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ON WEAK CONVERGENCE IMPLYING STRONG CONVERGENCE IN L_1-SPACES

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Recently, Visintin gave conditions under which weak convergence in $L_1(T;\mathbb{R}^N)$ implies strong convergence. Here we analyze such results in terms of associated Young measures and present an extension to $L_1(T;\Xi)$, where Ξ is a separable reflexive Banach space.

1. Introduction

Let $(T,\mathcal{T},\mu)$ be an abstract σ-finite measure space and denote by $L_1(T;\mathbb{R}^N)$ the space of all integrable $\mathbb{R}^N$-valued functions on T. Recently, Visintin proved the following result for any given sequence $\{f_k\}_0^\infty$ of functions in $L_1(T;\mathbb{R}^N)$ ([8] Theorem 1).

THEOREM. *Suppose that* $f_k \to f_0$ *weakly in* $L_1(T;\mathbb{R}^N)$ *and* $f_0(t)$ *is an extreme point of* $\overline{co}\,\{f_k(t) : k \geq 0\}$ a.e. *in* T. *Then* $f_k \to f_0$ *strongly in* $L_1(T;\mathbb{R}^N)$.

Further, Visintin demonstrated by means of a counterexample ([8], p. 445) that this theorem does not continue to hold if one replaces the image space $\mathbb{R}^N$ by a separable Hilbert space. Nevertheless, we shall show in this note that the above result can be extended and deepened. Let Ξ be a separable reflexive Banach space with norm $\|\cdot\|$; the topological dual of Ξ is denoted by Ξ^* , and the Borel σ-algebra on Ξ by $\mathcal{B}(\Xi)$. As usual, by the weak topology on Ξ we mean the topology $\sigma(\Xi,\Xi^*)$.

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Let $\{f_k\}_0^\infty$ be any given sequence in $L_1(T;\Xi)$, the space of all integrable functions from T into Ξ whose dual (in the prequotient sense) is known to be $L_\infty(T;\Xi^*)$ [5]. Our main result is as follows.

THEOREM 1. *Suppose that* $f_k \to f_0$ *weakly in* $L_1(T;\Xi)$ *and*

(1) $f_0(t)$ *is an extreme point of* $\bigcap_{n=1}^{\infty} \overline{co}\,\{f_k(t) : k \geq n\}$ a.e. *in* T.

Then $f_k \to f_0$ *limitedly in* $L_1(T;\Xi)$.

Here we say that $\{f_k\}_1^\infty$ converges *limitedly* to f_0 in $L_1(T;\Xi)$ if

(2) $$\int_T g(t, f_k(t) - f\ (t))\ \mu(dt) \to 0$$

for every $\mathcal{T} \times \mathcal{B}(\Xi)$-measurable function $g : T \times \Xi \to \mathbb{R}$ satisfying

(3) $$g(t,0) = 0 ,$$

(4) $g(t,\cdot)$ is sequentially weakly continuous on Ξ,

(5) $|g(t,\xi)| \leq C\,\|\xi\| + \phi(t)$ for some $C \geq 0$ and $\phi \in L_1(T;\mathbb{R})$.

Clearly, limited convergence is stronger than weak convergence in $L_1(T;\Xi)$, and is in general not equivalent to it. In fact, when $\Xi = \mathbb{R}^N$ limited convergence is equivalent to strong convergence. In one direction this is seen by taking $g(t,\xi) = |\xi|$ (Euclidean norm). In the other direction we note that any subsequence in (2) has a further subsequence for which (2) holds, since $\{f_k\}_1^\infty$ converges in measure to f_0 *a fortiori* and Fatou's lemma can be applied in an obvious way. Let us also note that in the infinite-dimensional case $g(t,\xi) = \|\xi\|$ does not satisfy (4) (it is merely weakly lower semicontinuous).

Our approach in proving the above generalization of Visintin's result differs considerably from the one used in [8]. We use Young measures (alias relaxed controls), which are known to be quite useful for the study of weak convergence; for example, see [2],[3],[6]. A *Young measure* (with respect to T and Ξ) is defined to be a transition probability with respect to $(T,\mathcal{T})$ and $(\Xi,\mathcal{B}(\Xi))$ ([7], III). To every $(\mathcal{T},\mathcal{B}(\Xi))$-measurable function $f : T \to \Xi$ corresponds its *relaxation* ε_f, the Young measure defined by

$$\varepsilon_f(t) \equiv \text{Dirac probability measure at } f(t) .$$

A sequence $\{\delta_k\}_1^\infty$ of Young measures is said to converge *narrowly* (or *weakly*) to a Young measure δ_0 if

$$(6) \qquad \liminf_k \int_T \int_\Xi g(t,\xi)\ \delta_k(t)(d\xi)\ \mu(dt) \geq \int_T \int_\Xi g(t,\xi)\ \delta_0(t)(d\xi)\ \mu(dt)$$

for every $T \times \mathcal{B}(\Xi)$-measurable function $g : T \times \Xi \to [0,+\infty]$ such that

$$(7) \qquad g(t,\cdot) \text{ is sequentially weakly lower semicontinuous on } \Xi$$

(such functions are known as *normal integrands*); see [2],[3].

2. Proof of Theorem 1.

Theorem 1 will be proven by means of the following two lemmas.

LEMMA 2. *Suppose that* $f_k \to f_0$ *weakly in* $L_1(T;\Xi)$. *Then for every subsequence* $\{k_j\}$ *of* $\{k\}$ *there exist a Young measure* δ_* *and a further subsequence* $\{k_i\}$ *of* $\{k_j\}$ *such that*

$$(8) \qquad \varepsilon_{f_{k_i}} \to \delta_* \ \text{narrowly} ,$$

$$(9) \quad \operatorname{bar} \delta_*(t) \equiv \int_\Xi \xi\ \delta_*(t)(d\xi) \ \text{exists and equals } f_0(t) \text{ a.e. in } T ,$$

$$(10) \qquad \delta_*(t) \text{ is supported by } \bigcap_{n=1}^\infty \overline{co}\,\{f_{k_i}(t) : k_i \geq n\} \text{ a.e. in } T .$$

This result follows from ([3], Theorem 3.1, Lemma 3.2), proven only for the case $\mu(T) < +\infty$; however, it carries over to the σ-finite case without any reservation. Let us remark that the reflexiveness of the Banach space Ξ is of crucial importance for the methods of [3] .

LEMMA 3. *Under* (1) *the Young measure* δ_* *of Lemma 3 is such that*

$$(11) \qquad \delta_*(t) = \varepsilon_{f_0}(t) \ \text{a.e. in } T .$$

Proof. Let $t \in T$ be such that (1),(9),(10) hold. We claim that for every closed convex subset D of $C \equiv \bigcap_{n=1}^\infty \overline{co}\,\{f_k(t) : k \geq n\}$ with

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$f_0(t) \notin D$ it is true that $\delta_*(t)(D) = 0$. For if not so, we would have, writing $\nu \equiv \delta_*(t)$, that $\nu = \nu(D)\ \nu_1 + (1-\nu(D))\ \nu_2$, and hence $f_0(t) = \nu(D)$ bar $\nu_1 + (1-\nu(D))$ bar ν_2 . Here ν_1 and ν_2 are the normalized restrictions of ν to D and its complement. By closedness and convexity of the supports bar $\nu_1 \in D$ and bar $\nu_2 \in C$. Hence, (1) implies that $f_0(t) =$ bar $\nu_1 \in D$, which gives the desired contradiction. In particular, it now follows that the closed balls of Ξ not containing $f_0(t)$ are ν-null sets, and hence so also are the open balls not containing $f_0(t)$ (note that $(\Xi, \sigma(\Xi,\Xi^*))$ and $(\Xi, \|\cdot\|)$ have the same Borel sets by separability of the latter space). Hence, by separability of $(\Xi, \|\cdot\|)$, all open sets of Ξ not containing $f_0(t)$ are ν-null sets, so (11) follows. QED

REMARK. If the support of $\delta_*(t)$ is compact, (11) follows by an application of ([1] Corollary I.4.2).

Combining Lemmas 2 and 3 we find

THEOREM 4. *Suppose that* $f_k \to f_0$ *weakly in* $L_1(T;\Xi)$ *and that* (1) *holds. Then* $\varepsilon_{f_k} \to \varepsilon_{f_0}$ *narrowly* .

Proof. Suppose that for some $g : T \times \Xi \to [0,+\infty]$ as in (7) we would have

$$\liminf_k \int_T g(t,f_k(t))\ \mu(dt) < \int_T g(t,f_0(t))\ \mu(dt) \equiv \beta_0 .$$

For some subsequence $\{k_j\}$ of $\{k\}$ the above $\liminf_k \beta_k$ equals $\lim_j \beta_{k_j}$. Let δ_* and $\{k_i\}$ be as in Lemma 2. By Lemma 3 and (6),(8) it then follows that $\liminf_k \beta_k = \lim_i \beta_{k_i} \geq \beta_0$, a contradiction. QED

REMARK. When $\Xi = \mathbb{R}^N$ one can prove easily that $\varepsilon_{f_k} \to \varepsilon_{f_0}$ narrowly if and only if $f_k \to f_0$ in measure on every $B \in \mathcal{T}$ with $\mu(B) < +\infty$. The same equivalence fails for infinite-dimensional Ξ , again because $\|\cdot\|$ is not weakly continuous.

Let us now give the proof of Theorem 1. First, we shall prove (2) under the simplifying hypothesis $\mu(T) < +\infty$. By ([4], Theorem 1) we have that $\sup_k \int_T \|f_k\| \, d\mu < +\infty$ and $\sup_k \int_{B_j} \|f_k\| \, d\mu \to 0$ whenever $B_j \downarrow \emptyset$ (*uniform* σ-*additivity*). Since μ is supposed to be finite, this implies that $\{\|f_k\|\}_1^\infty$, and by (5) also $\{|g(\cdot,f_k(\cdot)-f_0(\cdot))|\}_1^\infty$ are uniformly integrable ([7], Proposition II.5.2). Let $\varepsilon > 0$ be arbitrary; then there exists $C > 0$ such that $\sup_k \int_{A_k} |g(t,f_k(t)-f_0(t))|\mu(dt) \leqslant \varepsilon$, where A_k denotes the set of all $t \in T$ with $|g(t,f_k(t)-f_0(t)) > C$. Now we define $g^C \equiv \max(g,-C) + C$. Then, writing $g_k \equiv g(\cdot,f_k(\cdot)-f_0(\cdot))$, etc., we get

$$\int_T g_k \, d\mu \geqslant \int_{\{g_k \geqslant -C\}} g_k \, d\mu - \varepsilon = \int_T (g_k^C - C) \, d\mu - \varepsilon .$$

Since we may apply (6) to g^C , we obtain by virtue of Theorem 4

$$\liminf_k \int_T g_k \, d\mu \geqslant \int_T (g_0^C - C) \, d\mu - \varepsilon = -\varepsilon .$$

Hence, $\liminf_k \int_T g_k \, d\mu \geqslant 0$. By repeating the above for $-g$ we conclude that (2) has been proven for the case $\mu(T) < +\infty$. In the general case (μ σ-finite) there exists a sequence $\{T_n\}_1^\infty$ in T such that $\mu(T_n) < +\infty$ for all n and $T \setminus T_n \downarrow \emptyset$. Thus, by the uniform σ-additivity property mentioned above and (5), $\lim_n \sup_k \int_{T \setminus T_n} |g_k| \, d\mu = 0$. By the previous step we know that for every n , $\lim_k \int_{T_n} g_k \, d\mu = 0$. Hence, we conclude that $\lim_k \int_T g_k \, d\mu = 0$, that is (2) holds. This finishes the proof of Theorem 1.

3. Conclusions

We have demonstrated that when $f_k \to f_0$ weakly in $L_1(T;\Xi)$ the extreme point condition (1) forces the associated relaxations to show narrow convergence in the sense of Young measures (Theorem 4). Limited

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convergence of the original functions $\{f_k\}_1^\infty$ to f_0 is the manifestation of this underlying narrow convergence. When Ξ is finite-dimensional limited convergence coincides with strong convergence in $L_1(T;\Xi)$.

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