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A FLUID SELF-EXCITED DYNAMO

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ABSTRACT

The possibility that a simply connected perfectly conducting fluid body could generate an increasing external magnetic field by acting as a self-excited dynamo is demonstrated by exhibiting a cycle of motions that doubles the external field each cycle. The essential feature of the motion is that interior points become surface points. This requires points which were originally adjacent to become widely separated. The possibility of such motions is demonstrated and some of the conditions that might lead to their development are considered.

Ever since 1919 when Larmor [1] suggested that the magnetic fields of sunspots, the earth, and the sun might be maintained by self-excited dynamo action, there have been discussions as to whether such a dynamo was really possible in a simply connected body of fluid in which there is nothing that resembles a commutator. Cowling [2] showed that a self-maintaining dynamo is impossible in a medium of finite conductivity if one requires axial symmetry. Elsasser [3] and Bullard [4] have treated much more complicated motions described by series of harmonics with results that suggest, although they are not always regarded as demonstrating, that such a dynamo is possible. Parker [5] finds that in a sphere of fluid which rotates non-uniformly and has a convective zone the eddies provide an important contribution to the dynamo action. Since these models are quite complicated, it seems worthwhile to investigate some of the essential features of a self-excited dynamo with the aid of an over-simplified model.

One simplification is to regard the conductivity as being infinite. In this case any field that is once established is maintained if the fluid remains stationary: here the problem of interest is to find a motion that increases the field external to the body. Bondi and Gold [6] have shown the great importance in this connexion of the theorem [7] that in a perfectly conducting fluid the material inside a magnetic tube of force at one instant will always lie in a tube which may be regarded as the same tube of force.

They considered only motions in which, because of a continuity condition, the surface is always made up of the same particles. Hence the same tubes of force always end on the surface. Since their number cannot increase, one can get nothing resembling a self-excited dynamo in a simply connected body. One can also get nothing like a pair of sunspots that suddenly appear when a loop of magnetic field lines is pushed up through the surface. It therefore appears likely that Bondi and Gold's continuity condition need not always hold.

Let us consider more general fluid motions for which the continuity conditions are not quite so stringent. First consider a simple cycle [8] of fluid flow that doubles the dipole field of a sphere of perfectly conducting fluid. Increase by any power of two can be obtained by repeating the cycle the required number of times. Although the model is so over-simplified that the cycle is not closely related to the processes that maintain the earth's magnetic fields or the processes that occur in stars, it does seem to cast light on the necessary properties of more realistic models. Fig. 1 shows the motion schematically in cross-section, any parallel cross-section appearing the same except for scale. One starts with a uniform magnetic field inside, and hence a dipole field outside, all produced by surface currents. A counter-clockwise eddy in the upper hemisphere and a clockwise eddy in the lower carry the configuration from (*a*) through (*b*) to (*c*), where the fields in the two hemispheres are parallel but oppositely directed. Thus the field is now largely quadrupole. A rigid body rotation of the lower hemisphere through 180° then gives an internal field which has the same direction everywhere and twice the flux of the original field. If one were dealing with a cube the internal field would be uniform and the cycle would be complete. With a sphere the internal field is not uniform but can be made so by further motion in which the tubes of force are shifted while remaining parallel to themselves. The simplest such motion lies in the planes shown in Fig. 1, but it changes the fluid densities. If the fluid is incompressible, a motion involving four eddies in the plane normal to the diameter through 1 and 1′ can lead to a uniform field, although some further separation of adjacent points is required. The ultimate location of each tube of force must be such that its length is the same as that of the original tube of force of which it formed one half. The ultimate result is a uniform internal and dipole external field of twice the original strength and a different direction as shown in (*d*).

It is at once apparent that the essential reason for the success of this mechanism is the crowding of all of the original surface points into a fraction of the final surface and the appearance on the remainder of the surface of the ends of tubes of force that were originally joined in the interior. In the model considered, all the surface points of Fig. 1 (*a*) are carried to the left half of (*c*), and the right half of the surface of (*c*) is made

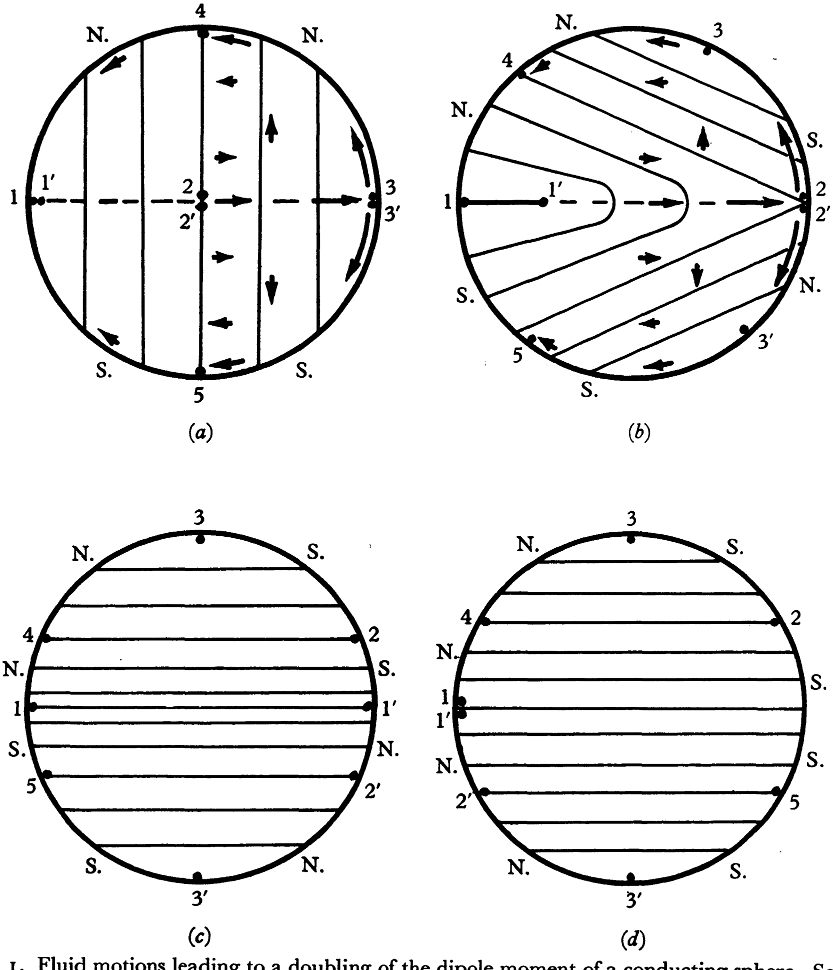

Fig. 1. Fluid motions leading to a doubling of the dipole moment of a conducting sphere. Solid lines represent tubes of force, the dotted line a plane along which the fluid will separate, numbers label certain particles of fluid, and arrows give the fluid velocities that will carry each configuration into the succeeding one. N. and S. give the nature of the effective surface poles. To go from *c* to *d* the bottom half of the system is rotated through 180° about the axis from 3′ to 3 and the tubes are shifted parallel to themselves to give a uniform internal field.

up of points which were on the equatorial plane of (*a*). Thus points such as 2 and 2′ must move through the stagnation point at the original location of 3 in a finite time; and points which were originally adjacent must become widely separated. Such motions do not occur at the usual stagnation points of a perfect fluid when there is no magnetic field because very close to these points the velocity is at most of the order of the distance from the stagnation point and a logarithmically infinite time is required for the fluid to pass through it.

It is now clear that if new tubes of force are to pass through the surface, as in a growing sunspot or a star with an increasing total magnetic field, there must be loci on the surface at each point of which the flow lines split, as at the extreme right of Fig. 1 (*a*). It is also clear that if r is the distance from such a point the velocity in its neighborhood must be of order r^p with $p < 1$ in order that a fluid particle may pass through the point in a finite time and allow room for interior points to spread out on the surface. It will be noted that the velocity and, if $\frac{1}{2} < p < 1$, the acceleration go to zero, not infinity, at the singularity. Thus the essential question is whether fluid motion of this character can take place. Such fluid motions can be described in terms of spherical harmonics following Elsasser [3] and Bullard [4], but the series converge very slowly (the coefficient of the nth harmonic in general being of order n^{-1}) and are hence difficult to use. If only a finite number of terms in the series are retained, p becomes unity and the motion is no longer of the required character. If there is a sharp groove in the surface at the splitting point so that the flow line turns through less than 90° at a single point, then the ordinary motion of an ideal fluid is all that is required. This provides a counter-example for the conjecture of Bondi and Gold [6] that the simple-connectivity of the volume containing the fluid prevents a separation of neighboring surface points to make room for interior points to rise to the surface.

Although it is relatively easy to describe a flow having the desired character, it is less easy to find forces that will produce it. An upwelling of the fluid at a point where the magnetic field lines are horizontal just below the surface is a necessary condition. If the surface is fixed, continued upwelling will compress the magnetic tubes of force and displace their fluid contents along the tubes away from the stagnation point. In this way the magnetic field is increased and ultimately the forces associated with it will become important. In an earlier paper [8] it was thought plausible that these forces would produce a fluid motion of the desired kind and that new tubes of force would break through the surface. But further analysis suggests that this is unlikely and that the magnetic forces are more likely to lead to a stoppage of the fluid motion. However, if the surface is not fixed by a rigid non-conducting barrier but is instead a region of continuously decreasing density as on a star, there seems likely to be no difficulty of this kind.

There is an alternative way to get the new tubes of force to intersect the surface. It has been assumed throughout that the conductivity is infinite. This really means that it is so large that the current density **i**, necessary to produce the magnetic fields, can flow without significant dissipation for times long compared to those considered. But an upwelling will increase the internal field strength without affecting the external field strength and this will lead to a large curl $\mathbf{H} = 4\pi\mathbf{i}/c$ on the surface. The infinite conductivity condition cannot be regarded as holding in the thin surface layer and the tubes of force move upward through and out of the fluid there.

Although the model described by Fig. 1 is so simplified that it would be expected to have little resemblance to any model that would explain the earth's magnetic field, it may be worth noting that in both there are pronounced secular variations and external quadrupole fields. However the main value of this model is in the light it can cast on the nature of the fluid motion required for the existence of a self-excited dynamo in a simply connected body, and in the proof that such dynamos can exist in a perfect conductor.

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Discussion

Buneman: The solution presented does not appear to be 'simply connected'. The dip, taken to its logical conclusion, together with a corresponding dip at the antipodes, i.e. a volcano together with an anti-volcano at the antipodes, are connected by a funnel right through the center of the earth. We now have a toroidal topology and therefore no violation of the Bondi–Gold theorem.

Davis: In the model there is no anti-volcano at the antipodes. This is necessary in order to avoid swallowing up there as many tube endings as are put to the surface by the upwelling. It does not seem to me that such a flow gives the sphere the connectivity of a torus unless one produces the volcano not by convection but by the insertion of a pump with fixed rigid pipes.

Gold: The Bondi–Gold theorem referred to must not be thought of as saying more than it did. But with its limitations—whether they approach physical reality or not—it may not go very far, but it is right. The external field of a

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simply connected body of a perfect conductor cannot increase as a result of a hydrodynamic motion in which neighboring fluid particles remain neighbors forever. If any of these limitations are dropped we no longer see any reason why the external field should not be caused to increase. Consequently, if the conductivity is finite, or if the connectivity of the body is changed by allowing the material to be cut at a dip as in Dr Davis' case, a dynamo could certainly be made.

Davis: I wish to raise no questions concerning the basic theory of Bondi and Gold or concerning the conditions for its validity, which they have carefully stated; and I think that we are in essential agreement as to the way in which the theory applies to the model under discussion here. But am I not right in feeling that you have now retreated slightly from the statement in the original paper that the absence of a 'tearing' of the fluid, i.e. a separation of originally adjacent points, 'is in any case implied if the motion is not to affect the topological connectivity of the body'? And do we not attach different meanings to 'connectivity'? I say that a pool of water remains simply connected as a knife is thrust into it until the knife touches the bottom; while you feel that the connectivity changes as soon as the knife enters the water.

I should also like to emphasize that a sharp edge of the groove gives no stagnation point of the flow even if the groove has a finite angle. Thus, the fluid 'turns around the corner' in a finite time.

Gold: I think that in the statement of the theorem neighbours must remain neighbours for two reasons, namely: (*a*) Without this the Bondi–Gold theorem would not be true. (*b*) We cannot even define the topological connectivity of the body if a splitting is allowed. This is normal formal hydrodynamics.

Swann: Does not your model have an infinite time constant for the decay of currents and fields because of the infinite conductivity?

Davis: The attempt here was to produce a mechanism which could defeat this infinite time constant by suitable motions. The time constant would depend on that of the hydrodynamic motions.

Ferraro: How does the external field fit the internal field?

Davis: The answer is given by the process described in Fig. 1. We start in Fig. 1 (*a*) with a uniform internal field and an external dipole field. The configuration is produced by suitable surface currents. At the last stage in Fig. 1 (*d*) the internal field strength is doubled. The flux out of the sphere should be conserved and the external field can be fitted to the internal by means of a suitably chosen distribution of surface currents.

Ferraro: Does that mean that the doubled field can also be doubled?

Davis: Yes.

Ferraro: I think this is unbelievable.

Davis: I do not see why this should be impossible. A doubled external field just implies doubled surface currents produced by the motion.