REMARKS ON THE UPPER CENTRAL SERIES OF A GROUP

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1. Introduction. Following, for example, Kuroš [8], we define the (transfinite) *upper central series* of a group G to be the series

$$1 = Z_0 \leqslant Z_1 \leqslant Z_2 \leqslant \dots \leqslant Z_\alpha \leqslant \dots$$

such that $Z_{\alpha+1}/Z_\alpha$ is the centre of G/Z_α, and if β is a limit ordinal, then $Z_\beta = \bigcup_{\alpha<\beta} Z_\alpha$. If α is the least ordinal for which $Z_\alpha = Z_{\alpha+1} = \dots$, then we say that the upper central series has length α, and that $Z_\alpha = H$ is the *hypercentre* of G. As usual, we call G *nilpotent* if $Z_n = G$ for some finite n.

By replacing the concept *central element of* G (i.e., one with only one conjugate in G) by the concept *FC-element of* G (i.e., one with only a finite number of conjugates in G), Haimo [5] has defined the *upper FC-series* of G to be the series

$$1 = F_0 \leqslant F_1 \leqslant F_2 \leqslant \dots \leqslant F_\alpha \leqslant \dots$$

where $F_{\alpha+1}/F_\alpha$ is the set of all FC-elements of G/F_α, and if β is a limit ordinal, then $F_\beta = \bigcup_{\alpha<\beta} F_\alpha$, If $F_\alpha = F_{\alpha+1} = \dots$, then we will call $F_\alpha = F$ the *hyper*-FC-*subgroup* of G.

It is clear that $F_\alpha \geqslant Z_\alpha$ for all α. In § 2, we investigate further the connection between the two series. The main result is

THEOREM 1. *If the centre Z_1 of G is torsion-free, then $F_\alpha \cap H = Z_\alpha$ for all α.*

(A group is said to be torsion-free if it contains no elements of finite order, other than 1.)

Now it is well known that H is locally nilpotent, but that not every locally nilpotent group coincides with its hypercentre. This leads us to consider the upper central series and upper FC-series of locally nilpotent groups. We find that in a locally nilpotent group, the hypercentre and hyper-FC-subgroup coincide, and hence deduce

COROLLARY 1. *If the centre of a locally nilpotent group G is torsion-free, then $F_\alpha = Z_\alpha$ for all α.*

In § 3 we investigate the hypercentre H and the hyper-FC-subgroup F of G under certain finiteness conditions. These include: FG, the property of being finitely generated; Max, the maximal condition for subgroups; Max-G, the maximal condition for subgroups which are normal in G; and the corresponding minimal conditions Min and Min-G. Baer [1] has shown that for H, the properties FG, Max and Max-G are equivalent; this is also true for F (Theorem 2). The corresponding result for the minimal conditions, that H satisfies Min-G only if H satisfies Min, does not hold, but the stronger condition that G satisfies Min-G is sufficient to imply that both H and F satisfy Min (Theorem 3). The results for the hyper-FC-subgroup are generalisations of some of the results on FC-nilpotent groups (groups for which $F_n = G$ for some finite n) in Duguid and McLain [4]. We may remark that examples of groups which coincide with their hyper-FC-subgroups are given by the infinite supersoluble groups defined by Baer [2]. Indeed, Theorems 1 and 2 of Baer's paper are both simple corollaries of our Theorem 2. Mal'cev [9] has proved the existence of groups with lower central series of arbitrary length. In § 4 we do the same for the upper central series. Explicitly, for any ordinal α, there exists a group G with upper central series of length α, terminating in G.†

† *Added in proof.* This has also been proved by V. M. Gluškov, *Mat. Sb.*, **31** (1952), 491–496, and by S. Moran (to appear). I am grateful to Dr. K. A. Hirsch for these references.

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2. The upper central and upper FC-series.

LEMMA 1. *If x belongs to the hypercentre, but not to the centre of G, and x^n belongs to some term of the upper central series to which x does not belong, then there is an element g of G such that $y=[x, g]$ and y^n also belong to different terms of the upper central series.*

(Here, $[x, g]$ denotes, as usual, the commutator $x^{-1}g^{-1}xg$.)

Proof: If $x^n \neq 1$, let α be the least ordinal for which $x^n \epsilon Z_\alpha$. Then α is not a limit ordinal, and so $x^n \notin Z_{\alpha-1}$. If $x^n=1$, we let $\alpha=1$.

In each case the hypothesis asserts that $x \notin Z_\alpha$. Hence there exists $g \epsilon G$ such that $y=[x, g] \notin Z_{\alpha-1}$. Now y belongs to some term in the upper central series (since x does), so there exists a least ordinal β such that $y \epsilon Z_\beta$. Then β is not a limit ordinal, and $y \notin Z_{\beta-1}$. Now y belongs to Z_β, but not to $Z_{\alpha-1}$, so $\beta > \alpha-1$. Therefore $\beta \geqslant \alpha$, and $x^n \epsilon Z_\beta$. Thus both $y=[x, g]$ and x^n belong to Z_β, and this is the centre of G modulo $Z_{\beta-1}$. Hence

$$y^n=[x, g]^n \equiv [x^n, g] \equiv 1 \mod Z_{\beta-1}.$$

So y does not belong to $Z_{\beta-1}$, but y^n does. This completes the proof of the lemma.

This lemma implies that $Z_{\alpha+1}/Z_\alpha$ can contain an element of order n only if Z_1 contains an element, not 1, of order dividing n. In particular,

LEMMA 2. *If the centre Z_1 of a group is torsion-free, then so is every factor group $Z_{\alpha+1}/Z_\alpha$ of the upper central series.*

Proof: If the lemma is false, let α be the least ordinal such that $Z_{\alpha+1}/Z_\alpha$ contains a periodic element xZ_α, not the identity. Then $x \epsilon Z_{\alpha+1}$, $x \notin Z_\alpha$ and $x^n \epsilon Z_\alpha$ for some integer n. Let $y=[x, g]$ be chosen as in Lemma 1, so that, if β is the least ordinal for which $y \epsilon Z_\beta$, then β is not a limit ordinal, $y \notin Z_{\beta-1}$ but $y^n \epsilon Z_{\beta-1}$. Therefore $Z_\beta/Z_{\beta-1}$ contains a periodic element $yZ_{\beta-1}$. Clearly $\alpha \geqslant \beta > \beta-1$, and this contradicts the definition of α. Therefore the lemma is true.

Proof of Theorem 1: We use transfinite induction, and assume that $F_\beta \cap H=Z_\beta$ for all ordinals β less than α. (The assertion is trivial when $\beta=0$, since $F_0=Z_0=1$.)

If α is a limit ordinal, then

$$Z_\alpha=\bigcup_{\beta<\alpha} Z_\beta=\bigcup_{\beta<\alpha}(F_\beta \cap H)=\bigcup_{\beta<\alpha} F_\beta \cap H=F_\alpha \cap H.$$

If $\alpha=\beta+1$ for some β, let x be an arbitrary element of $F_\alpha \cap H$, and suppose that $x \notin Z_\alpha$. Then there exists an element g of G such that $[x, g] \notin Z_\beta$. Now $[x, g]$ belongs to H, since x does, so there exists an ordinal γ such that $[x, g] \epsilon Z_\gamma$ and $[x, g] \notin Z_{\gamma-1}$. Clearly $\gamma > \beta$, so $Z_{\gamma-1} \geqslant Z_\beta$.

Now x has only a finite number of conjugates mod F_β, so among the elements $g^{-n}xg^n$, $(n=0, 1, 2, \ldots)$, two must be conjugate mod F_β. Therefore $x \equiv g^{-n}xg^n$ mod F_β, for some $n>0$. Hence $[x, g^n]=x^{-1}g^{-n}xg^n \epsilon F_\beta$. But $x \epsilon H$, so $[x, g^n] \epsilon H$. Therefore

$$[x, g^n] \epsilon F_\beta \cap H=Z_\beta \leqslant Z_{\gamma-1}.$$

But $[x, g]$ lies in the centre of G mod $Z_{\gamma-1}$. Hence

$$[x, g]^n \equiv [x, g^n] \equiv 1 \mod Z_{\gamma-1}.$$

Thus we find that $[x, g]$ belongs to Z_γ but not to $Z_{\gamma-1}$, and $[x, g]^n$ belongs to $Z_{\gamma-1}$. This contradicts the result of Lemma 2, that $Z_\gamma/Z_{\gamma-1}$ is torsion-free.

Therefore $x \epsilon Z_\alpha$, and so $Z_\alpha \geqslant F_\alpha \cap H$. But in any group, $Z_\alpha \leqslant F_\alpha \cap H$, so $Z_\alpha=F_\alpha \cap H$, and the proof of Theorem 1 is complete.

LEMMA 3. *If G is a locally nilpotent group, and x is an FC-element of G, then x belongs to Z_n for some finite n.*

Proof: Let $x=x_1, x_2, \ldots, x_m$ be the conjugates of x in G. Any inner automorphism of G

permutes the $\{x_i\}$, so there is a finite number (at most $m!$) of elements $g_1, \ldots, g_r$ of G such that any element g of G permutes the $\{x_i\}$ in the same way as one of the g_j.

Let $K = \mathrm{Gp}\{x, g_1, \ldots, g_r\}$. K is finitely generated, and hence nilpotent: $Z_n(K) = K$ for some n.

If $X = \mathrm{Gp}_G\{x\} = \mathrm{Gp}\{x_1, \ldots, x_m\}$ is the least normal subgroup of G containing x, then we will prove by induction that $Z_k \geqslant X \cap Z_k(K)$, $(k = 1, 2, \ldots)$. This is true if $k = 0$, so we assume that

$$Z_{k-1} \geqslant X \cap Z_{k-1}(K)$$

and that $y = x_u^a x_v^b \ldots x_w^c$ is an arbitrary element of $X \cap Z_k(K)$. Let $g \epsilon G$, and let g_j permute the $\{X_i\}$ in the same way as g. Since $y \epsilon Z_k(K)$, it commutes with g_j mod $Z_{k-1}(K)$. Hence

$$[y, g] = [y, g_j] \in Z_{k-1}(K).$$

Also, $[y, g] \epsilon X$, so $[y, g]$ belongs to $X \cap Z_{k-1}(K)$, and hence to Z_{k-1}. As this is true for all $g \epsilon G$, y belongs to Z_k. Hence

$$Z_k \geqslant X \cap Z_k(K) \quad (k = 1, 2, 3, \ldots).$$

In particular, $x \in X \cap Z_n(K)) \leqslant Z_n$, and the lemma is proved.

LEMMA 4. *If G is locally nilpotent, then $Z_\alpha \leqslant F_\alpha \leqslant Z_{\omega\alpha}$ for all α. (Here ω denotes the first limit ordinal.)*

This follows from Lemma 3 by transfinite induction. We omit the details.

Corollary 1, stated in the introduction, follows immediately, since if G is locally nilpotent, then $F_\alpha \leqslant H$ by Lemma 4, so $F_\alpha = F_\alpha \cap H$, and if the centre of G is torsion-free, then Theorem 1 states that $F_\alpha \cap H = Z_\alpha$.

3. The finiteness conditions.

THEOREM 2. *The following properties of the hyper-FC-subgroup F of G are equivalent:*

(a) F satisfies Max-G,

(b) F satisfies Max,

(c) F is FG,

(d) F is a finite extension of an FG nilpotent group.

Proof: If F satisfies Max-G, let K be a normal subgroup of G, maximal among those which are contained in F and which satisfy Max, and suppose that $K \neq F$. Now F/K contains an element $x_1K \neq K$ which is FC in G/K; let its conjugates be $x_2K, x_3K, \ldots, x_rK$. Then

$$L = \mathrm{Gp}\{x_1, \ldots, x_r, K\}$$

is a normal subgroup of G, and $F \geqslant L > K$. Now L/K is an FG, FC-group, and so satisfies Max (for example, [4]). Since Max is a poly property (see P. Hall [7]), L satisfies Max, and this contradicts the definition of K. Thus (*a*) implies (*b*), so (*a*) and (*b*) are equivalent.

Now (*b*) implies (*c*) in any group, and (*d*) implies (*b*), since every FG nilpotent group satisfies Max, and, as we noted above, Max is a poly property.

To prove that (*c*) implies (*d*), we use a result in [4] that every finitely generated FC-nilpotent group is a finite extension of a finitely generated nilpotent group. Hence we have only to prove that a FG group with upper FC-series

$$1 = F_0 < F_1 < \ldots < F_\alpha = G$$

actually has $F_n = G$ for some finite n. Let $S = \{g_1, \ldots, g_{2m}\}$ be the set of generators of G and their inverses, and let x be one of the $g_i \epsilon S$. We define a finite set $X_r \leqslant G$ inductively. Let X_1

contain only x. If we have defined X_r, let α_r be the least ordinal such that $X_r \leqslant F_{\alpha_r}$. If $F_{\alpha_r} \neq 1$, then α_r is not a limit ordinal (since X_r is finite). Hence X_r is FC in G mod F_{α_r-1}. Therefore there is a finite set $X_r^* \geqslant X_r$ such that, for any $y_i \epsilon X_r^*$ and $g_j \epsilon S$, there is an element $y_k \epsilon X_r^*$ for which $y_{ij} = y_k^{-1} g_j^{-1} y_i g_j \epsilon F_{\alpha_r-1}$. If we define X_{r+1} to be the set of all these y_{ij}, then X_{r+1} is finite.

Now $\alpha_1, \alpha_2, \ldots, \alpha_r, \ldots$ is a set of ordinals with the property that if $\alpha_r \neq 0$ then $\alpha_r > \alpha_{r+1}$. Hence $\alpha_n = 0$ for some n. Thus $X_n = X_n^* = 1 = F_0$.

Assume now that $X_{n-r}^* \leqslant F_r$. If $X_{n-r-1}^* = \{y_1, \ldots, y_s\}$, then for any $y_i \epsilon X_{n-r-1}^*$, $g_j \epsilon S$, there exists a $y_k \epsilon X_{n-r-1}^*$ such that

$$y_k^{-1} g_j^{-1} y_i g_j \in X_{n-r} \leqslant X_{n-r}^* \leqslant F_r.$$

Hence $g_i^{-1} y_i g_j \equiv y_k \bmod F_r$, for some k. Since any element of G is expressible as a product of elements of S, this shows that X_{n-r-1}^* is a complete set of conjugates mod F_r, and hence that any $y_i \epsilon X_{n-r-1}^*$ is FC mod F_r. Therefore $X_{n-r-1}^* \leqslant F_{r+1}$. By induction, this is true for all r. In particular,

$$x = X_1 \leqslant X_1^* \leqslant F_{n-1}.$$

Thus each element g_i of S belongs to F_r for some finite r, and therefore $G = F_r$.

This completes the proof of the theorem.

The corresponding result for the minimal conditions, that if F (or even H) satisfies Min-G then it satisfies Min, is not true, as the following example shows.

Example. Let H be the direct product of the cyclic groups of order a prime p,

$$H = \{z_1\} \times \{z_2\} \times \ldots .$$

Let $G_i = \text{Gp}\{H, g_i\}$ $(i = 2, 3, \ldots)$, where g_i commutes with all of the z_j except z_i, and

$$g_i^{-1} z_i g_i = z_{i-1} z_i.$$

Now let G be the free product of the groups G_i, amalgamating the subgroup H.

It is clear that the rth term of the upper central series of G is given by

$$Z_r = \{z_1\} \times \{z_2\} \times \ldots \times \{z_r\},$$

and that H is the hypercentre. (It is also the hyper-FC-subgroup of G.) Now the only subgroups of H which are normal in G are the Z_r, since if a normal subgroup N of G contains $x = z_1^{r_1} z_2^{r_2} \ldots z_s^{r_s}$, $r_s \neq 0$, then it contains $[x, g_s] = z_{s-1}^{r_s}$, and consequently $\{z_{s-1}\}, \{z_{s-2}\}, \ldots, \{z_1\}$, and therefore also $\{z_s^{r_s}\} = \{z_s\}$, i.e., $N \geqslant Z_s$. Since the Z_r form a well ordered chain, H satisfies Min-G. But H, being an infinite direct product, does not satisfy Min.

THEOREM 3. *If G satisfies Min-G (the minimal condition for normal subgroups), then H and F both satisfy Min.*

Proof: Since F contains H, it is sufficient to show that F satisfies Min.

We use the lemma proved in McLain [10], that if G satisfies Min-G and K is the (unique) minimal normal subgroup of finite index in G, then $Z_1(K)$ satisfies Min, and $Z_2(K) = Z_1(K)$.

Suppose that $F \frown K$ is not contained in $Z_1(K)$. Let α be the least ordinal for which there exists an $x \in F_\alpha \frown K$ such that $x \notin Z_1(K)$. Then α is not a limit ordinal, and x has only a finite number, say r, of conjugates mod $F_{\alpha-1}$. Clearly, x has only r conjugates mod $F_{\alpha-1} \frown K$, and hence has at most r conjugates mod $Z_1(K)$. Therefore the centraliser of x in G mod $Z_1(K)$ has index at most r in G, and so contains K. (Any subgroup of finite index in G contains K.)

Thus x belongs to the second centre $Z_2(K)$ of K and so to $Z_1(K)$. This contradiction proves that

$$F \frown K \leqslant Z_1(K).$$

Hence $F \frown K$ satisfies Min. But $F/(F \frown K) \cong KF/K$ which is a subgroup of the finite group G/K. Hence $F/(F \frown K)$ is finite and since Min is also a poly property, F satisfies Min.

COROLLARY 2. *If G satisfies Min-G, then the upper central series has length less than $\omega 2$, and the upper FC-series has length at most 2.*

Proof: P. Hall's strict inclusion theorem for the finite upper central series of a group [**6**] may be easily extended to the transfinite case to read: If N is a normal subgroup of G, and $N \frown Z_\alpha = N \frown Z_{\alpha+1}$, then $N \frown Z_\beta = N \frown Z_{\beta+1}$ for all $\beta > \alpha$. (Detailed proof is not given. We may remark that the corresponding result for the FC-series is also true, but will not be required.)

Since $Z_1(K)$ is the direct product of a finite number of groups of type (p^∞) with a finite group, any element x of $H \frown K \leqslant F \frown K \leqslant Z_1(K)$ is contained in a finite characteristic subgroup X of $Z_1(K)$. X is normal in G, so we can apply the strict inclusion theorem. Thus, if X is not contained in Z_r, then $X \frown Z_r > X \frown Z_{r-1} > \ldots > X \frown Z_0 = 1$, so, since X is finite, $X \leqslant Z_r$ for some finite r. Hence $H \frown K = H \frown Z_1(K) \leqslant Z_\omega$, and so H/Z_ω is finite. Therefore $H = Z_{\omega+n}$ for some finite n.

The centraliser of $Z_1(K)$ contains K, and so has finite index n in G. Hence any element of $Z_1(K)$ has at most n conjugates in G, and so $Z_1(K) \leqslant F_1$. Thus we can sharpen the equation $F \frown K \leqslant Z_1(K)$ to the equation $F_1 \frown K = Z_1(K)$. Therefore F/F_1 is finite, and so $F = F_2$.

4. A group with transfinite upper central series.

Construction: Let $\Lambda = \{\lambda, \mu, \nu, \ldots\}$ be a partially ordered set (henceforth a "poset"), and let L be the set of all pairs (μ, ν) for which $\mu < \nu$. Denote by G_L the set of all elements of the form $g = 1 + \Sigma a_{\mu\nu} e_{\mu\nu}$, where each $(\mu, \nu) \epsilon L$, the $a_{\mu\nu}$ belong to a field $\mathfrak{K}$ and only a finite number are different from zero. G_L becomes a multiplicative group if we define multiplication by

$$e_{\kappa\lambda} e_{\mu\nu} = \begin{cases} e_{\kappa\nu} \text{ if } \lambda = \mu, \\ 0 \quad \text{otherwise.} \end{cases}$$

We call a subset S of L a normal partition if, for every $(\mu, \nu) \epsilon S$, S also contains each (κ, λ) with $\kappa \leqslant \mu < \nu \leqslant \lambda$. For such an S, let G_S denote the set of all elements of G_L such that $a_{\mu\nu} \neq 0$ only if $(\mu, \nu) \epsilon S$. It is easy to see that G_S is a subgroup of G_L, and is generated by the set of all $1 + ae_{\mu\nu}$, $a \epsilon \mathfrak{K}$, $(\mu, \nu) \epsilon S$. (The details are omitted.) In particular,

$$G_L = \mathrm{Gp}\{1 + ae_{\mu\nu},\ a \epsilon \mathfrak{K},\ (\mu,\nu) \epsilon L\}.$$

If $(\mu,\nu) \epsilon S$, $(\kappa,\lambda) \epsilon L$, then

$$[1 + ae_{\mu\nu}, 1 + be_{\kappa\lambda}] = \left\{ \begin{array}{ll} 1 + abe_{\mu\lambda} & \text{if } \nu = \kappa, \\ 1 - abe_{\kappa\nu} & \text{if } \lambda = \mu, \\ 1 & \text{otherwise} \end{array} \right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and belongs to G_S in each case. Hence G_S is a normal subgroup of G_L.

The group G_L is a generalisation of the case when Λ is a chain of n elements. G_L is then the well known group of all $n \times n$ unitriangular matrices over $\mathfrak{K}$, and the normal partition subgroups are the normal partition subgroups as defined by Weir [**11**].

LEMMA 5. *The centre of G_L modulo the normal partition group G_S is the normal partition group G_T, where T is the set of all $(\lambda, \mu) \epsilon L$ such that $\kappa < \lambda$ implies $(\kappa, \mu) \epsilon S$ and $\mu < \nu$ implies $(\lambda, \nu) \epsilon S$.*

Proof: By equation (1), the generators of G_L and G_T commute mod G_S, so G_T is contained in the centre of G_L mod G_S.

Let $g = 1 + \Sigma a_{\lambda\mu} e_{\lambda\mu}$ be in the centre of G_L mod G_S, and let $(\xi, \zeta) \epsilon L$. Then the coefficient of $e_{\xi\mu}$ in $h = [g, 1 - e_{\xi\zeta}]$ is $a_{\zeta\mu}$. Since $h \epsilon G_S$, this implies that if $a_{\lambda\mu} \neq 0$ and $\kappa < \lambda$, then $(\kappa, \mu) \epsilon S$. Similarly, if $a_{\lambda\mu} \neq 0$ and $\nu > \mu$, then $(\lambda, \nu) \epsilon S$. Hence $g \epsilon G_T$, and the proof of the lemma is complete.

Now the union of a tower of normal partition subgroups is itself a normal partition subgroup, so all the terms of the upper central series of G_L are normal partition groups. If $Z_\alpha(G_L) = G_T$, then we donote T by S^α.

THEOREM 4. *For any transfinite ordinal α, there exists a poset $\Lambda = \Lambda_\alpha$ such that $Z_\alpha(G_L) = G_L$, but $Z_\beta(G_L) \neq G_L$ if $\beta < \alpha$.*

Proof: If α is finite, we can take Λ_α a chain of $\alpha + 1$ elements. G_L is then nilpotent of class α. We use transfinite induction, and assume that, for all $\beta < \alpha$, there is a poset Λ_β such that

(*a*) $S_\beta^\beta = L_\beta$, but $S_\beta^\gamma \neq L_\beta$ if $\gamma < \beta$,

(*b*) Λ_β satisfies the ascending and descending chain conditions,

(*c*) if β is a limit ordinal, then Λ_β is the cardinal sum of all the Λ_γ, $\gamma < \beta$ (see Birkhoff [**3**], p. 7, for the definition of the cardinal sum of posets), and

(*d*) if $\beta = \gamma + 1$, then there exists a unique minimal element ξ of Λ_β, and another element $\lambda \epsilon \Lambda_\beta$ for which $(\xi, \lambda) \notin S_\beta^\gamma$.

Case 1. *If α is a limit ordinal*, we take $\Lambda = \Lambda_\alpha$ as the cardinal sum of the Λ_β, $\beta < \alpha$. G_L is the direct product of the G_{L_β}, and the induction hypotheses are satisfied.

We now require two lemmas.

LEMMA 6. *If Λ_1 and Λ_2 are two posets $(\lambda_1, \mu_1) \epsilon L_1$, $(\lambda_2, \mu_2) \epsilon L_2$ and there is a one-one, order-preserving mapping between the sections of the Λ_i less than λ_i and between the sections greater than μ_i, then $(\lambda_1, \mu_1) \epsilon S_1^\alpha$ if and only if $(\lambda_2, \mu_2) \epsilon S_2^\alpha$.*

This follows immediately from Lemma 5.

LEMMA 7. *If Λ_1 satisfies (a), ..., (d), and Λ_2 is formed by the addition of one element ξ less than every element of Λ_1, then $(\mu, \nu) \epsilon S_1^\alpha$ implies $(\mu, \nu) \epsilon S_2^{\alpha+1}$.*

Proof, by transfinite induction: The lemma is true when $\alpha = 1$, so assume that it holds for all $\beta < \alpha$ and that $(\mu, \nu) \epsilon S_1^\alpha$.

If α is a limit ordinal, then $(\mu, \nu) \epsilon S_1^\beta$ for some $\beta < \alpha$, and so $(\mu, \nu) \epsilon S_2^{\beta+1} \leqslant S_2^{\alpha+1}$.

If $\alpha = \beta + 1$, suppose that $(\mu, \nu) \notin S_2^{\alpha+1}$. Then, by Lemma 5, there is either a $(\nu, \rho) \epsilon L_2$ with $(\mu, \rho) \notin S_2^\alpha$, or else a $(\lambda, \mu) \epsilon L_2$ with $(\lambda, \nu) \notin S_2^\alpha$. In the first case, and in the second case if $\lambda \neq \xi$, the induction hypothesis asserts that (μ, ρ) or $(\lambda, \nu) \notin S^\beta$. Hence, by Lemma 5, $(\mu, \nu) \notin S_1^{\beta+1}$, which is a contradiction. In the second case if $\lambda = \xi$, we may assume that μ is a minimal element of Λ_1 (otherwise the previous argument holds). Applying Lemma 6, $(\xi, \nu) \notin S_\beta^\alpha$ implies that $(\mu, \nu) \notin S_1^\alpha$. This is a contradiction, and Lemma 7 is proved.

Case 2. *If $\alpha = \beta + 1$, where β is a limit ordinal*, we form $\Lambda = \Lambda_\alpha$ by adding two elements $\xi < \zeta$ less than Λ_β.

Suppose first that $(\xi, \zeta) \epsilon S^\beta$. Then $(\xi, \zeta) \epsilon S^\gamma$ for some $\gamma < \beta$, and so $(\xi, \mu) \epsilon S^\gamma$ for all $\mu \epsilon \Lambda_\beta$. For any $\mu \epsilon \Lambda_{\gamma+1}$, let λ be the minimal element of $\Lambda_{\gamma+1}$. We can apply Lemma 6 to (λ, μ) in $\Lambda_{\gamma+1}$ and (ξ, μ) in Λ, to find that $(\lambda, \mu) \epsilon S_{\gamma+1}^\gamma$. This contradicts property (*d*) of $\Lambda_{\gamma+1}$. Hence $(\xi, \zeta) \notin S^\beta$.

For any $(\lambda, \mu)\epsilon L_\beta$, $(\lambda, \mu)\epsilon S^\gamma_\beta$ for some $\gamma<\beta$; so, applying Lemma 7 twice,

$$(\lambda, \mu)\epsilon S^{\gamma+2}\leqslant S^\beta.$$

Also (ξ, μ) and (ζ, μ) belong to $S^{\gamma+2}$, as this is a normal partition. If λ is a minimal element of Λ_β, then $\lambda\epsilon\Lambda_\gamma$ for some $\gamma<\beta$, and the above argument shows that, for any $\mu>\lambda$, $(\xi, \mu)\epsilon S^{\gamma+2}$. Then, by Lemma 5, $(\xi, \lambda)\epsilon S^{\gamma+3}$ and $(\zeta, \lambda)\epsilon S^{\gamma+4}$.

Thus (ξ, ζ) is the only element of L not in S^β, so, by Lemma 5, $S^{\beta+1}=L$, and Λ satisfies $(a), \ldots, (d)$.

Case 3. *If* $\alpha=\beta+2$ *for some* β, we form $\Lambda=\Lambda_\alpha$ by adding one element ξ less than $\Lambda_{\beta+1}$.

Let λ be the minimal element of $\Lambda_{\beta+1}$, and $(\lambda, \mu)\notin S^\beta_{\beta+1}$. By Lemma 6, $(\xi, \mu)\notin S^\beta$. Therefore (Lemma 5) $(\xi, \lambda)\notin S^{\beta+1}$. Now for any $\mu\epsilon\Lambda_{\beta+1}$, $\mu\neq\lambda$, $(\lambda, \mu)\epsilon S^{\beta+1}_{\beta+1}$. So, by Lemma 6, $(\xi, \mu)\epsilon S^{\beta+1}$. Also, if $\rho<\sigma<\tau$ all belong to $\Lambda_{\beta+1}$, then $(\rho, \tau)\epsilon S^\beta_{\beta+1}$ (otherwise, by Lemma 5, (ρ, σ) would not belong to $S^{\beta+1}_{\beta+1}$), and so, by Lemma 7, $(\rho, \tau)\epsilon S^{\beta+1}$.

Thus $S^{\beta+1}$ contains every element of L except (ξ, λ), and possibly some $(\rho, \tau)\epsilon L_{\beta+1}$ for which there exists no element $\sigma\epsilon\Lambda_{\beta+1}$ such that $\rho<\sigma<\tau$. By Lemma 5, $S^{\beta+2}$ must be the whole of L. Hence Λ satisfies conditions $(a), \ldots, (d)$.

This completes the proof of Theorem 4.

Remark. If k has prime characteristic p, then G_L is a locally finite p-group. If $\mathfrak{K}$ has zero characteristic, then G_L is torsion-free. In the latter case, by Theorem 1, the upper FC-series of G_L coincides with the upper central series, and so has length α.

Added in proof. Another group with upper FC-series of arbitrary length has recently been constructed by A. M. Duguid.

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