J. Austral. Math. Soc. (Series A) **46** (1989), 296–301

AN ALGORITHM FOR THE EAR DECOMPOSITION OF A 1-FACTOR COVERED GRAPH

C. H. C. LITTLE AND F. RENDL

(Received 7 August 1987; revised 14 March 1988)

Communicated by Louis Caccetta

Abstract

We give a constructive proof for the theorem of Lovász and Plummer which asserts the existence of an ear decomposition of a 1-factor covered graph.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 05 C 99.

1. Introduction

A *1-factor* in a graph G is a set F of edges such that each vertex is incident with exactly one edge of F. We say that G is *1-factor covered* if for every $e \in E(G)$ there exists a 1-factor which contains e. In this paper we confine our attention to such graphs.

We identify paths and circuits with their edge sets. A circuit is *alternating* with respect to two given 1-factors if it is contained in their symmetric difference. Note that if G is a 1-factor covered graph and $|E(G)| > 1$, then for each edge e there exists an alternating circuit containing e.

An *ear* is a path of odd cardinality.

Let H be a 1-factor covered subgraph of a 1-factor covered graph G. Let A be an alternating circuit in G which includes $E(G) - E(H)$ and meets $E(H)$. Then an $A\overline{H}$-arc (or an $\overline{H}$-arc) is a subpath of $E(G) - E(H)$, of maximal length, whose internal vertices are in $V(G) - V(H)$. If there are n such arcs, and each is an ear, then we say that G is obtained from H by an *n-ear adjunction*. An *ear*

https://doi.org/10.1017/S1446788700030767 Published online by Cambridge University Press

decomposition of G is a sequence $G_0, G_1, \ldots, G_t$ of 1-factor covered graphs such that $|E(G_0)| = 1$, $G_t = G$ and, for each $i > 0$, G_i is obtained from G_{i-1} by an n-ear adjunction with $n = 1$ or $n = 2$. (Note that the definition in [4] permits a 2-ear adjunction only if neither ear can be used for a 1-ear adjunction.) It has been shown by Lovász and Plummer [1] (see also [2]) that such a decomposition exists, and an algorithm for its construction appears in [4]. Our purpose here is to give an elementary constructive proof of the result of Lovász and Plummer.

2. Proof of the theorem

Throughout this section we fix a 1-factor F in a 1-factor covered graph G. An *alternating path* in G (with respect to F) is a path P in which each internal vertex is incident with an edge of $P \cap F$. We adopt as a lemma the following statement which is proved in [4].

LEMMA 1. *Let F be a 1-factor in a connected 1-factor covered graph G. Let $v \in V(G)$ and $w \in V(G)$. Then there is an alternating path P joining v and w such that an edge of $P \cap F$ is incident on v.*

The proof of this assertion given in [4] furnishes an efficient algorithm for the construction of such a path.

If u and v are distinct vertices in a path P, then we denote by $P[u, v]$ the subpath of P joining them.

LEMMA 2. *Let F be a 1-factor in a graph G. Let C be a circuit in G which contains a unique vertex v not incident with an edge of $F \cap C$. Let $e \in C$, and let e join vertices x and x', where $x \neq v$. Let R be a path in $G - \{e\}$ which is alternating with respect to F, joins v to a vertex $w \in V(C) - \{v\}$ and has its terminal edges in F. Suppose that $C'[w, x] \cap R = \emptyset$, where $C' = C - \{a\}$ for some edge $a \in C$ incident on v. Then $C \cup R$ includes a circuit which is alternating with respect to F and contains e.*

PROOF. We use induction on the number n of $R\overline{C}$-arcs (that is, maximal subpaths of R whose edges and internal vertices are not in C).

Let $C^* = C - \{e\}$, and let g be the edge of R incident on w. By symmetry we can assume that $g \in C^*[v, x]$. If $C^*[v, x'] \cap R = \phi$, then $R \cup C^*[w, x] \cup \{e\} \cup C^*[x', v]$ is the required alternating circuit. Thus in particular the lemma holds if $n = 1$.

We may now suppose that $n > 1$, that the lemma holds whenever the number of $R\overline{C}$-arcs is less than n, and that there exists $h \in C^*[v, x'] \cap R$. Let h join vertices y and y', where $h \in C^*[x', y']$. We may assume h chosen so that $|C^*[x', y]|$

https://doi.org/10.1017/S1446788700030767 Published online by Cambridge University Press

is minimised. Thus $h \in F$. If $h \in R[v, y]$, then the lemma holds by the induction hypothesis applied to the path $R[v, y]$. (Note that $R[y, w]$ contains an $R\overline{C}$-arc since $e \notin R$.) Suppose therefore that $h \in R[v, y']$. Then the required alternating circuit is $R[y', w] \cup C'[w, y']$.

We are now equipped for our proof of the "2-ear theorem" of Lovász and Plummer.

THEOREM 1. *Let F be a 1-factor in a 1-factor covered connected graph G. Let H be a 1-factor covered connected proper subgraph of G such that $E(H) \neq \phi$ and $F \cap E(H)$ is a 1-factor of H. Then G contains a circuit A which is alternating with respect to F and admits just one or two $A\overline{H}$-arcs.*

REMARK. The $A\overline{H}$-arcs constitute the ears featured in one step of an ear decomposition of G.

PROOF. As the theorem is vacuous if $|E(G)| \leq 1$, we assume that each edge of G belongs to a circuit which is alternating with respect to F. In particular, let A be such an alternating circuit which contains an edge of $E(G) - E(H)$ incident on a vertex of H. Then there exists an $A\overline{H}$-arc.

We now assume that A is chosen as a circuit, alternating with respect to F, which has an $A\overline{H}$-arc but as few $A\overline{H}$-arcs as possible subject to this requirement. If A has no more than two $A\overline{H}$-arcs, then the theorem holds, and so we suppose that A has at least three.

Let P_1, P_2, P_3 be $A\overline{H}$-arcs, and let P_i join vertices u_i and v_i for each $i \in \{1, 2, 3\}$. We may assume that these vertices occur on A in the cyclic order $u_1, v_1, u_2, v_2, u_3, v_3$. They are distinct, for each is incident on an edge of F which must belong to $A \cap E(H)$. For each $i \in \{1, 2, 3\}$, we let $P'_i = A - P_i$.

By Lemma 1 there exists a path Q_0 in H which is alternating with respect to F and joins vertices in distinct components of the graph spanned by $E(H) \cap A$. Without loss of generality we can therefore assume the existence of a subpath Q of Q_0 joining a vertex $q_1 \in V(P'_3[v_1, u_2])$ to a vertex $q_2 \in V(P'_1[v_2, u_3])$ such that $Q \cap A = \phi$ and $V(Q) \cap V(A) = \{q_1, q_2\}$. Let b_1 and b_2 be the edges of F incident on q_1 and q_2 respectively. If $\{b_1, b_2\} \subset P'_2[q_1, q_2]$ then the choice of A is contradicted by the circuit $Q \cup P'_2[q_1, q_2]$. Similarly $\{b_1, b_2\} \not\subset P'_1[q_1, q_2]$. We may therefore assume without loss of generality that $b_1 \in P'_1[q_1, q_2]$ and $b_2 \in P'_2[q_1, q_2]$.

By Lemma 1, there exists a path R in H, alternating with respect to F, which has q_1 as a terminal vertex and has b_1 and the edge of F incident on u_1 as its terminal edges. Choose $g \in P'_2[u_1, v_3] \cap R$, and let g join vertices w and w', where $g \in R[q_1, w]$. We may assume that g is chosen to minimise $|R[q_1, w]|$. If $g \in P'_2[u_1, w']$, then choose an edge $e \in P_1$; otherwise choose $e \in P_3$. Applying Lemma 2 to the circuit $P'_2[q_1, q_2] \cup Q$ and the alternating path $R[q_1, w]$, we deduce

https://doi.org/10.1017/S1446788700030767 Published online by Cambridge University Press

that $P_2'[q_1, q_2] \cup Q \cup R[q_1, w]$ includes a circuit which is alternating with respect to F and contains e. This circuit must include either P_1 or P_3 but not P_2, and thereby contradicts the choice of A.

3. Constructing an ear decomposition

The proof of Theorem 1 suggests the following method for finding an n-ear adjunction of G_j to obtain G_{j+1} such that the sequence $\{G_i\}$ is an ear decomposition of G.

Suppose G is a 1-factor covered connected proper subgraph of G. Let F be a given 1-factor in G_j that can be extended to a 1-factor F' in G.

Step 1. Use the following procedure to find a circuit A, alternating with respect to F', having at least one $A\overline{G}_j$-arc. By assumption there exists $v \in V(G_j)$ and $e \in E(G) - E(G_j)$ such that v meets e. Find a 1-factor K in G such that $e \in K$ and let A be the alternating circuit in $F' + K$ that contains e. Let $n(A)$ be the number of $A\overline{G}_j$-arcs. We will write n instead of $n(A)$ if there is no risk of confusion.

Step 2. If $n = 1$ then the $A\overline{G}_j$-arc gives a 1-ear adjunction.

Step 3. If $n = 2$ then the $A\overline{G}_j$-arcs give a 2-ear adjunction.

(It is easy to test whether either of these ears can be used as a 1-ear adjunction. We need merely test whether $G_j \cup P_1$ or $G_j \cup P_2$ is 1-factor covered where P_1 and P_2 are the ears. This can be done by determining whether $G_j \cup P_i$, $i \in \{1, 2\}$, contains a 1-factor that uses a terminal edge of P_i.)

Step 4. If $n \geq 3$ then apply the steps implicit in the proof of Theorem 1 and Lemma 2 to transform A into an alternating circuit A' having one or two $A'\overline{G}_j$-arcs. This is done as follows: as in the proof of Theorem 1 let P_1, P_2, P_3 be $A\overline{G}_j$-arcs and let P_i join vertices u_i, v_i for $i \in \{1, 2, 3\}$, where the vertices appear on A in the cyclic order $u_1, v_1, u_2, v_2, u_3, v_3$.

(a) Using the labelling technique described in [4] find a path Q_0 in G_j which is alternating with respect to F and joins v_1 and v_2. Let Q be a subpath of Q_0 such that $Q \cap A = \varnothing$ and $V(Q) \cap V(A) = \{q_1, q_2\}$, where q_1 and q_2 are as in the proof of the theorem. Denote the edges of F incident on q_1 and q_2 by b_1 and b_2 respectively.

(b) If $\{b_1, b_2\} \subset P_2'[q_1, q_2]$ then let $A' = Q \cup P_2'[q_1, q_2]$. Similarly if $\{b_1, b_2\} \subset P_1'[q_1, q_2]$, then let $A' = Q \cup P_1'[q_1, q_2]$. Continue with Step 2, replacing A by A'. (Note that in these cases $n(A) > n(A') \geq 1$.)

(c) Without loss of generality assume $b_1 \in P_1'[q_1, q_2]$. Find an alternating path R in G_j starting at q_1, containing b_1 and having the edge of F incident on u_1 as terminal edge. Choose $g \in P_2'[u_1, v_3] \cap R$ where g joins vertices w and w', $g \in R[q_1, w]$, and $|R[q_1, w]|$ is minimised. If $g \in P_2'[u_1, w']$ choose an edge $e \in P_1$; otherwise choose $e \in P_3$.

(d) Apply the steps implicit in the proof of Lemma 2 to the circuit $P_2'[q_1, q_2] \cup Q$, the alternating path $R[q_1, w]$ and the edge e to obtain an alternating circuit A'. Again $n(A) > n(A') \geq 1$. Go to Step 2 with A replaced by A'.

LEMMA 3. *An n-ear adjunction for a 1-factor covered connected subgraph H of a 1-factor covered graph G, where $n \in \{1, 2\}$, can be found in $O(|V(G)|\,|E(G)|)$ worst case time.*

PROOF. Due to Theorem 1, performing Steps 1-4 produces the required n-ear adjunction. Analysing the computational effort we find that Steps 1 and 3 essentially require the computation of one or two 1-factors in G, and each step is invoked no more than once. These 1-factors can be found in $O(|V(G)|^{1/2}|E(G)|)$ time [3].

In Step 4 we first have to find two alternating paths Q_0 and R in G_j. This is done in $O(|E(G_j)|)$ time by the labelling process described in [4]. In Step 4(d) we have to apply the steps implicit in the proof of Lemma 2. This amounts to determining an edge $h \in R$ closest to a certain vertex on the circuit $P_2'[q_1, q_2] \cup Q$ and can be done in $O(|R|)$ time. Thus one execution of Step 4 requires only $O(|E(G)|)$ time. To complete the proof, note that Step 4 is invoked no more than $O(|V(G)|)$ times.

REMARK. It should be pointed out that the method described in [4] to find an n-ear adjunction requires one to find $O(|E(G)|)$ minimum weight 1-factors, and therefore has running time $O(|E(G)|\,|V(G)|^3)$.

Acknowledgement

This paper was written while the authors were visiting the University of Waterloo, and they wish to thank the Department of Combinatorics and Optimization for its hospitality during this time. In addition, the first author wishes to thank M. D. Plummer for helpful conversations on this topic. The authors also thank the referee for several helpful comments.

References

[1] L. Lovász and M. D. Plummer, 'On bicritical graphs', *Infinite and finite sets* II, Colloq. Keszthely, Hungary, 1973, edited by A. Hajnal, R. Rado and V. T. Sós, pp. 1051–1079 (Colloq. Math. Soc. János Bolyai, 10, North-Holland, Amsterdam, 1975).

[2] L. Lovász and M. D. Plummer, *Matching theory* (Ann. Discrete Math. **29**, North-Holland Mathematics Studies 121, North-Holland, Amsterdam, 1986).

[3] S. Micali and V. V. Vazirani, 'An $O(\sqrt{|V|} \cdot |E|)$ algorithm for finding maximal matchings in general graphs', *Proc. 21st IEEE Symp. on Foundations of Computer Science*, 1980, pp. 17–27.

[4] D. J. Naddef and W. R. Pulleyblank, 'Ear decompositions of elementary graphs and GF_2-rank of perfect matchings', *Bonn workshop on combinatorial optimisation*, pp. 241–260, (Ann. Discrete Math. **16**, North-Holland Mathematics Studies, North-Holland, Amsterdam).

Department of Mathematics
and Statistics
Massey University
Palmerston North
New Zealand

Institut für Mathematik
Technische Universtät Graz
Kopernikusgasse 24
A-8010 Graz
Austria

https://doi.org/10.1017/S1446788700030767 Published online by Cambridge University Press