}(h); \partial h(x) \neq \varnothing\}, \quad \mathbf{R}(\partial h) = \bigcup\{\partial h(x) : x \in \mathrm{dom}(h)\},$$
$$\mathbf{G}(\partial h) = \{(x, x^*) \in X \times X^* : x^* \in \partial h(x)\}$$

are called, respectively, the *effective domain*, the *range* and the *graph* of the subdifferential mapping ∂h. The *Fenchel conjugate* of h is the proper convex w^*-l.s.c. function $h^*\colon X^* \to (-\infty, \infty]$, given by $h^*(y^*) = \sup(y^* - h)(X)$. We say that h is *supercoercive* if

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = \infty$$

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and that h is *cofinite* if h^* is finite (and hence continuous) on X^*. It is easy to see that the following implications hold:

$$\text{dom}(h) \text{ is bounded} \Longrightarrow h \text{ is supercoercive} \Longrightarrow h \text{ is cofinite.}$$

Let T be a topological space . A multifunction $F\colon T \to 2^X$ is called:

(i) *upper semicontinuous* (u.s.c.) if $F^{-1}(D) := \{t \in T; F(t) \cap D \neq \varnothing\}$ is closed whenever D is a closed subset of X;

(ii) *lower semicontinuous* (l.s.c.) if $F^{-1}(U)$ is open whenever U is an open subset of X.

If T is a Hausdorff paracompact topological space and $F\colon T \to 2^X$ is a l.s.c. multifunction with nonempty closed convex values, then the *Michael selection theorem* [12] ensures that F admits a continuous selection, that is, a continuous function $f\colon T \to X$ such that $f(t) \in F(t)$, for each $t \in T$.

As L. Vesely has pointed out in a private communication (2008), [15, Proposition 2.2] still holds if T is a Hausdorff paracompact space. To see this, we just have to use the following lemma instead of [15, Lemma 1.2], in the proofs of [15, Theorem 1.3] and [15, Proposition 2.2].

Lemma 1.1 *Let T_0 be a closed set in a Hausdorff paracompact topological space T, X a Banach space, $D \subset X$ a nonempty convex set, and $f\colon T \times X \to (-\infty, \infty]$ a function. Suppose that*

(i) *for each $t \in T$, the function $f(t, \cdot)$ is convex and lower semicontinuous with $\text{dom}(f(t, \cdot)) = D$;*

(ii) *for each $x \in D$, the function $f(\cdot, x)$ is continuous on T.*

Let $\varepsilon\colon T \to \mathbb{R}^+$ be a continuous function and $g\colon T \to \mathbb{R}$ a l.s.c. function such that $g(t) \geq \inf f(t, D) > -\infty$ for each $t \in T$. Let $\varphi\colon T_0 \to D$ be a continuous function such that $f(t, \varphi(t)) \leq g(t) + \varepsilon(t)$ whenever $t \in T_0$. Then φ admits a continuous extension $\widetilde{\varphi}\colon T \to D$ such that $f(t, \widetilde{\varphi}(t)) \leq g(t) + \varepsilon(t)$ for each $t \in T$.

Proof Define $F\colon T \to 2^X$ by $F(t) = \{x \in X;\ f(t,x) \leq g(t)+\varepsilon(t)\}$. We claim that F is l.s.c. In fact, fix $t_0 \in T$, $x_0 \in F(t_0)$ and $\eta > 0$. If $f(t_0, x_0) < g(t_0) + \varepsilon(t_0)$, put $z_0 = x_0$. If else, there exists $x_1 \in x_0 + \eta B_X$ such that $f(t_0, x_1) < g(t_0) + \varepsilon(t_0)$ (observe that $g(t) \geq \inf f(t, D) > -\infty$ whenever $t \in T$); in this case put $z_0 = x_1$. In both cases, $z_0 \in x_0 + \eta B_X$ and $f(t_0, z_0) < g(t_0) + \varepsilon(t_0)$. Now, the function $f(\cdot, z_0) - g(\cdot) - \varepsilon(\cdot)$ is upper semicontinuous by our assumptions; moreover, $f(t_0, z_0) - g(t_0) - \varepsilon(t_0) < 0$. Then there exists a neighborhood $\mathcal{U}$ of t_0 such that $f(t, z_0) - g(t) - \varepsilon(t) < 0$ for each $t \in \mathcal{U}$. This proves our claim.

For $t \in T \setminus T_0$ define $\widetilde{F}(t) = F(t)$ and for $t_0 \in T_0$ define $\widetilde{F}(t_0) = \{\varphi(t_0)\} \subset F(t_0)$. Since T_0 is closed, $\widetilde{F}\colon T \to 2^X$ is l.s.c. too. Moreover, $\widetilde{F}$ assumes nonempty convex closed values. By the Michael selection theorem, we can find a continuous selection $\widetilde{\varphi}\colon T \to D$ of $\widetilde{F}$. Obviously, $\widetilde{\varphi}$ satisfies the conclusion of the lemma. ∎

The following proposition is an easy modification of [15, Proposition 2.2].

Proposition 1.2 *Let T be a paracompact Hausdorff topological space and T_0 a closed subset of T. Let X be a Banach space, $h\colon X \to (-\infty, \infty]$ a proper l.s.c. convex function* *and $\varepsilon\colon T \to \mathbb{R}^+$ a continuous function. Suppose that $\varphi\colon T_0 \to X$ and $\varphi^*\colon T \to X^*$ are continuous mappings such that $(\varphi(t_0), \varphi^*(t_0)) \in \mathbf{G}(\partial h)$ whenever $t_0 \in T_0$. If h is cofinite, then there exists a continuous function $y_0\colon T \to \operatorname{dom}(h)$ such that $y_0|_{T_0} = \varphi$, and*

$$\big[h - \varphi^*(t)\big]\big(y_0(t)\big) \le \inf\big[h - \varphi^*(t)\big]\big(\operatorname{dom}(h)\big) + \varepsilon(t) \tag{1.1}$$

for each $t \in T$.

Moreover, for each such y_0 and each continuous function $\lambda\colon T \to \mathbb{R}^+$, there exist $v\colon T \to \operatorname{dom}(h)$ and $v^*\colon T \to X^*$ such that

(i) *$v|_{T_0} = \varphi$ and $v^*|_{T_0} = \varphi^*|_{T_0}$;*
(ii) *$v^*(t) \in \partial h(v(t))$ for each $t \in T$;*
(iii) *$\|v(t) - y_0(t)\| < \lambda(t)$ and $\|v^*(t) - \varphi^*(t)\| \le \frac{6\varepsilon(t)}{\lambda(t)}$ for each $t \in T$;*
(iv) *v is continuous;*
(v) *if X is Fréchet smooth, then v^* is continuous;*
(vi) *if X is Gâteaux smooth, then v^* is w^*-continuous.*

Proof Put $D = \operatorname{dom}(h)$ and $f(t,x) = [h - \varphi^*(t)](x)$. Consider the function $g(t) := \inf f(t,D) = \inf f(t,X) = -h^*(\varphi^*(t))$, where h^* is the Fenchel conjugate of h. Suppose that h is cofinite, then g is continuous. Moreover, for $t_0 \in T_0$, we have $f(t_0, \varphi(t_0)) = [h - \varphi^*(t_0)](\varphi(t_0)) = g(t_0)$. Then we can proceed as in the proof of [15, Proposition 2.2]. ■

Denote by I_C the indicator function of a nonempty closed convex set C in X. Let $(x, x^*) \in C \times X^*$, then it is not difficult to see that $x^* \in \partial I_C(x)$ if and only if $x^*x = \sup x^*(C)$. If we put $h = I_C$ in the proposition above, we obtain the following corollary.

Corollary 1.3 *Let T, T_0, and ε be as in the proposition above, let C be a nonempty closed convex subset of X, and suppose that $\varphi\colon T_0 \to C$, $\varphi^*\colon T \to X^*$ are continuous mappings such that $\varphi^*(t_0)\varphi(t_0) = \sup_C \varphi^*(t_0)$, whenever $t_0 \in T_0$. Suppose that C is bounded, then there exists a continuous function $y_0\colon T \to C$ such that $y_0|_{T_0} = \varphi$ and*

$$\varphi^*(t)y_0(t) \ge \sup_C \varphi^*(t) - \varepsilon(t)$$

for each $t \in T$.

Moreover, for each such y_0 and each continuous function $\lambda\colon T \to \mathbb{R}^+$, there exist $v\colon T \to C$ and $v^*\colon T \to X^*$ such that

(i) *$v|_{T_0} = \varphi$ and $v^*|_{T_0} = \varphi^*|_{T_0}$;*
(ii) *$v^*(t)v(t) = \sup_C v^*(t)$ for each $t \in T$;*
(iii) *$\|v(t) - y_0(t)\| < \lambda(t)$ and $\|v^*(t) - \varphi^*(t)\| \le \frac{6\varepsilon(t)}{\lambda(t)}$ for each $t \in T$;*
(iv) *v is continuous;*
(v) *if X is Fréchet smooth, then v^* is continuous;*
(vi) *if X is Gâteaux smooth, then v^* is weak*-continuous.*

In the sequel, we will use the following well-known result about extensions of continuous functions several times.

Theorem 1.4 (Dugundji Extension Theorem [5]) *Suppose that T_0 is a closed subset of a metric space T. Suppose that V is a locally convex linear topological space. Then every continuous function $f\colon T_0 \to V$ admits a continuous extension $F\colon T \to$* $\operatorname{conv} f(T_0)$.

Lemma 1.5 *Suppose that T_0 is a closed subset of a metric space T. Suppose that X is infinite-dimensional. If $f\colon T_0 \to X \setminus \{0\}$ is a continuous function, then there exists $\widetilde{f}\colon T \to X \setminus \{0\}$ a continuous extension of f.*

Proof Let $g\colon T_0 \to \mathbb{R}^+$ and $h\colon T_0 \to S_X$ be the continuous functions defined by $g(t_0) = \|f(t_0)\|$ and $h(t_0) = f(t_0)/\|f(t_0)\|$, for $t_0 \in T_0$. Let us extend g to a continuous function $\widetilde{g}\colon T \to \mathbb{R}^+$. By the Dugundji extension theorem and by [5, Theorem 6.2], h admits a continuous extension $\widetilde{h}\colon T \to S_X$. The function $\widetilde{f}$, defined by $\widetilde{f}(t) = \widetilde{g}(t)\widetilde{h}(t)$ $(t \in T)$, is continuous and extends f. ■

Lemma 1.6 *Let C be a nonempty closed convex subset of X, let $x \in X \setminus C$ and $S \subset S_{X^*}$. Suppose that, for each $x^* \in S$,*

$$\sup x^*(C) \le \inf x^*\left(x + \tfrac{1}{2}\operatorname{dist}(x, C)B_X\right).$$

Then $y^* \in \operatorname{conv}(S)$ implies $\|y^*\| \ge \frac{1}{2}$.

Proof Let us suppose that $y^* = \sum_{i=i}^n a_i x_i^*$, where $x_i^* \in S$, $a_i \ge 0$, for $i = 1, \dots, n$, and $\sum_{i=i}^n a_i = 1$. Fix $\alpha > 0$ and choose any $y \in C$ such that $\|y - x\| \le \operatorname{dist}(x, C) + \alpha$. Put $w = y - x$, we have

$$x_i^* w \le \inf x_i^*\left(x + \tfrac{1}{2}\operatorname{dist}(x, C)B_X\right) - x_i^* x = -\tfrac{1}{2}\operatorname{dist}(x, C) \quad (i = 1, \dots, n).$$

Then

$$y^*\left(\frac{-w}{\|w\|}\right) \ge \frac{1}{2}\frac{\operatorname{dist}(x, C)}{\operatorname{dist}(x, C) + \alpha},$$

and the proof is complete, as $\alpha > 0$ can be chosen arbitrarily small. ■

Lemma 1.7 *Let C be a nonempty closed convex subset of X.*

(i) *The multifunction $F\colon X \setminus C \to 2^X$, defined by*

$$F(x) = [x + \frac{3}{2}\operatorname{dist}(x, C)B_X] \cap C,$$

is l.s.c. and assumes nonempty convex closed values;

(ii) *The multifunction $F^*\colon X \setminus C \to 2^{X^*}$, defined by*

$$F^*(x) = \{x^* \in B_{X^*}; \sup x^*(C) \le \inf x^*\left(x + \tfrac{1}{2}\operatorname{dist}(x, C)B_X\right)\},$$

is l.s.c. and assumes nonempty convex closed values. Moreover, the multifunction $G^*\colon X \setminus C \to 2^{X^*}$, defined by $G^*(x) = \overline{\operatorname{conv}}\,(F^*(x) \cap S_{X^*})$, is l.s.c. too, and, for each $x \in X \setminus C$, we have $\operatorname{dist}\big(0, G^*(x)\big) \ge 1/2$.

Proof (i) Fix $x \in X \setminus C$ and an open subset V of X such that there exists $k \in F(x) \cap V$. By the definition of F, $\|x-k\| \leq \frac{3}{2}\operatorname{dist}(x,C)$; moreover, we can choose $k' \in C$ such that $\|x-k'\| < \frac{3}{2}\operatorname{dist}(x,C)$. Then there exists $k'' \in [k,k'] \cap V$ such that $\|x-k''\| < \frac{3}{2}\operatorname{dist}(x,C)$. Now, by the continuity of the function $\operatorname{dist}(\,\cdot\,,C)$, there exists W, an open neighborhood of x, such that $k'' \in F(y)$, for each $y \in W$.

(ii) Let us observe that, for $x \in X \setminus C$,

$$F^*(x) = \{x^* \in B_{X^*}; \sup x^*(C) - \inf x^*\left(x + \tfrac{1}{2}\operatorname{dist}(x,C)B_X\right) \leq 0\}.$$

Then, for each $x \in X \setminus C$, $F^*(x)$ is the intersection between B_{X^*} and a nonempty convex closed cone. Let us prove that F^* is l.s.c.. Suppose $x \in X \setminus C$ and $x^* \in F^*(x) \cap V$, where V is an open subset of X^*. By the Hahn–Banach theorem, we can choose $y^* \in B_{X^*}$ such that $\sup y^*(C) < \inf y^*\left(x+\frac{1}{2}\operatorname{dist}(x,C)B_X\right)$. Then there exists $z^* \in [x^*, y^*] \cap V$ such that

$$\sup z^*(C) < \inf z^*\left(x + \tfrac{1}{2}\operatorname{dist}(x,C)B_X\right). \tag{1.2}$$

Moreover, if $x' \in X \setminus C$ and $\|x-x'\|$ is small enough, (1.2) still holds if x is replaced by x', that is, $z^* \in F^*(x') \cap V$. This proves that F^* is l.s.c.

The lower semicontinuity of the multifunction G^* follows easily by the lower semicontinuity of F^*. Now, suppose $x^* \in \operatorname{conv}\left(F^*(x) \cap S_{X^*}\right)$, by Lemma 1.6, $\|x^*\| \geq 1/2$. Then

$$\operatorname{dist}\left(0, G^*(x)\right) = \operatorname{dist}\left(0, \operatorname{conv}[F^*(x) \cap S_{X^*}]\right) \geq 1/2. \qquad \blacksquare$$

Fact 1.8 *Let K be a compact convex set contained in a closed convex subset C of X. Suppose that $0 \in K$ and that $\operatorname{int} C = \varnothing$. Then 0 is not in the interior of the closed convex set $C - K$.*

Proof Suppose that $B_X \subset m(C-K)$ for some $m \in \mathbb{N}$. By the compactness of K, we have $mK \subset \bigcup_{i=1}^n (x_i + \frac{1}{2}B_X)$ for some $x_1, \dots, x_n \in mK$. Since $K \subset C$ and hence $0 \in (mC - x_i)$ $(i = 1, \dots, n)$, we get

$$B_X \subset \bigcup_{i=1}^n (mC - x_i) + \tfrac{1}{2}B_X \subset (mC - x_1) + \dots + (mC - x_n) + \tfrac{1}{2}B_X.$$

Then $B_X \subset [nmC - (x_1 + \dots + x_n)] + \frac{1}{2}B_X$, and since C is closed and convex, $\frac{1}{2}B_X \subset [nmC - (x_1 + \dots + x_n)]$, which is a contradiction, since $\operatorname{int} C = \varnothing$. $\blacksquare$

Lemma 1.9 *Let T_0 be a closed σ-compact subset of a metric space T. Suppose that C is a nonempty closed convex subset of X and that $\operatorname{int} C = \varnothing$. Then each continuous function $f\colon T_0 \to C$ admits a continuous extension $\Phi\colon T \to X$ such that $\Phi(T \setminus T_0) \subset X \setminus C$.*

Proof Without any loss of generality we can suppose that $0 \in C$. By the Dugundji extension theorem, we can extend f to a continuous function $\widetilde{F}\colon T \to \operatorname{conv}[f(T_0)]$.

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Since there exists a sequence $(K_n)_n$ of compact subsets of T such that $T_0 = \bigcup_{n\in\mathbb{N}} K_n$, we have

$$\widetilde{F}(T) \subset \operatorname{conv}\Big[\bigcup_{n\in\mathbb{N}} f(K_n)\Big] \subset \bigcup_{n\in\mathbb{N}} A_n,$$

where, for $n \in \mathbb{N}$, A_n is the compact convex set $\overline{\operatorname{conv}}\,[\bigcup_{i=1}^{n} f(K_i) \cup \{0\}]$.

We claim that $X \neq \bigcup_{n,m\in\mathbb{N}} m(C - A_n)$. Suppose this is not the case. By the Baire theorem, for some $n, m \in \mathbb{N}$, there exists $w \in \operatorname{int}[m(C - A_n)]$. Since $m'(C - A_{n'}) \supset m(C - A_n)$, if $m' \geq m$ and $n' \geq n$, and $\bigcup_{n,m\in\mathbb{N}} m(C - A_n) = X$, we can suppose, without any loss of generality, that $-w \in m(C - A_n)$ and hence that $0 \in \operatorname{int}[m(C - A_n)]$. By Fact 1.8, we get a contradiction, and the claim is proved.

Now, fix $x_0 \in X \setminus \bigcup_{n,m\in\mathbb{N}} m(C - A_n)$ and take a continuous function $\lambda\colon T \to [0, 1]$ such that $\lambda(t) = 0$ if and only if $t \in T_0$. For each $t \in T$, put

$$F(t) = \lambda(t)x_0 + \big[1 - \lambda(t)\big]\widetilde{F}(t).$$

Then F is a continuous extension of f; moreover, $F(T \setminus T_0) \subset X \setminus C$. In fact, fix $t \in T \setminus T_0$, put $\lambda = \lambda(t)$, $a = \widetilde{F}(t) \in A_n$, for some $n \in \mathbb{N}$, and suppose that $\lambda x_0 + (1 - \lambda)a = c \in C$. Then

$$x_0 = \tfrac{1}{\lambda}[c - (1 - \lambda)a] \in \tfrac{1}{\lambda}(C - A_n) \subset m(C - A_n),$$

for some $m \in \mathbb{N}$. This is a contradiction, by the choice of x_0. ■

Remark 1.10 Let T_0 be a closed subset of a metric space T. Suppose that C is a closed convex subset of X and that $\operatorname{int}(C - C) = \varnothing$. Then we get the same conclusion as Lemma 1.9. In fact, in this case, by the Baire theorem, we can find $x_0 \in X \setminus \bigcup_{n\in\mathbb{N}} n(C - C)$, and we can proceed as in the last part of the proof of Lemma 1.9.

2 Connectedness Properties of $\operatorname{supp}(C)$, $\operatorname{Supp}(C)$, $\Sigma(C)$ and $\mathbf{D}(\partial h)$, $\mathbf{R}(\partial h)$, $\mathbf{G}(\partial h)$

Definition 2.1 Let S be a metric space and $n \in \mathbb{N} \cup \{0\}$. If every continuous function $f\colon S_{\mathbb{R}^{n+1}} \to S$ admits a continuous extension $F\colon B_{\mathbb{R}^{n+1}} \to S$, then we say that S is *n-connected.*

Let $\mathcal{P}$ be a class of metric spaces. We say that S is an *Absolute Retract for the class* $\mathcal{P}$ (AR($\mathcal{P}$)) if, for each $T \in \mathcal{P}$ and for each closed subset T_0 of T, every continuous function $f\colon T_0 \to S$ admits a continuous extension $F\colon T \to S$. In the case where $\mathcal{P}$ is the class of all metric spaces, we say that S is an *Absolute Retract* (AR).

Suppose that C is a nonempty closed convex subset of X; let us denote $\Pi(C) = \{x^* \in X^*; \sup x^*(C) < \infty\}$ and $S^n = S_{\mathbb{R}^{n+1}}$. Proposition 2.3 is an easy modification of the following well-known result (see, for instance, [6, Chapter XVI]).

Fact 2.2 *Let $m > n$. Then any continuous map from S^n to S^m can be continuously extended to $B_{\mathbb{R}^{n+1}}$.*

Proposition 2.3 *Let C be a nonempty closed convex subset of X.*

(i) *Suppose that C does not contain any closed affine subset of codimension $(n+1)$. Then $\Pi(C)\setminus\{0\}$ is n-connected; that is, each continuous map $f\colon S^n \to \Pi(C)\setminus\{0\}$ admits a continuous extension $F\colon B_{\mathbb{R}^{n+1}} \to \Pi(C)\setminus\{0\}$. Moreover, if*

$$\sup_{t\in S^n}\sup_C f(t) < \infty,$$

then it is possible to choose F in such a way that $\sup_{t\in B_{\mathbb{R}^{n+1}}}\sup_C F(t) < \infty$.

(ii) *Suppose that C does not contain any closed affine subset of codimension $(n+1)$ (respectively, of finite codimension). Then ∂C is n-connected (respectively, ∂C is* AR*).*

Before starting the proof of Proposition 2.3, we recall some elementary facts and definitions about spherical simplexes and homotopy (see *e.g.,* [6, Chapter XVI]). Let E be a normed finite dimensional linear space and let $\{p_0,\dots,p_n\}$ be any set of $(n+1)$-points on S_E. If this set has diameter < 1, then its convex hull does not contain the origin of E, and it can be radially projected from there into S_E to give $\sigma = (p_0,\dots,p_n)$, the *spherical n-simplex* with *vertices* $\{p_0\dots,p_n\}$. We say that σ is *degenerate* if and only if its vertices lie on a 1-codimensional subspace of E. For $k < n$ and $\{q_0,\dots,q_k\} \subset \{p_0,\dots,p_n\}$ we say that the spherical k-simplex $\sigma' = (q_0,\dots,q_k)$ is a *k-face* of σ.

By a *triangulation* $\mathcal{T}$ of S^n we mean a decomposition of S^n into finitely many non-degenerate non-overlapping spherical n-simplexes such that each $(n-1)$-face of an n-simplex of $\mathcal{T}$ is the common $(n-1)$-face of exactly two n-simplexes. By the compactness of S^n we have the following fact.

Fact 2.4 *Let (S,d) be a metric space and $f\colon S^n \to S$ a continuous map. Then there exists a triangulation $\mathcal{T}$ of S^n such that* diam $f(\sigma) < 1$ *whenever $\sigma \in \mathcal{T}$.*

Let Y and Z be two topological spaces and $I = [0,1]$ the unit interval. Two continuous maps $f,g\colon Z \to Y$ are called *homotopic* (in Y) if there exists a continuous map $\Phi\colon Z\times I \to Y$ such that $\Phi(z,0) = f(z)$ and $\Phi(z,1) = g(z)$ for each $z \in Z$ (written: $f \overset{\Phi}{\sim} g$ or $f \sim g$). If $f\colon Z \to Y$ is homotopic to a constant function, we say that f is null-homotopic ($f \sim 0$).

Remark 2.5 (a) If $f \overset{\Phi}{\sim} g$ and $g \overset{\Psi}{\sim} h$, then $f \overset{\Delta}{\sim} h$, where $\Delta = \Psi\circ\Phi$ is defined by

$$\Delta(z,t) = \begin{cases} \Phi(z,2t) & 0 \le t \le 1/2;\\ \Psi(z,2t-1) & 1/2 \le t \le 1.\end{cases}$$

(b) Let $f\colon S^n \to Y$ then, if $f \overset{\Phi}{\sim} 0$, the formula $\Phi(z,t) = F([1-t]z)$ defines a continuous extension $F\colon B_{\mathbb{R}^{n+1}} \to Y$ of f. Vice versa, if f admits a continuous extension $F\colon B_{\mathbb{R}^{n+1}} \to Y$, the same formula shows that f is null-homotopic.

Proof of Proposition 2.3 (i) By the compactness of S^n and by the continuity of f, we can assume that $\inf\|f(S^n)\| \ge 1$. By Fact 2.4 there exists a triangulation $\mathcal{T}$ of S^n

such that $\operatorname{diam}\big(f(\sigma)\big) < 1$ whenever $\sigma \in \mathcal{T}$. Suppose $t \in \sigma = (p_0, \dots, p_n) \in \mathcal{T}$, then there exists a unique choice of $\lambda_i \geq 0$ $(i = 0, \dots, n)$ such that

$$\sum_{i=0}^{n} \lambda_i = 1 \quad \text{and} \quad t = \Big(\sum_{i=0}^{n} \lambda_i p_i\Big) \Big/ \Big\| \sum_{i=0}^{n} \lambda_i p_i \Big\|.$$

Since $\mathcal{T}$ is a triangulation of S^n and since $\operatorname{diam}\big(f(\sigma)\big) < 1$ whenever $\sigma \in \mathcal{T}$, the map $\varphi\colon S^n \to X^*$, defined by $\varphi(t) = \sum_{i=0}^{n} \lambda_i f(p_i)$ $(t \in \sigma = (p_0, \dots, p_n) \in \mathcal{T})$, is well defined, continuous, and assumes values in $\Pi(C) \setminus \{0\}$ (remember that we assumed $\inf \|f(S^n)\| \geq 1$). Moreover, since $\operatorname{diam}\big(f(\sigma)\big) < 1$, we can continuously deform f in φ (in $\Pi(C) \setminus \{0\}$) using the homotopy defined by

$$h_1(\lambda, t) = \lambda\varphi(t) + (1 - \lambda) f(t) \quad (\lambda, t) \in [0, 1] \times S^n.$$

We claim that φ is homotopic to a constant function in $\Pi(C) \setminus \{0\}$. Without any loss of generality we can suppose that $0 \in C$ and hence that $\dim(X/L_C) > n + 1$, where L_C denotes the maximal closed subspace contained in C. Now, fix $\sigma = (p_0, \dots, p_n) \in \mathcal{T}$ and observe that

$$\dim\big(\operatorname{span}(\varphi(\sigma))\big) \leq n + 1 < \dim(X/L_C) = \dim \Pi(C).$$

Then there exists $x_0^* \in \Pi(C) \setminus \bigcup_{\sigma \in \mathcal{T}} \operatorname{span}(\varphi(\sigma))$. For each $(\lambda, t) \in [0, 1] \times S^n$, put $h_2(\lambda, t) = \lambda x_0^* + (1 - \lambda)\varphi(t)$. Since $\varphi(t) \neq 0$ and $x_0^* \notin \mathbb{R}\varphi(t)$ whenever $t \in S^n$, h_2 is well defined and deforms φ in the constant function x_0^* continuously (in $\Pi(C) \setminus \{0\}$). By Remark 2.5, $f \overset{h_2 \circ h_1}{\sim} 0$, and we can continuously extend f to $F\colon B_{\mathbb{R}^{n+1}} \to \Pi(C) \setminus \{0\}$. This proves our claim and concludes the first part of the proof.

Now, suppose that $\sup_{t \in S^n} \sup_C f(t) < \infty$. Looking at the first part of the proof, we can observe that $F(B_{\mathbb{R}^{n+1}}) \subset \operatorname{conv}(f(S^n) \cup \{x_0^*\})$. In particular, this implies that $\sup_{t \in B_{\mathbb{R}^{n+1}}} \sup_C F(t) \leq \max\{\sup_{t \in S^n} \sup_C f(t), \sup_C x_0^*\} < \infty$.

(ii) Suppose that $\operatorname{int} C \neq \varnothing$; otherwise the proof is trivial.

Suppose that C does not contain any closed affine subset of codimension $(n + 1)$. By [3, Theorem 1.3], either there exists a homeomorphism of ∂C onto a closed hyperplane of X or there exists a homeomorphism of ∂C onto the product of a closed subspace of X by S^m, for some $m > n$. Then the thesis follows by Fact 2.2.

If C does not contain any closed affine subset of finite codimension, by [3, Theorem 1.3], there exists a homeomorphism of ∂C onto a closed subspace of X, and hence ∂C is AR. ■

The following two propositions follow easily from Proposition 1.2 and Corollary 1.3.

Proposition 2.6 *Let X be a Banach space and $h\colon X \to (-\infty, \infty]$ a proper l.s.c. convex function.*

(i) *If $h_{|\operatorname{dom}(h)}$ is continuous,* $\mathbf{D}(\partial h)$ *is* AR.

Moreover, if X admits an equivalent Fréchet smooth norm:

(ii) $\mathbf{G}(\partial h)$ *is* AR*;*

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(iii) *if* h *is cofinite,* $\mathbf{R}(\partial h)$ *is* AR.

Proof To prove (i), let S_0 be a closed subset of a metric space S and suppose $\varphi\colon S_0 \to \operatorname{dom}(\partial h)$ is a continuous function. Let us extend φ to a continuous function $y_0\colon S \to \operatorname{conv}\varphi(S_0)$ and observe that $\operatorname{conv}(\varphi(S_0)) \subset \operatorname{dom}(h)$. Now, put $T = S \setminus S_0$ and fix $x^* \in X^*$ such that $\inf(h - x^*)(X) > -\infty$; since $h_{|\operatorname{dom}(h)}$ is continuous, the function $\varepsilon\colon T \to (0,\infty)$, defined by

$$\varepsilon(t) = (h - x^*)\big(y_0(t)\big) - \inf(h - x^*)(X) + 1,$$

is continuous, too. Hence, we can apply the second part of Proposition 1.2, with $T_0 = \varnothing$, $\varphi^* \equiv x^*$ and with $\lambda(t) = \operatorname{dist}(t, S_0)$ $(t \in T)$, in order to obtain a continuous function $v\colon T \to \operatorname{dom}(\partial h)$ such that $\|v(t) - y_0(t)\| < \operatorname{dist}(t, S_0)$. Now, we can continuously extend v to S by defining $v(s_0) = y_0(s_0) = \varphi(s_0)$, for each $s_0 \in S_0$.

To prove (iii), let S_0 and S be as above and suppose that $y^*\colon S_0 \to \mathbf{R}(\partial h)$ is a continuous function. Let us extend y^* to a continuous function $\varphi^*\colon S \to X^*$. By Proposition 1.2, with $T = S \setminus S_0$, $T_0 = \varnothing$, $\lambda \equiv 6$ and $\varepsilon(t) = \operatorname{dist}(t, S_0)$ $(t \in T)$, we get a continuous function $v^*\colon T \to \mathbf{R}(\partial h)$ such that $\|v^*(t) - \varphi^*(t)\| \le \operatorname{dist}(t, S_0)$ for each $t \in T$. As above, we can continuously extend v^* to S by defining $v^*(s_0) = y^*(s_0)$, for each $s_0 \in S_0$.

To prove (ii), let $g\colon X \to (-\infty, \infty]$ be a proper l.s.c. convex function. Let T_0 be a closed subset of a metric space T. Let $\varphi\colon T_0 \to X$ and $y^*\colon T_0 \to X^*$ be continuous mappings such that $(\varphi(t_0), y^*(t_0)) \in \mathbf{G}(\partial g)$ whenever $t_0 \in T_0$.

Suppose that $\|\cdot\|$, the norm of X, is Fréchet smooth, then, by Šmulyan's lemma, we can consider the continuous map $J\colon X \to X^*$ defined by

$$\partial(\|\cdot\|^2/2)(x) = \{x^* \in X^*;\ x^*x = \|x\|^2 = \|x^*\|^2\} = \{J(x)\}.$$

If we put $h(x) = g(x) + \|x\|^2/2$, for each $x \in X$, then $\operatorname{dom}(h) = \operatorname{dom}(g)$ and, by [14, Theorem 3.16], $\partial h(x) = J(x) + \partial g(x)$, for each $x \in \operatorname{dom}(g)$. If we define $\varphi^*(t_0) = y^*(t_0) + J(\varphi(t_0))$, for each $t_0 \in T_0$, then $\big(\varphi(t_0), \varphi^*(t_0)\big) \in \mathbf{G}(\partial h)$ whenever $t_0 \in T_0$. Moreover, by the Dugundji extension theorem, we can assume that φ^* is continuous and defined on T with values in X^*.

Now, it is easy to see that h is supercoercive. Then we can put $\varepsilon(t) = 1$, for each $t \in T$, and apply the first part of Proposition 1.2 to get y_0 as in (1.1). Put $\lambda(t) = 1$, for each $t \in T$. By the second part of Proposition 1.2, we obtain continuous functions $v\colon T \to \operatorname{dom}(h)$ and $v^*\colon T \to X^*$ such that $v|_{T_0} = \varphi$, $v^*|_{T_0} = \varphi^*$ and $v^*(t) \in \partial h(v(t))$, for each $t \in T$. Put $w^*(t) = v^*(t) - J(v(t))$, for each $t \in T$. Then $w^*|_{T_0} = y^*$ and $w^*(t) \in \partial g(v(t))$, for each $t \in T$. ■

Proposition 2.7 *Let* C *be a nonempty closed convex subset of* X *and suppose that* X *admits an equivalent Fréchet smooth norm.*

(i) *If* C *is bounded and* X *is infinite-dimensional, then* $\operatorname{Supp}(C)$ *is* AR.
(ii) *If* C *does not contain any closed affine subset of codimension* $(n+1)$*, then* $\operatorname{Supp}(C)$ *is* n*-connected.*
(iii) *If* C *is bounded and* X *is infinite-dimensional, then* $\Sigma(C)$ *is* AR.

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Proof (i) Let T_0 be a closed subset of a metric space T. Let $f = (\varphi, \theta^*)$, where $\varphi\colon T_0 \to \operatorname{supp}(C)$ and $\theta^*\colon T_0 \to X^* \setminus \{0\}$ are continuous mappings such that $\theta^*(t)\varphi(t) = \sup_C \theta^*(t)$ whenever $t \in T_0$. We just have to prove that f admits a continuous extension on T with values in $\operatorname{Supp}(C)$.

Extend θ^* to a continuous map $\varphi^*\colon T \to X^* \setminus \{0\}$ using Lemma 1.5. Applying Corollary 1.3, with $\lambda = 12$ and $\varepsilon = \|\varphi^*(t)\|$, for each $t \in T$, we get $v\colon T \to X$ and $v^*\colon T \to X^*$ such that

(a) $v|_{T_0} = \varphi$ and $v^*|_{T_0} = \theta^*$;
(b) $v^*(t)v(t) = \sup_C v^*(t)$ for each $t \in T$;
(c) $\|\varphi^*(t) - v^*(t)\| \leq \|\varphi^*(t)\|/2$ (and then $v^*(t) \neq 0$) whenever $t \in T$;
(d) v and v^* are continuous.

Now, put $F = (v, v^*)$. By (a), F extends f; by (b) and (c), $F(t) \in \operatorname{Supp}(C)$ whenever $t \in T$; (d) concludes the proof.

(ii) Let $T_0 := S^n \subset B_{\mathbb{R}^{n+1}} =: T$ and $f = (\varphi, \theta^*)$, where φ and θ^* are defined as in the proof of (i). Using the Dugundji extension theorem, we can extend φ to a continuous map $y_0\colon T \to \operatorname{conv}\big(\varphi(T_0)\big)$. Since the map $t \mapsto \theta^*(t)\varphi(t) = \sup_C \theta^*(t)$ $(t \in T_0)$ is continuous, by the compactness of T_0, we have $\sup_{t\in T_0} \sup_C \theta^*(t) < \infty$; hence, by Proposition 2.3(i), we can extend θ^* to a continuous map $\varphi^*\colon T \to \Pi(C) \setminus \{0\}$ such that $\sup_{t\in T} \sup_C \varphi^*(t) < \infty$.

By the compactness of T and by the continuity of the map $t \mapsto \varphi^*(t)y_0(t)$, we get $\inf_{t\in T} \varphi^*(t)y_0(t) > -\infty$. Hence there exists $\varepsilon > 0$ such that

$$\varphi^*(t)y_0(t) \geq \sup_C \varphi^*(t) - \varepsilon,$$

whenever $t \in T$. Now we just have to apply Corollary 1.3, with $\lambda(t) = 12\varepsilon/\|\varphi^*(t)\|$ $(t \in T)$, and proceed as in the proof of (i).

(iii) Let S_0 be a closed subset of a metric space S. Suppose $f\colon S_0 \to \Sigma(C)$ is a continuous function. By Lemma 1.5, we can extend f to a continuous function $\varphi^*\colon S \to X^* \setminus \{0\}$. Let us consider the metric space $T = S \setminus S_0$ and define $\varepsilon\colon T \to \mathbb{R}^+$ by $\varepsilon(t) = \min\{\operatorname{dist}(t, S_0), \|\varphi^*(t)\|/2\}$, for each $t \in T$. We can apply Corollary 1.3, with $T_0 = \varnothing$ and $\lambda \equiv 6$ to obtain a continuous function $v^*\colon T \to \Sigma(C)$ such that, for each $t \in T$,

$$\|v^*(t) - \varphi^*(t)\| \leq \operatorname{dist}(t, S_0). \tag{2.1}$$

Now, by (2.1), we can continuously extend v^* to S by defining $v^*(s_0) = f(s_0)$, for each $s_0 \in S_0$. ∎

Now, we investigate connectedness properties of the set of all support points of a convex closed set in X.

Lemma 2.8 *Let S_0 be a closed subset of a metric space S. Suppose that C is a closed convex subset of X. Let $f\colon S_0 \to \operatorname{supp}(C)$ be a continuous function that admits a continuous extension $\Phi\colon S \to X$ such that $\Phi(S \setminus S_0) \subset X \setminus C$. Then f admits a continuous extension $v\colon S \to \operatorname{supp}(C)$.*

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Proof Suppose that the multifunctions F and G^* are defined as in Lemma 1.7 and put $T = S \setminus S_0$. By the Michael selection theorem, the continuity of Φ and Lemma 1.7, we can find continuous selections $\varphi\colon T \to C$ and $\varphi^*\colon T \to B_{X^*}$ of the multifunctions $T \ni t \mapsto F(\Phi(t))$ and $T \ni t \mapsto G^*(\Phi(t))$, respectively. By the definition of F and G^*, we have, for each $t \in T$:

$$\|\varphi^*(t)\| \geq 1/2 \quad \text{and} \quad \varphi^*(t)\varphi(t) \geq \sup_C \varphi^*(t) - \operatorname{dist}(\Phi(t), C).$$

We are going to apply the second part of Corollary 1.3, with $\varepsilon(t) = \operatorname{dist}(\Phi(t), C)$ and $\lambda(t) = 24\varepsilon(t)$ $(t \in T)$. In this way, we obtain mappings $v\colon T \to C$ and $v^*\colon T \to X^*$ such that

(a) v is continuous;
(b) $\|v(t) - \varphi(t)\| < 24\varepsilon(t)$, for each $t \in T$;
(c) $\|v^*(t) - \varphi^*(t)\| \leq 1/4$, for each $t \in T$, and then $\|v^*(t)\| \geq 1/4$, for each $t \in T$;
(d) $v^*(t)v(t) = \sup_C v^*(t)$, for each $t \in T$, and then $v(t) \in \operatorname{supp}(C)$, for each $t \in T$.

To conclude the proof, we just have to observe that the continuous function $v\colon T \to \operatorname{supp}(C)$ can be continuously extended to S, by putting $v(s_0) = f(s_0)$, for each $s_0 \in S_0$. ■

The following theorem is the main result of this section.

Theorem 2.9 *Suppose that C is a nonempty closed convex subset of X.*

(i) *If C does not contain any closed affine subset of codimension $(n+1)$, then* $\operatorname{supp}(C)$ *is n-connected.*
(ii) *If C does not contain any closed affine subset of finite codimension, then* $\operatorname{supp}(C)$ *is* AR(σ-compact) *and contractible.*
(iii) *If* $\operatorname{int}(C - C) = \varnothing$, *then* $\operatorname{supp}(C)$ *is* AR. *In particular, if C has a central symmetry (*i.e., *there exists $x_0 \in C$ such that $x_0 - C = C - x_0$) and C does not contain any closed affine subset of finite codimension, then* $\operatorname{supp}(C)$ *is* AR.

Proof (i) If $\operatorname{int} C \neq \varnothing$, the thesis holds by Proposition 2.3(ii) and by the fact that, in this case, $\operatorname{supp}(C) = \partial C$. Suppose that $\operatorname{int} C = \varnothing$ and let $f\colon S^n \to \operatorname{supp}(C)$ be a continuous function. By Lemma 1.9, f admits a continuous extension $\Phi\colon B_{\mathbb{R}^{n+1}} \to X$ such that $\Phi(B_{\mathbb{R}^{n+1}} \setminus S^n) \subset X \setminus C$. The thesis holds, by Lemma 2.8.

(ii) The proof that $\operatorname{supp}(C)$ is AR(σ-compact) is the same as in (i). Let us prove that $\operatorname{supp}(C)$ is contractible, that is, $Id_{\operatorname{supp}(C)}$, the identity map on $\operatorname{supp}(C)$, is homotopic null in $\operatorname{supp}(C)$.

If $\operatorname{supp}(C) = C$, the proof is trivial, since in this case $\operatorname{supp}(C)$ is a convex set. So, without any loss of generality, we can suppose that $0 \in C \setminus \operatorname{supp}(C)$. Put $\widetilde{C} = \overline{C + B_X}$. We claim that $\mu_{\widetilde{C}}$, the Minkowski functional of the closed convex set $\widetilde{C}$, is positive on the set $\operatorname{supp}(C)$. If this is not the case, there exists $x \in \operatorname{supp}(C)$ such that the half-line $\mathbb{R}^+x$ is contained in $\widetilde{C}$. Then, for each $n \in \mathbb{N}$, there exists $c_n \in C$ such that

$$2 \geq \|nx - c_n\| = \tfrac{n}{2}\|2x - \tfrac{2}{n}c_n\|.$$

Since C is closed and convex, if we let $n \to \infty$, we obtain $2x \in C$. But x is a support point of C and $0 \in C \setminus \operatorname{supp}(C)$, so we get a contradiction and our claim is proved. Then $0 < \mu_{\widetilde{C}}(s) < 1$ for each $s \in \operatorname{supp}(C)$.

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Put $f(s) = \frac{s}{\mu_{\widetilde{C}}(s)}$ for each $s \in \operatorname{supp}(C)$ and consider the continuous map

$$\theta_0 \colon \operatorname{supp}(C) \times [0,1] \longrightarrow X,$$

defined by $\theta_0(s,\lambda) = (1-\lambda)s + \lambda f(s)$. It is easy to see that

(a) $\theta_0(\,\cdot\,,0) = Id_{\operatorname{supp}(C)}$;
(b) $\theta_0(s,\lambda) \notin C$ if $\lambda \in (0,1]$ and $s \in \operatorname{supp}(C)$;
(c) $\theta_0(s,1) = f(s) \in \partial\widetilde{C}$ for each $s \in \operatorname{supp}(C)$.

It is easy to see that, since C does not contain any closed affine subset of finite codimension, $\widetilde{C}$ does not contain any closed affine subset of finite codimension either. Then, by Proposition 2.3(ii), there exists $x_0 \in \partial\widetilde{C}$ and a continuous map $\theta_1 \colon \operatorname{supp}(C) \times [0,1] \to \partial\widetilde{C}$ such that

(a) $\theta_1(\,\cdot\,,0) = f$;
(b) $\theta_1(s,\lambda) \in \partial\widetilde{C}$ if $\lambda \in [0,1]$ and $s \in \operatorname{supp}(C)$;
(c) $\theta_0(s,1) = x_0$ for each $s \in \operatorname{supp}(C)$.

Now, by the Bishop–Phelps theorem (see *e.g.*, [14, Proposition 3.20]), there exists $s_0 \in \operatorname{supp}(C)$ such that $[s_0, x_0] \cap C = \{s_0\}$. Let us define the continuous map $\theta_2 \colon \operatorname{supp}(C) \times [0,1] \to X$, defined by $\theta_2(s,\lambda) = (1-\lambda)x_0 + \lambda s_0$, and let us define $\Phi = \theta_2 \circ (\theta_1 \circ \theta_0)$, the product of the homotopies described above. Then

(a) $\Phi(\,\cdot\,,0) = Id_{\operatorname{supp}(C)}$;
(b) $\Phi(s,\lambda) \notin C$ if $\lambda \in (0,1)$ and $s \in \operatorname{supp}(C)$;
(c) $\Phi(s,1) = s_0 \in \operatorname{supp}(C)$ for each $s \in \operatorname{supp}(C)$.

To conclude the proof, we can apply Lemma 2.8 with $S = \operatorname{supp}(C) \times [0,1]$ and $S_0 = (\operatorname{supp}(C) \times \{0\}) \cup (\operatorname{supp}(C) \times \{1\})$.

(iii) The proof of the first part is similar to the proof of (i). We just have to use Remark 1.10 instead of Lemma 1.9. The proof of the second part follows by the second part of Proposition 2.3 and by the obvious fact that, if C has a central symmetry, $\operatorname{int}(C) = \varnothing$ if and only if $\operatorname{int}(C - C) = \varnothing$. ∎

Definition 2.10 (*cf.* [9, Definition 1.1, § 6]) Let V and W be topological spaces. A map $f \colon V \to W$ is called compact if $f(V)$ is contained in a compact subset of W.

Corollary 2.11 *Let $C \in \mathcal{BCC}(X)$. Suppose that X is infinite-dimensional and that we are in one of the following cases:*

(i) $W = \operatorname{supp}(C)$*;*
(ii) $W = \operatorname{Supp}(C)$ *and X is Fréchet smooth;*
(iii) $W = \Sigma(C)$ *and X is Fréchet smooth.*

Then every continuous compact map $f \colon W \to W$ has a fixed point.

Proof We just prove (i), as the proofs of (ii) and (iii) are similar. Let K be a compact subset of $\operatorname{supp}(C)$ such that $f(\operatorname{supp}(C)) \subset K$. Since K is a compact subset of C, by the first part of Theorem 2.9(ii), we can extend Id_K, the identity map on K, to a continuous function $j \colon C \to \operatorname{supp}(C)$. Now, $f \circ j \colon C \to C$ is a compact map and hence, by [9, Theorem 3.2, § 6], $f \circ j$ has a fixed point $x_0 \in f \circ j(C) \subset K$. Then $j(x_0) = x_0$ is a fixed point for f. ∎

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Remark 2.12 (a) In the case X admits an equivalent norm that is Gâteaux smooth, similar results to those stated in Proposition 2.6(ii), (iii), and (iv), and in Proposition 2.7 can be obtained in the w^*-topology, using Proposition 1.2(vi), and Corollary 1.3(vi). For example, the following results hold.

> Suppose that T_0 is a closed subset of a metric space T, that $C \in \mathcal{BCC}(X)$, that X is infinite-dimensional, and that is Gâteaux smooth. Then every $\|\cdot\|$-continuous map $\varphi^*\colon T_0 \to \Sigma(C)$ admits an extension $\nu^*\colon T \to \Sigma(C)$ that is w^*-continuous on T.
>
> Let C be a nonempty closed convex subset of a Gâteaux smooth Banach space X. If C does not contain any closed affine subset of codimension $(n+1)$, then every $\|\cdot\| \times \|\cdot\|$-continuous map $\varphi\colon S^n \to \operatorname{Supp}(C)$ admits an extension $\Phi\colon B_{\mathbb{R}^n} \to \operatorname{Supp}(C)$ that is $\|\cdot\| \times w^*$-continuous.

(b) All the results presented in this section are stated in the setting of metric spaces. Some of these results (more precisely, Proposition 2.6(ii), and Proposition 2.7(i)) can be easily straightened in the more general setting of Hausdorff paracompact spaces. In fact, we can use [1, Corollary 7.5] instead of the Dugundji extension theorem, in the proof of Proposition 2.6(ii). Moreover we can observe that Lemma 1.5 still holds in the case where T is paracompact. However, not all the results presented in Section 2 can be generalized, in this way, to the setting of Hausdorff paracompact spaces. In fact, in the proof of Proposition 2.6(i) and (iii), Proposition 2.7(iii), Theorem 2.9, and Lemma 1.9, we use the following fact. If T_0 is a closed subset of a metric space T, then $T \setminus T_0$ is a metric (and hence a Hausdorff paracompact) space and there exists a continuous function $\varepsilon\colon T \to [0,\infty)$ such that $\varepsilon^{-1}(0) = T_0$. In general, this is not true if T is just a paracompact space. Moreover, in the proof of Lemma 1.9, we cannot use [1, Corollary 7.5] instead of the Dugundji extension theorem.

3 Finite-dimensional and Finite-codimensional Subspaces

Definition 3.1 Let W be a subspace of X and $K \in \mathcal{BCC}(X)$. Let us denote

$$\operatorname{supp}_W(K) = \{x \in K; y^*(x) = \sup y^*(K) \text{ for some } y^* \in S_{X^*} \cap W^\perp\}.$$

Let $q\colon X \to X/W$ be the quotient map and denote $\widehat{K} = q(K)$. Then $\widehat{K}$ is a bounded convex set in X/W. Suppose that $y^* \in W^\perp = (X/W)^*$, then we have

$$\sup y^*(\widehat{K}) = \sup y^*(K+W) = \sup y^*(K).$$

The above observation easily implies the following corollary.

Corollary 3.2 *In the above notations, if we suppose that $\widehat{K}$ is closed in X/W, we get*

$$\widehat{K} \in \mathcal{BCC}(X/W), \quad \operatorname{supp}_W(K) = q^{-1}\big(\operatorname{supp}(\widehat{K})\big) \cap K, \quad \Sigma(K) \cap W^\perp = \Sigma(\widehat{K}).$$

Proposition 3.3 *Let V, W be closed subspaces of X. Suppose that V is finite-codimensional in X, $W \subset V$ and V/W has at least dimension two. For each $K \in \mathcal{BCC}(X)$ such that $V \cap K \neq \varnothing$, we have:*

(i) $\operatorname{supp}(K) \cap V$ *is pathwise connected;*
(ii) *if W is finite-dimensional (respectively, W is reflexive),* $\operatorname{supp}_W(K) \cap V$ *is connected (respectively,* $\operatorname{supp}_W(K) \cap V$ *is connected in the w-topology);*
(iii) *if X is a dual space, W, K are closed in the w^*-topology, then* $\operatorname{supp}_W(K) \cap V$ *is connected in the w^*-topology;*
(iv) *if W is reflexive, then $\Sigma_1(K) \cap W^\perp$ is uncountably dense in $S_{X^*} \cap W^\perp$ and $\Sigma(K) \cap W^\perp$ is connected in the w^*-topology. Moreover, if X admits an equivalent Fréchet (respectively, Gâteaux) smooth norm, $\Sigma(K) \cap W^\perp$ is pathwise connected (respectively, pathwise connected in the w^*-topology).*

Proof (i) By [2, Lemma 4], $\operatorname{supp}(K) \cap V = \operatorname{supp}(K \cap V)$, where the last set denotes the support points of $K \cap V$ in the Banach space V. Apply [4, Theorem 3.3].

(ii) First of all, observe that, if W is reflexive, then $\widehat{K} = q(K)$ is a closed set in X/W. Indeed, if $\widehat{x} \in \overline{q(K)}$, we have

$$0 = \operatorname{dist}\big(\widehat{x}, q(K)\big) = \operatorname{dist}\big(K, q^{-1}(\widehat{x})\big).$$

Since $q^{-1}(\widehat{x})$ is a boundedly w-compact affine set in X, there exists $x \in q^{-1}(\widehat{x}) \cap K$, which means that $\widehat{x} \in q(K)$.

Now, suppose that W is finite dimensional (respectively, W is reflexive), and consider the multifunction $F\colon \widehat{K} \to \mathcal{BCC}(X)$, given by $F(\widehat{x}) = q^{-1}(\widehat{x}) \cap K$. We claim that F is u.s.c. (respectively w-u.s.c.). If this is not the case, there exist a closed (respectively, a w-closed) set $D \subset X$ and a sequence $\{\widehat{x}_n\} \subset \widehat{K}$ such that $\widehat{x}_n \to \widehat{x} \in \widehat{K}$, $F(\widehat{x}_n) \cap D \neq \varnothing$, and $F(\widehat{x}) \cap D = \varnothing$. Since q^{-1} admits a continuous selection by the Bartle–Graves theorem [8, Corollary 7.56], there exist $x_n \in q^{-1}(\widehat{x}_n)$ and $w_n \in W$ such that $x_n \to x \in q^{-1}(\widehat{x})$ and $x_n + w_n \in K \cap D$ for each $n \in \mathbb{N}$. Observe that the sequence $\{w_n\}$ is bounded and contained in W. Thus we can suppose that $w_n \to w \in W$ (respectively, $w_n \to w \in W$ in the w-topology). Now, since $K \cap D \ni x_n + w_n \to x + w, x + w \in q^{-1}(\widehat{x}) \cap D \cap K = F(\widehat{x}) \cap D$. This contradiction proves our claim.

Let $x_0, x_1 \in \operatorname{supp}_W(K) \cap V$. Then the points $\widehat{x}_i = q(x_i)$ $(i = 0, 1)$ belong to $\operatorname{supp}(\widehat{K}) \cap q(V)$. Let us observe that $q(V)$ is a closed finite-codimensional subspace of X/W. By (i), there exists a continuous mapping $\gamma\colon [0, 1] \to \operatorname{supp}(\widehat{K}) \cap q(V)$ such that $\gamma(i) = \widehat{x}_i$ $(i = 0, 1)$. By Corollary 3.2, by our claim and since $W \subset V$, $F \circ \gamma$ is an u.s.c. (respectively, a w-u.s.c.) multifunction with nonempty convex closed values contained in $\operatorname{supp}_W(K) \cap V$. It follows (*e.g.,* by [13, Lemma 6]) that the image $(F \circ \gamma)([0, 1])$ is connected (respectively, is connected in the w-topology), is contained in $\operatorname{supp}_W(K) \cap V$ and contains x_0, x_1. This completes the proof, since x_0, x_1 are arbitrary points of $\operatorname{supp}_W(K) \cap V$.

For the proof of (iii), we just observe that, since W is w^*-closed, the quotient map $q\colon X \to X/W$ is w^*-w^*-continuous and $\widehat{K} = q(K)$ is w^*-compact. In particular, $\widehat{K}$ is closed, and we can then proceed as in the proof of (ii).

(iv) The first part follows immediately from Corollary 3.2, [4, Theorem 3.4], and the following claim.

Claim Let C be a nonempty bounded closed convex subset of a real, at least two dimensional, Banach space X. Then $\operatorname{Supp}(C) \subset (X, \|\cdot\|) \times (X^*, w^*)$ is connected and hence $\Sigma(C)$ is connected in the w^*-topology.

Proof of the claim We just have to prove that if $(x_i, x_i^*) \in \text{Supp}(C)$ $(i = 0, 1)$, then there exists a $\|\cdot\| \times w^*$-connected set $\Gamma \subset \text{Supp}(C)$ joining (x_0, x_0^*) and (x_1, x_1^*). Without any loss of generality, we can suppose that $x_0^* \neq -x_1^*$. Put $\varphi^*(t) = (1-t)x_0^* + tx_1^*$, for each $t \in [0,1]$. Let $\varepsilon > 0$ be such that the w^*-compact set $B = [x_0^*, x_1^*] + \varepsilon B_{X^*}$ does not contain the origin of X^*. Apply Corollary 1.3, with $T_0 = \{0,1\} \subset T = [0,1]$ and $\lambda = 6$, to get $v\colon [0,1] \to C$ and $v^*\colon [0,1] \to X^*$ such that

(a) $\big(v(i), v^*(i)\big) = (x_i, x_i^*)$ $(i = 0, 1)$;
(b) $v^*(t)v(t) = \sup_C v^*(t)$ for each $t \in T$;
(c) $\|v^*(t) - \varphi^*(t)\| \leq \varepsilon$ for each $t \in [0,1]$;
(d) v is continuous.

It is not difficult to see (*cf.* the proof of [15, Lemma 2.5]) that the map $\vartheta\colon [0,1] \to 2^{(X, \|\cdot\|) \times (X^*, w^*)}$, defined by

$$\vartheta(t) = \{v(t)\} \times \big[B \cap \partial I_C\big(v(t)\big)\big],$$

is u.s.c. and assumes nonempty connected values. It follows (*e.g.*, by [13, Lemma 6]) that the set $\Gamma = \bigcup_{t \in [0,1]} \vartheta(t) \subset \text{Supp}(C)$ is connected in $(X, \|\cdot\|) \times (X^*, w^*)$. Moreover, $(x_i, x_i^*) \in \Gamma$ $(i = 0, 1)$, and the proof of the claim is complete. ■

The latter part follows from Corollary 3.2, [15, Theorem 3.2(c)] and the fact that since W is reflexive, X/W admits an equivalent Fréchet smooth norm (respectively, from Corollary 3.2, Remark 2.12(a), and the fact that, since W is reflexive, X/W admits an equivalent Gâteaux smooth norm). ■

Proposition 3.3(iv), says, in particular, that, if H is a finite-codimensional w^*-closed subspace of the topological dual X^* of an infinite dimensional Banach space X and $K \in \mathcal{BCC}(X)$, then $\Sigma_1(K)$ is uncountably dense in H. It seems to be not completely clear if it is possible, in general, to omit the assumption H is w^*-closed. However, this can be done if X admits an equivalent Fréchet smooth norm (*cf.* the proposition below).

Definition 3.4 Let X be a Banach space and $n \in \mathbb{N}$, suppose that $X = H \oplus V$ and that V is n-dimensional. Let $\Theta\colon H \to V$ be a continuous function. Then we say that

$$L = \big\{\big(h, \Theta(h)\big) \in H \oplus V;\ h \in H\big\}$$

is a continuous n-codimensional surface of X.

The following lemma is a particular case of the *Leray-Schauder continuation principle* (see, *e.g.*, [9, Theorem 6.5, § 12]).

Lemma 3.5 *Let V be a finite-dimensional normed space and let $f\colon B_V \times [0,1] \to V$ be a continuous function such that*

(i) $f(v, \mu) \neq 0$, *for each* $(v, \mu) \in S_V \times (0,1)$;
(ii) $f(v, 0) = v$, *for each* $v \in B_V$;
(iii) $f(v, 1) = 0$ *if and only if* $v = 0$.

Then there exists a continuum $S \subset f^{-1}(0)$ *joining* $(0,0)$ *and* $(0,1)$.

Proposition 3.6 *Let X be an infinite-dimensional Banach space that admits an equivalent Fréchet smooth norm, L a continuous n-codimensional surface of X^*, and $C \in \mathcal{BCC}(X)$. Then*

(i) *$\Sigma(C) \cap L$ is dense in L;*
(ii) *$\Sigma(C) \cap L$ is connected and locally connected.*

Proof Let $X^* = H \oplus V$ and let us denote by P_H and P_V the linear bounded projections onto H and V, respectively. Suppose that V is n-dimensional and that L is a continuous n-codimensional surface of X^* defined by a continuous function $\Theta\colon H \to V$.

If $x_0 = (h_0, v_0) = (h_0, \Theta(h_0)) \in L$ and $\alpha, \beta > 0$, define

$$W(x_0, \alpha, \beta) = \left\{ (h, v) \in H \oplus V;\ \|h - h_0\| < \alpha, \|v - v_0\| < \beta \right\}.$$

We have the following claim.

Claim Put $W = W(x_0, \alpha, \beta)$. Suppose that $\beta \le 1/2$, that $\|h - h_0\| < \alpha$ implies $\|\Theta(h) - v_0\| < \beta$, that $0 \notin W$, and fix $x_1 = (h_1, v_1) = (h_1, \Theta(h_1)) \in L \cap W$. Then there exists a connected set $\Gamma \subset L \cap W$ such that $x_i \in \Gamma$ $(i = 0, 1)$ and such that $\Gamma \setminus \{x_0, x_1\} \subset \Sigma(C)$.

Proof of the claim. Let us define a continuous function $\varphi^*\colon (2B_V) \times [0,1] \to H \oplus V$ by $\varphi^*(v, \mu) = ((1-\mu)h_0 + \mu h_1, (1-\mu)v_0 + \mu v_1 + v)$. Fix any $c \in C$ and let us observe that, by the compactness of $(2B_V) \times [0,1]$ and by the continuity of the map $x^* \mapsto \sup_C x^*$, there exists $\varepsilon > 0$ such that

$$\varphi^*(t)c \ge \sup_C \varphi^*(t) - \varepsilon$$

for each $t \in T := (2B_V) \times (0,1)$. Now we can apply the second part of Corollary 1.3, with $T_0 = \varnothing$, with $\varphi \equiv c$, and with $\lambda\colon T \to (0, \infty)$ a continuous function such that, for each $t = (v, \mu) \in T$,

$$6\varepsilon/\lambda(t) < \min\{\alpha\mu(1-\mu)/\|P_H\|, 1/\|P_V\|\},$$

in order to obtain a continuous function $v^*\colon T \to \Sigma(C)$ such that

$$\|v^*(t) - \varphi^*(t)\| < \min\{\alpha\mu(1-\mu)/\|P_H\|, 1/\|P_V\|\}$$

for each $t = (v, \mu) \in T$. Then it is not difficult to see that $\|P_H v^*(t) - h_0\| < \alpha$ and hence that $\|\Theta\big(P_H v^*(t)\big) - v_0\| < \beta$, for each $t \in T$. Define a function $f\colon (2B_V) \times [0,1] \to V$ by $f(v, \mu) = P_V v^*(v, \mu) - \Theta(P_H v^*(v, \mu))$, for each $\mu \in (0,1)$ and $v \in 2B_V$, and by $f(v, \mu) = v$ if $\mu \in \{0, 1\}$, $v \in 2B_V$. Since for $i = 0, 1$ and $v \in 2B_V$,

$$f(v, i) = P_V \varphi^*(v, i) - \Theta\big(P_H \varphi^*(v, i)\big),$$

and since, for $t = (v, \mu) \in T$, $\|v^*(t) - \varphi^*(t)\| < \alpha\mu(1-\mu)/\|P_H\|$, f is continuous on $2B_V \times [0,1]$. Moreover, for each $(v, \mu) \in (2S_V) \times (0,1)$,

$$\|f(v, \mu) - v\| \le \|P_V[v^*(v, \mu) - \varphi^*(v, \mu)]\| + \beta + \mu\beta < 2 = \|v\|.$$

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Hence, $f(v,\mu) \neq 0$ for each $(v,\mu) \in (2S_V) \times (0,1)$. By Lemma 3.5, there exists a continuum $S \subset f^{-1}(0)$ joining $(0,0)$ and $(0,1)$. Moreover, by our construction, $S \setminus \{(0,0),(0,1)\} \subset T$. Then the set $\Gamma = v^*(S \setminus \{(0,0),(0,1)\}) \cup \{x_0, x_1\}$ satisfies the conclusion of the claim. ■

The claim above immediately implies that $\Sigma(C) \cap L$ is dense in L and that $\Sigma(C) \cap L$ is locally connected.

Let us prove that $\Sigma(C) \cap L$ is connected. If $x_0 = (h_0, v_0) \in \Sigma(C) \cap L$, put $\beta_0 = 1/2 \min\{1, \|x_0\|\}$ and define:

$$\alpha_0 = 1/2 \sup\big\{ \alpha \in \mathbb{R}^+ ; [\|h - h_0\| < \alpha \Rightarrow \|\Theta(h) - v_0\| < \beta_0],\ 0 \notin W(x_0, \alpha, \beta_0) \big\}.$$

Put $W_{x_0} = W(x_0, \alpha_0, \beta_0)$. It is easy to see that $\{W_{x_0} \cap L\}_{x_0 \in \Sigma(C) \cap L}$ is an open cover of $L \setminus \{0\}$.

Now, if we fix $x_0, x_1 \in \Sigma(C) \cap L$, since $L \setminus \{0\}$ is connected, by [7, Problem 6.3.1], there exist $y_1, \dots, y_k \in \Sigma(C) \cap L$ such that $x_0 \in W_{y_1} \cap L$, $x_1 \in W_{y_k} \cap L$ and, for $i, j \in \{1, \dots, k\}$, $W_{y_i} \cap W_{y_j} \cap L \neq \varnothing$ if and only if $|i - j| \leq 1$. By our claim, $\bigcup_{i=1}^{k} W_{y_i} \cap \Sigma(C) \cap L$ is a connected set containing x_0 and x_1. ■

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