I. Higuchi and M. Itô
Nagoya Math. J.
Vol. 53 (1974), 127-135

ON THE THEOREM OF KISHI FOR A CONTINUOUS FUNCTION-KERNEL

ISAO HIGUCHI AND MASAYUKI ITÔ

1. Introduction

In the potential theory with respect to a non-symmetric function-kernel, the following theorem is obtained by M. Kishi ([3]).

Let X be a locally compact Hausdorff space and G be a lower semi-continuous function-kernel on X. Assume that $G(x, x) > 0$ for any x in X and that G and the adjoint kernel $\check{G}$ of G satisfy "the continuity principle". Then the following four statements are equivalent.

(1) G satisfies the domination principle.
(2) $\check{G}$ satisfies the domination principle.
(3) G satisfies the balayage principle.
(4) $\check{G}$ satisfies the balayage principle.

In the class of lower semi-continuous function-kernels on X, the subclass of continuous function-kernels on X is essential for the continuity principle. We remark that the continuity principle follows from a certain maximum principle.

A function G defined in the product space $X \times X$ is called a continuous function-kernel if G is non-negative, continuous in the extended sense and finite outside the diagonal set of $X \times X$.

In this paper, we shall prove the continuity principle for $\check{G}$ follows from the domination principle for G under a certain additional condition. On the other hand, it is well-known that the domination principle for G implies the continuity principle for G itself. Using our result, we shall obtain that the above theorem is valid in the case of a continuous function-kernel on X without the assumption that G and $\check{G}$ satisfy the continuity principle. In the proof, we shall use a result of one of the authors (cf. [2]), and the following proposition will be essential in our proof.

Received August 8, 1973.

Let G be a continuous function-kernel on X satisfying the domination principle. Assume that every point of X is non-isolated and that $G(x, x) > 0$ for any x in X and for any non-empty open set ω in X, the G-capacity of ω is positive. Then $G(x, y) > 0$ in $X \times X$.

We remark that M. Kishi first proved the above equivalence for a strictly positive function-kernel (cf. [3]).

2. Preliminaries

Let X be a locally compact Hausdorff space and Δ denote the diagonal set of the product space $X \times X$. A function G defined everywhere in $X \times X$ is called a function-kernel if G is non-negative and Borel measurable ([2]). The function-kernel $\check{G}$ on X, defined by $\check{G}(x, y) = G(y, x)$ for any (x, y) in $X \times X$, is called the adjoint kernel of G. For a positive Radon measure μ in X, the potential $G\mu$ and the adjoint potential $\check{G}\mu$ of μ are defined by

$$G\mu(x) = \int G(x, y)d\mu(y) \quad \text{and} \quad \check{G}\mu(x) = \int \check{G}(x, y)d\mu(y) .$$

These are Borel functions on X and $0 \leqq G\mu(x) \leqq +\infty$, $0 \leqq \check{G}\mu(x) \leqq +\infty$ in X. The G-energy of μ is defined by $\int G\mu d\mu$. Evidently the G-energy of μ is equal to the $\check{G}$-energy of μ.

We denote by M_0 the family of all positive Radon measures in X with compact support and by $E_0 = E_0(G)$ the subfamily of M_0 constituted by positive Radon measures with finite G-energy. We have evidently $E_0(G) = E_0(\check{G})$.

For a compact set K in X, we set

$$\text{cap}_G (K) = \sup\left\{\mu(K)\,;\, \mu \in E_0, S\mu \subset K, \int G\mu d\mu \leqq \int d\mu\right\} ,$$

where $S\mu$ denotes the support of μ. For a subset A of X, we denote by $\text{cap}_G (A)$ the quantity $\sup\{\text{cap}_G (K)\,;\, K : \text{compact} \subset A\}$, and we call it the G-capacity of A[1]. Evidently $\text{cap}_G (A) = \text{cap}_{\check{G}} (A)$. For a subset A of X, $\text{cap}_G (A) = 0$ if and only if $\{\mu \in E_0\,;\, S\mu \subset A\} = \{0\}$. We say that a property holds G-*p.p.p.* on a subset A of X if $\text{cap}_G (B) = 0$, where B is the set of points in A where the property fails to hold.

Let us define the potential theoretical principles for a function-kernel G on X.

[1] This is usually called the inner G-capacity of A.

https://doi.org/10.1017/S0027763000016081 Published online by Cambridge University Press

(I) G satisfies the restrained domination principle if, for μ, ν in E_0, $G\mu(x) \leqq G\nu(x)$ G-$p.p.p.$ in X whenever $G\mu(x) \leqq G\nu(x)$ G-$p.p.p.$ on $S\mu$.

PROPOSITION 1 ([2]). *Let G be a strictly positive function-kernel on X. Then G satisfies the restrained domination principle if and only if $\check{G}$ satisfies it.*

(II) G satisfies the domination principle if, for μ in E_0 and ν in M_0, an inequality $G\mu(x) \leqq G\nu(x)$ on $S\mu$ implies the same inequality in the whole space.

PROPOSITION 2. *If G satisfies the domination principle, then G satisfies the restrained domination principle.*

In fact, if for μ, ν in $E_0, G\mu(x) \leqq G\nu(x)$ G-$p.p.p.$ on $S\mu$, then there exists an increasing sequence $(K_n)_{n=1}^{\infty}$ of compact sets contained in $S\mu$ such that $\mu(CK_n) \downarrow 0$ with $n \uparrow +\infty$ and $G\mu(x) \leqq G\nu(x)$ on K_n. Then, by the domination principle for G, $G\mu_n(x) \leqq G\nu(x)$ everywhere in X, where μ_n is the restriction of μ to K_n. Letting $n \uparrow +\infty$, we have $G\mu(x) \leqq G\nu(x)$ in X.

(III) G satisfies the balayage principle if, for a given compact set K in X and a given μ in M_0, there exists a positive Radon measure μ' in M_0 supported by K such that

$$G\mu'(x) \leqq G\mu(x) \quad \text{on } X \quad \text{and} \quad G\mu'(x) = G\mu(x) \quad G\text{-}p.p.p. \quad \text{on } K .$$

This measure μ' is called a G-balayaged measure of μ on K.

PROPOSITION 3. *If G satisfies the balayage principle, then $\check{G}$ satisfies the domination principle.*

This can be proved in the same way as in [3].

(IV) G satisfies the complete maximum principle if for μ in E_0 and ν in M_0, an inequality $G\mu(x) \leqq G\nu(x) + 1$ on $S\mu$ implies the same inequality in the whole space.

It is evident that the complete maximum principle for G implies the domination principle for G.

(V) G satisfies the classical maximum principle if for μ in M_0, an inequality $G\mu(x) \leqq 1$ on $S\mu$ implies the same inequality on X.

(IV) G satisfies the continuity principle if for a μ in M_0, the finite continuity of the restriction of $G\mu$ to $S\mu$ implies that $G\mu$ is finite continuous in the whole space.

https://doi.org/10.1017/S0027763000016081 Published online by Cambridge University Press

When we discuss the continuity principle, it is natural to assume that a function-kernel is lower semi-continuous or continuous in the extended sense. A function-kernel G on X is said to be lower semi-continuous if G is a lower semi-continuous function in $X \times X$. G is called a continuous function-kernel on X if G is continuous in the extended sense in $X \times X$ and $G(x, y) < +\infty$ for any (x, y) in $X \times X - \varDelta$. The following proposition is well-known and can be proved by the same way as in the classical case (i.e., the continuity principle for the Newton kernel).

PROPOSITION 4. *If a continuous function-kernel G on X satisfies the domination principle or the classical maximum principle, then G satisfies the continuity principle.*

3. The positivity of a continuous function-kernel and the continuity principle

Throughout this section, G is a continuous function-kernel on X. We say that G satisfies the condition $(*)$ if:

$(*)$ For any non-empty open set ω in X, $\mathrm{cap}_G(\omega) > 0$.

Remark. The condition $(*)$ is very natural in the potential theory. Let us observe it for a continuous composition kernel on a locally compact abelian group.

Let $G(x, y) = k(x - y)$ be a continuous composition kernel on a locally compact abelian group X, where k is continuous in the extended sense and finite outside the origin. Suppose that G satisfies the domination principle. Then G satisfies the condition $(*)$ if and only if k is ξ-summable in a certain neighborhood of 0, where ξ is a Haar measure on X.

If k is ξ-summable in a certain neighborhood of 0, then for any finite continuous function f in X with compact support, the convolution $k * f$ is defined in X and finite continuous. Therefore we obtain that the "if" part is valid. We shall show that the "only if" part is valid. By the domination principle for G, k is identically equal to 0 if $k(0) = 0$, and hence we may assume $k(0) > 0$. The condition $(*)$ implies $E_0 \neq \{0\}$. We can find a λ $(\neq 0)$ in E_0 supported in $C\{0\} \cap \{x \in X; k(x) > 0\}$. We may assume that $G\lambda$ is bounded on $S\lambda$. By virtue of the domination principle for G, there exists a constant $c > 0$ such that $G\lambda(x) \leqq ck(x)$ on

https://doi.org/10.1017/S0027763000016081 Published online by Cambridge University Press

X, and hence $G\lambda$ is locally bounded on X. Therefore, for any finite continuous function f in X with compact support,

$$+\infty > \int G\lambda(x)\,|f(x)|\,d\xi(x) = \int k(x)(\check{\lambda} * |f|)(x)d\xi(x) ,$$

where $\check{\lambda}$ is the measure defined by $\check{\lambda}(e) = \lambda(-e)$ for any Borel set e. Consequently k is locally ξ-summable.

Let us consider our continuous function-kernel on X. Our first theorem is the following

THEOREM 1. *Let G be a continuous funtion-kernel on X satisfying the domination principle and the condition $(*)$. Assume that $G > 0$ on Δ and every point in X is not isolated. Then $G(x, y) > 0$ on $X \times X$.*

Proof. Suppose that there exists a point (x_0, y_0) in $X \times X$ where G vanishes. Then $(x_0, y_0) \in X \times X - \Delta$. Put $g(y) = G(x_0, y)$. Then g is defined in X and continuous in the extended sense. Every point in X being not-isolated, we can find y_1 in the support of g satisfying $y_1 \neq x_0$ and $g(y_1) = 0$. By $G(y_1, y_1) > 0$, there exists an open neighborhood V of y_1 such that $G(x, y) > 0$ in $V \times V$. Set $\omega = V \cap \{y \in X\,;\, g(y) > 0\}$. Then $\omega \neq \emptyset$, and there exists a positive Radon measure λ $(\neq 0)$ contained in E_0 and supported in ω. By virtue of the Lusin theorem and the continuity principle for G (cf. Proposition 4), we may assume that $G\lambda$ is finite continuous in X. Then we can find a positive constant a such that $G\lambda(x) \leqq aG\varepsilon_{y_1}(x)$ on $S\lambda$, where ε_{y_1} is the unit measure at y_1, because $G\varepsilon_{y_1}(x) > 0$ on $S\lambda$. By the domination principle for G, we have $G\lambda(x) \leqq aG\varepsilon_{y_1}(x)$ on X. Hence

$$0 = aG(x_0, y_1) = aG\varepsilon_{y_1}(x_0) \geqq G\lambda(x_0) = \int g(y)d\lambda(y) > 0 .$$

This is a contradiction. Consequently $G(x, y) > 0$ in $X \times X$. This completes the proof.

We discuss the continuity principle for a continuous function-kernel on X. For a closed subset X' of X, we denote by G' the restriction of G to $X' \times X'$. Then G' is evidently a continuous function-kernel on X'.

LEMMA 1. *If G satisfies the domination principle, then G' satisfies it.*

This follows from the fact that for any positive Radon measure μ in $X', G\mu(x) = G'\mu(x)$ on X'.

LEMMA 2. *Suppose that G is strictly positive in $X \times X$. If G satisfies the restrained domination principle and the condition* (*), *then G satisfies the continuity principle.*

This lemma can be shown by the same manner as in the usual case. Let us give the proof. Suppose that for a μ in M_0, the restriction of $G\mu$ to $S\mu$ is finite continuous. It sufficies to show $\lim_{x \to x_0} G\mu(x) = G\mu(x_0)$ for every boundary point x_0 of $S\mu$.

If $\mu(\{x_0\}) > 0, G(x_0, x_0) < +\infty$, because $G\mu(x_0) < +\infty$, and hence our desired equality follows immediately from the finite continuity of G at (x_0, x_0).

Suppose $\mu(\{x_0\}) = 0$. By $G\mu(x_0) < +\infty$, for a given positive number ε, there exists an open neighborhood V of x_0 such that $\int_V G(x_0, y)d\mu(y) < \varepsilon$. The function $\int_V G(x, y)d\mu(y)$ of x being finite continuous as a function on $S\mu \cap V$, we can choose another open neighborhood W of x_0 which satisfies $W \subset V$ and

$$\int_W G(x, y)d\mu(y) \leqq \int_V G(x, y)d\mu(y) < \int_V G(x_0, y)d\mu(y) + \varepsilon$$

for any x in $S\mu \cap W$. Denote by μ' the restriction of μ to W. Then $G\mu' < 2\varepsilon$ on $S\mu'$. We may assume $\overline{V} \neq X$. By the condition (*) and $G(x, y) > 0$ in $X \times X$, there exists a ν in E_0 such that $S\nu \cap \overline{V} = \emptyset$ and $G\nu(x) > 1$ on $S\mu'$. By virtue of the restrained domination principle for G and the continuity of $G\mu'$ in $CS\mu'$, we have

$$G\mu'(x) \leqq 2\varepsilon G\nu(x) \qquad \text{in } W .$$

On the other hand, $G(\mu - \mu')$ is finite and continuous at x_0, and hence

$$\overline{\lim_{x \to x_0}}\, G\mu(x) \leqq G\mu(x_0) + 2\varepsilon G\nu(x_0) .$$

$G\mu$ being lower semi-continuous and ε being arbitrary, we have $\lim_{x \to x_0} G\mu(x) = G\mu(x_0)$. This completes the proof.

By Proposition 1, we have the following

LEMMA 3. *Under the same assumptions as in Lemma 2, $\check{G}$ satisfies the continuity principle.*

THEOREM 2. *Let G be a continuous function-kernel on X satisfying* *the condition* $(*)$. *If* $G > 0$ *on* Δ *and* G *satisfies the domination principle, then* G *and* $\check{G}$ *satisfy the continuity principle.*

Proof. We denote by X' the closed subset of X constituted by all points which are not isolated and by G' the restriction of G to $X' \times X'$. By Lemma 1, G' is a continuous function-kernel on X' satisfying the domination principle. Evidently G' satisfies the condition $(*)$ and every point of X' is not isolated. Consequently $\check{G}'$ satisfies the continuity principle by Proposition 2 and Lemma 3. The continuity principle for G is well-known, and we shall prove only the continuity principle for $\check{G}$. Suppose that for $\mu \in M_0$, the restriction of $G\mu$ to $S\mu$ is finite continuous. Denote by μ' the restriction of μ to X'. Then, as a function on $S\mu'$, $G'\mu'$ is finite continuous, because $G'\mu' = G\mu - G(\mu - \mu')$ on X', and hence $G'\mu'$ is finite continuous in X'. By $G\mu' = G'\mu'$ on X' and the fact that $X - X'$ is discrete, $G\mu'$ is finite continuous on X. On the other hand, $S(\mu - \mu')$ is discrete, and then $G(\mu - \mu')$ is finite continuous on X, i.e., $G\mu$ is finite continuous on X. This completes the proof.

4. Remarks on Kishi's theorem

Let us start the following theorem.

THEOREM 3. *Let* G *be a continuous function-kernel on* X. *Assume* $G(x, x) > 0$ *on* Δ. *Then the following four statements are equivalent.*

(1) G *satisfies the domination principle.*

(2) $\check{G}$ *satisfies the domination priciple.*

(3) G *satisfies the balayage principle.*

(4) $\check{G}$ *satisfies the balayage principle.*

Proof. By Proposition 3, we know (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1). The implication (2) $\Rightarrow$ (4) is the dual form of (1) $\Rightarrow$ (3). Therefore it suffices to show the implication (1) $\Rightarrow$ (3). Suppose that (1) is valid. Put $\Omega = \cup \{\omega\colon \text{open}, \operatorname{cap}_G(\omega) = 0\}$ and $X' = X - \Omega$. Let G' the restriction of G to $X' \times X'$. Then G' is a continuous function-kernel on X' satisfying the domination principle and the condition $(*)$. Therefore, by Theorem 2, G' satisfies the continuity principle. Let us remember the existence theorem of Kishi ([4]).

PROPOSITION 5. *Let* $\overline{G}$ *be a lower semi-continuous function-kernel on a locally compact Hausdorff space* Y. *Assume that* $\overline{G}(x, x) > 0$ *for any*

https://doi.org/10.1017/S0027763000016081 Published online by Cambridge University Press

x in Y and $\check{G}$ satisfies continuity principle. For a given compact set F in Y and a given non-negative finite continuous function u on F, there exists a positive Radon measure λ in Y supported by F such that $\overline{G}\lambda(x) \geqq u(x)$ G-p.p.p. on F and $\overline{G}\lambda(x) \leqq u(x)$ λ-a.e. on Y.

We continue our proof. Let K be a compact set in X and μ be a positive Radon measure in X with compact support. First we suppose $S\mu \cap K = \emptyset$. Let μ' be a positive Radon measure in X obtained in Proposition 5 for the case of $\overline{G} = G', F = K \cap X'$ and $u = G\mu$. We have evidently $G\mu'(x) = G'\mu'(x)$ on X' and $\mu' \in E_0(G)$, and "G'-p.p.p. on $K \cap X'$" is equal to "G-p.p.p. on $K \cap X'$". By virtue of the domination principle for G and the inequality $G\mu'(x) \leqq G\mu(x)$ μ'-a.e. in X, we obtain, by the usual way, the inequality $G\mu'(x) \leqq G\mu(x)$ everywhere in X. Consequently μ' is a G-balayaged measure of μ on $K \cap X'$. For an arbitrary ν in $E_0(G)$ supported by K, ν is supported by $K \cap X'$. In fact, if $\nu(\Omega) > 0$, there exists an open set $\omega \subset \Omega$ such that $\mathrm{cap}_G(\omega) = 0$ and $\nu(\omega) > 0$, which is a contradiction. Hence "G-p.p.p. on K" is equivalent to "G-p.p.p. on $K \cap X'$". Therefore μ' is a G-balayaged measure of μ on K.

We remark here that, to show immediately the existence of a G-balayaged measure of μ on K in the case of $S\mu \cap K \neq \emptyset$, it is necessary that $\check{G}$ satisfies the continuity principle (cf. [3]). But, by the above discussion, we obtain that $\check{G}$ satisfies the domination principle. In fact, suppose that for a ν in $E_0(G)$ and a λ in M_0, $\check{G}\nu(x) \leqq \check{G}\lambda(x)$ on $S\nu$. For an arbitrary y in $CS\nu$, there exists a G-balayaged measure ε'_y of ε_y on $S\nu$, and hence we have

$$\begin{aligned}\check{G}\nu(y) &= \int G\varepsilon_y(x)d\nu(x) = \int G\varepsilon'_y(x)d\nu(x) = \int \check{G}\nu(x)d\varepsilon'_y(x) \\ &\leqq \int \check{G}\lambda(x)d\varepsilon'_y(x) = \int G\varepsilon'_y(x)d\lambda(x) \leqq \int G\varepsilon_y(x)d\lambda(x) = \check{G}\lambda(y)\,,\end{aligned}$$

i.e., $\check{G}\nu \leqq \check{G}\lambda$ on X. Consequently $\check{G}$ satisfies the domination principle, and hence $\check{G}$ satisfies the continuity principle (cf. Proposition 4). In the same way as in [3], we obtain that G satisfies the balayage principle. This completes the proof.

By the present theorem and Proposition 4, we have the following

COROLLARY. *Let G be a continuous function-kernel on X satisfying $G > 0$ on Δ. If G satisfies the domination principle, then G and $\check{G}$ satisfy the continuity principle.*

M. Kishi discussed other potential theoretical properties for G implied by the domination principle for G under the condition that $\check{G}$ satisfies the continuity principle (cf. [3], [4], & c.). By the above corollary, in these cases, we can omit the continuity principle for $\check{G}$. In particular, the following theorem is fundamental.

THEOREM 4. *Let G be a continuous function-kernel on X satisfying $G > 0$ on Δ. Then G satisfies the complete maximum principle if and only if G satisfies the domination principle and the classical maximum principle.*

REFERENCES

[1] R. Durier: Thèse, Faculté des Sciences d'Orsay, Université de Paris, 1969.
[2] M. Itô: Sur les principes divers du maximum et le type positif, Nagoya Math. J., **44**, 1971, p. 133–164.
[3] M. Kishi: Maximum principles in the potential theory, Nagoya Math. J., **23**, 1963, p. 165–187.
[4] ——: An existence theorem in potential theory, Nagoya Math. J., **27**, 1966, p. 133–137.

Suzuka College of Technology
and
Nagoya University

https://doi.org/10.1017/S0027763000016081 Published online by Cambridge University Press