BOUNDING THE MINIMUM ORDER OF A π-BASE

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Abstract

Let X be a topological space. A family $\mathcal{B}$ of nonempty open sets in X is called a π-base of X if for each open set U in X there exists $B \in \mathcal{B}$ such that $B \subset U$. The order of a π-base $\mathcal{B}$ at a point x is the cardinality of the family $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$ and the order of the π-base $\mathcal{B}$ is the supremum of the orders of $\mathcal{B}$ at each point $x \in X$. A classical theorem of Shapirovskiĭ ['Special types of embeddings in Tychonoff cubes', in: *Subspaces of Σ-Products and Cardinal Invariants*, Topology, Coll. Math. Soc. J. Bolyai, 23 (North-Holland, Amsterdam, 1980), pp. 1055–1086; 'Cardinal invariants in compact Hausdorff spaces', *Amer. Math. Soc. Transl.* **134** (1987), 93–118] establishes that the minimum order of a π-base is bounded by the tightness of the space when the space is compact. Since then, there have been many attempts at improving the result. Finally, in ['The projective π-character bounds the order of a π-base', *Proc. Amer. Math. Soc.* **136** (2008), 2979–2984], Juhász and Szentmiklóssy proved that the minimum order of a π-base is bounded by the 'projective π-character' of the space for any topological space (not only for compact spaces), improving Shapirovskiĭ's theorem. The projective π-character is in some sense an 'external' cardinal function. Our purpose in this paper is, on the one hand, to give bounds of the projective π-character using 'internal' topological properties of the subspaces on compact spaces. On the other hand, we give a bound on the minimum order of a π-base using other cardinal functions in the frame of general topological spaces. Open questions are posed.

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1. Introduction

We start with some definitions. A family $\mathcal{B}$ of nonempty open sets in X is called a *π-base of X at a point $x \in X$* if, for any open set U such that $x \in U$, there exists $B \in \mathcal{B}$ such that $B \subset U$. Observe that it is not necessary that x belongs to B. The family $\mathcal{B}$ is a *π-base in X* if it is a π-base at every $x \in X$. The *π-weight of a space X* is the minimal cardinality of a π-base. The *π-character of a space X at a point* $x \in X$, $\pi_\chi(x, X)$, is the minimal cardinality of a π-base at x and the *π-character of the space X* is given by the cardinal number $\pi_\chi(X) = \sup\{\pi_\chi(x, X) : x \in X\}$. Inequalities involving the π-character of a space X can be found in [4], where this cardinal number is used to establish bounds on weight, the cardinality of a space and the collection of all regular open sets in a space. Let X be a space and $\mathcal{B}$ a family of

subsets of X. The *order of the family* $\mathcal{B}$ *at a point* $x \in X$, ord($\mathcal{B}, x$), is defined as the cardinality of the family $\{B \in \mathcal{B} : x \in B\}$. The *order of the family* $\mathcal{B}$ is defined as ord($\mathcal{B}$) = sup$\{$ord($x, \mathcal{B}$) : $x \in X\}$. Using the terminology of [5], the π-*separating weight of a space* X, $\pi s w(X)$, is the minimum order of a π-base of X. Examples of first countable spaces whose π-separating weight is as large as you wish can be found in [6].

Our historic starting point is a classical theorem of Shapirovskiĭ. In [13, 14] Shapirovskiĭ proved the inequality $\pi s w(K) \leq t(K)$ where K is a compact space. A different proof of this result was given in [17] using free sequences. More related results can be found in [6, 15, 16]. Recently, Juhász and Szentmiklóssy [7, Theorem 2], using a cardinal function called the projective π-character, proved that any Tychonoff space has a π-base of order at most the projective π-character of the space. This result is stronger for compact spaces than the theorem of Shapirovskiĭ because it replaces tightness with projective π-character that is smaller, and also the new theorem extends the result to all Tychonoff spaces. In fact, Juhász and Szentmiklóssy showed in the same paper a compact space K satisfying the strict inequality $p\pi_\chi(K) < t(K)$; see [7, Example 1].

Now we wish to know something else about the projective π-character. The idea arises from the original theorem of Shapirovskiĭ. The proof uses the fact that on a compact space K the tightness $t(K)$ is equal to the hereditarily π-character $h\pi_\chi(K)$ of the space K; that is, the supremum of the π-character of the subspaces of K [12]. In Section 3 we define cardinal functions in order to establish bounds of the projective π-character using in these definitions topological properties of the subspaces on the class of compact spaces. Also open related questions are posed. In Section 4 we consider topological spaces and we give a bound on the order of a π-base, using in this case 'internal' cardinal functions. Some natural questions arise and are posed.

2. Notation and terminology

All spaces X are assumed to be Tychonoff (completely regular) spaces. Our basic references are [2, 3, 8]. A *cardinal number* $\mathfrak{m}$ is the set of all ordinals which precede it. In particular, $\mathfrak{m}$ is a set of cardinality $\mathfrak{m}$. For each cardinal number $\mathfrak{m}$ there exists a smallest cardinal number larger than $\mathfrak{m}$ denoted by $\mathfrak{m}^+$. To every well-ordered set X an ordinal α is assigned, called the *order type of* X. An ordinal number λ is *limit* if there is no ordinal number immediately preceding λ, that is, if for every $\xi < \lambda$ there exists an ordinal number α such that $\xi < \alpha < \lambda$. If the ordinal number ξ immediately precedes α, then we say that ξ is the *predecessor* of α, α is the *successor* of ξ, and we write $\alpha = \xi + 1$. If the set of all ordinal numbers smaller than a limit number λ contains a subset A of type α such that for every $\xi < \lambda$ there exists $\xi' \in A$ such that $\xi < \xi' < \lambda$, then we say that the ordinal number α is *cofinal* with λ. The *cofinality* of a limit ordinal λ, cof(λ), is the least of the ordinal numbers which are cofinal with λ.

We assume that to all the cardinal functions we add the countable cardinal ω. The *weight of* X, $w(X)$, is the smallest infinite cardinal for a base of the topology of X.

The *tightness of a point x* in a topological space X, $t(x, X)$, is the smallest infinite cardinal number $\mathfrak{m}$ with the property that if $x \in \overline{A}$, where $A \subset X$ is a set, then there exists $A_0 \subset A$ such that $|A_0| \leq \mathfrak{m}$ and $x \in \overline{A_0}$. The *tightness of a topological space* X, $t(X)$ is the supremum of all numbers $t(x, X)$ for $x \in X$. Let T be a set; then 2^T denotes the set of all subsets of T. A cardinal function ϕ is *monotone* if $\phi(Y) \leq \phi(X)$ for every subspace Y of X. Monotone cardinal functions are cardinality and weight. On the other hand, density, π-weight, π-character, ... are not. For each cardinal function ϕ that is not monotone, the *hereditarily cardinal function $h\phi$* is defined by $h\phi(X) = \sup\{\phi(Y) : Y \subseteq X\}$.

3. The projective π-character in compact spaces

The following definition can be found in [7, Theorem 2]. For a cardinal function ϕ defined on a class $\mathcal{C}$ of topological spaces, the *projective function associated $p\phi$* is defined as the supremum of the values $\phi(Y)$ where Y ranges over all continuous images of X belonging to $\mathcal{C}$. In particular, *the projective π-character* defined on the class of the Tychonoff spaces is the supremum of the values of $\pi_\chi(Y)$ for each Y Tychonoff space such that there exists a continuous onto map $f : X \to Y$.

We define the cardinal function called *the hereditarily closed π-character of a space X* as the supremum of all $\pi_\chi(F)$ such that F is a closed subspace of X, that is,

$$hc\pi_\chi(X) := \sup\{\pi_\chi(F) : F \subseteq X \text{ is closed in } X\}.$$

By definition, $h\pi_\chi(X) \geq hc\pi_\chi(X)$. This cardinal function give us the first inequality. Before proving this we need some other results.

The proof of the lemma that follows includes a standard argument using Zorn's lemma. A function $f : K \to Y$ continuous and onto is called *irreducible with respect to a subset $A \subset K$* if for every closed subset F of K such that $A \subset F$ and $F \neq K$ we have that $f(F) \neq Y$; see [1].

LEMMA 3.1. *Let $f : K \to Y$ be a continuous map from a compact space K onto a space Y. For every subset $A \subset K$ there is a closed subspace $F \subset K$ such that $A \subset F$, $f(F) = Y$, and the restriction of f to F is irreducible with respect to A.*

PROOF. Fix $A \subset K$. Let us denote by $\mathcal{F}$ the family of all closed subsets F of K such that $f(F) = Y$ with $A \subset F$. The family $\mathcal{F}$ is not empty and closed under intersections of decreasing chains, hence by Zorn's lemma $\mathcal{F}$ contains a minimal member F. Now f restricted to F is irreducible with respect to A by the construction (no proper closed subset of F is mapped onto Y).   □

PROPOSITION 3.2. *Let K be a compact space. Then $p\pi_\chi(K) \leq hc\pi_\chi(K)$.*

PROOF. Let $f : K \to Y$ be a continuous onto map. Since Lemma 3.1, for each $y \in Y$ we consider the pair (x_y, F_y) where $x_y \in f^{-1}(y)$, $F_y \subset K$ is a closed subspace with $x_y \in F_y$ and $f : F_y \to Y$ is irreducible with respect to x_y. We now claim that

$\pi_\chi(x_y, F_y) \geq \pi_\chi(y, Y)$. Let $\mathcal{B}$ be a local π-base at x_y in F_y. Then the family $\{Y \setminus f(F_y \setminus B) : B \in \mathcal{B}\}$ is a π-base at y in Y for each $y \in Y$. Indeed, let V be an open set such that $y \in V$. The set $f^{-1}(V)$ is open and $x_y \in f^{-1}(V)$, thus there exists $B \in \mathcal{B}$ such that $B \subset f^{-1}(V)$. Now $Y \setminus f(F_y \setminus B) \subset V$. Finally, we observe that the following inequality holds for every $y \in Y$:

$$\pi_\chi(F_y) \geq \pi_\chi(x_y, F_y) \geq \pi_\chi(y, Y).$$

Also, for every Y with the initial conditions the following chain holds:

$$\begin{aligned} hc\pi_\chi(K) &= \sup\{\pi_\chi(F) : F \subset K \text{ closed in } K\} \\ &\geq \sup\{\pi_\chi(F_y) : y \in Y\} \\ &\geq \sup\{\pi_\chi(y, Y) : y \in Y\} \\ &= \pi_\chi(Y). \end{aligned}$$

Hence, $hc\pi_\chi(K) \geq p\pi_\chi(K)$ and the proof is complete. □

The other bound that we stated follows easily by considering the following definitions.

Let X be a topological space. We define the *infimum π-character of a space X* as

$$\inf \pi_\chi(X) := \inf\{\pi_\chi(x, X) : x \in X\}.$$

We now consider the *infimum hereditarily closed π-character of a space X* as the cardinal function

$$\inf hc\pi_\chi(X) := \sup\{\inf \pi_\chi(F) : F \subset X, F \text{ is closed in } X\}.$$

Observe that $\inf hc\pi_\chi(X) \leq hc\pi_\chi(X)$.

PROPOSITION 3.3. *Let K be a compact space. Then*

$$\inf hc\pi_\chi(K) \leq p\pi_\chi(K).$$

We consider the following results, whose combination will provide the proof of Proposition 3.3. Firstly, we recall that *the Hilbert cube of weight* $\mathfrak{m}$ is the space $I^\mathfrak{m}$, that is, the Cartesian product $\prod_{s \in S} I_s$ where $I_s = I$ is unit interval for every $s \in S$ and $|S| = \mathfrak{m}$.

LEMMA 3.4. *Let $\mathfrak{m}$ be an infinite cardinal number; then $\pi_\chi(I^\mathfrak{m}) = \mathfrak{m}$.*

PROOF. We prove that $\pi_\chi(I^\mathfrak{m}) \geq \mathfrak{m}$. Let $(x_s)_{s \in S} \in I^\mathfrak{m} = \prod_{s \in S} I_s$ where $I_s = I$ for every $s \in S$ and $|S| = \mathfrak{m}$. We assume that $(x_s)_{s \in S}$ has a local π-base $\mathcal{B}$ consisting of canonical open sets with $|\mathcal{B}| < \mathfrak{m}$, that is, $B \in \mathcal{B}$ is an open set of the form $\prod_{s \in S} U_s$ where each U_s is open in I and $U_s = I$ for all but finitely many $s \in S$. For each $B \in \mathcal{B}$, let J_B be the set given by $J_B := \{s \in S : U_s \neq I\}$. Since $|\mathcal{B}| < \mathfrak{m}$, the set $J = \bigcup\{J_B : B \in \mathcal{B}\}$ verifies that $|J| < \mathfrak{m}$. Thus, there exists $s_0 \in S \setminus J$. Let $U_{s_0} \neq I$ be an open set such that $x_{s_0} \in U_{s_0}$ and consider the set $V = \prod_{s \in S} V_s$ defined as $V_s = I$ if $s \neq s_0$ and $V_{s_0} = U_{s_0}$. Now $B \not\subset V$ for any $B \in \mathcal{B}$, which is a contradiction. □

The following result can be found in [5, Theorem 3.18].

THEOREM 3.5. *Let K be a compact space. The following statements are equivalent.*

(1)   *There is a closed set $F \subset K$ with $\pi_\chi(x, F) \geq \mathfrak{m}$ for each $x \in F$.*
(2)   *K can be mapped continuously onto $I^\mathfrak{m}$.*

Summarizing, for each compact space K we have that

$$\inf hc\pi_\chi(K) \leq p\pi_\chi(K) \leq hc\pi_\chi(K) \leq t(K) = h\pi_\chi(K),$$

where the last equality was proved by Shapirovskiĭ [12].

QUESTION 3.6. Is the inequality $p\pi_\chi(K) \leq hc\pi_\chi(K)$ strict on compact spaces? Are these cardinal functions different (on compact spaces)?

QUESTION 3.7. Is the inequality $\inf hc\pi_\chi(K) \leq p\pi_\chi(K)$ strict on compact spaces? Are these cardinal functions different (on compact spaces)?

In general, for each topological space X the inequality $t(X) \leq h\pi_\chi(X)$ holds [4, Theorem 3.8]. On the other hand, $hc\pi_\chi(X) \leq h\pi_\chi(X)$ also holds. In [5] examples of spaces X such that $t(X) < p\pi_\chi(X)$ are given.

Some questions arise for noncompact spaces.

QUESTION 3.8. Does the inequality $t(X) \leq p\pi_\chi(X)$ hold for X a topological space?

QUESTION 3.9. What happens between $t(X)$ and $hc\pi_\chi(X)$? Are they comparable for X a topological space?

QUESTION 3.10. Are the cardinal functions $hc\pi_\chi(X)$ and $p\pi_\chi(X)$ comparable for X a topological space?

4. The π-separating weight of a topological space

In this section we give a result in terms of cardinal functions different from the projective π-character.

DEFINITION 4.1. Let X be a Tychonoff space. The *Nagami index* of X, $\mathrm{Nag}(X)$, is the smallest infinite cardinal number $\mathfrak{m}$ such that there are a family of compact sets $\mathcal{C} = \{C : C \subset X\}$ and a family of closed sets $\mathcal{T} = \{T : T \subset X\}$ satisfying:

(1) $X = \bigcup\{C : C \in \mathcal{C}\}$;
(2) $|\mathcal{T}| \leq \mathfrak{m}$;
(3)   for each compact set $C \in \mathcal{C}$ and each open set U with $C \subset U$ there exists $T \in \mathcal{T}$ such that $C \subset T \subset U$.

For more about the Nagami index, see [9–11].

Using the sketch of the original proof given by Shapirovskiĭ in [14], Tkachuk proved in [16, Theorem 3.1] that for Lindelöf Σ-spaces the order of a π-base is at most the supremum $\sup\{t(X), \pi_\chi(X)\}$. The same result also is true for each topological

space bounding now with the Nagami index, see inequality (4.1) below. The proof mimics the proof given first by Shapirovskiĭ and then by Tkachuk, being careful with cardinalities. Let $\mathcal{T}$ be a subset of 2^T; we denote by $\bigwedge \mathcal{T}$ the family of all finite intersections of elements of $\mathcal{T}$.

THEOREM 4.2. *Let X be a topological space. Then*

$$\pi s w(X) \leq \sup\{\text{Nag}(X), t(X), \pi_\chi(X)\}. \tag{4.1}$$

PROOF. We can assume without loss of generality that X is a topological space without isolated points. Indeed, if S is the set of isolated points of X, the family $\mathcal{S} = \{\{x\} : x \in S\}$ is a disjoint π-base at every point of $\overline{S}$ and the space $\overline{X \setminus \overline{S}}$ has no isolated points. We define $\mathfrak{m} := \sup\{\text{Nag}(X), t(X), \pi_\chi(X)\}$. Now for any $x \in X$ there exists a π-base $\mathcal{B}_x$ such that $|\mathcal{B}_x| \leq \mathfrak{m}$ and $x \notin \overline{U}$ for any $U \in \mathcal{B}_x$. Since $\text{Nag}(X) \leq \mathfrak{m}$ there exist a family $\mathcal{C}$ of compact sets covering the space X and a family $\mathcal{T}$ of closed sets in X with $|\mathcal{T}| \leq \mathfrak{m}$ satisfying the conditions of Definition 4.1. We will use transfinite induction. For a cardinal number τ let P_τ be the following statement.

For every $S \subset X$ such that $|S| \leq \tau$ there exists a family $I(S)$ of open subsets of X such that:

(a) $I(S)$ is a π-base in X for each $x \in S$;
(b) $\text{ord}(I(S)) \leq \mathfrak{m}$;
(c) $|I(S)| \leq |S| \cdot \mathfrak{m}$.

Since, by [4, Theorem 3.8(b)], $\pi w(X) = d(X) \cdot \pi_\chi(X)$, the statement P_τ holds for $\tau \geq d(X)$. We will show induction on cardinal numbers with $\tau < d(X)$. If $\tau \leq \mathfrak{m}$ then P_τ holds. Now assume that $\nu > \mathfrak{m}$ and that P_τ has been proved for each $\tau < \nu$. Fix a set S with $|S| = \nu$ and well-order S as follows:

$$S = \{x_\alpha : \alpha \text{ is not a limit ordinal and } \alpha < \nu\}.$$

We assume that for all $\alpha < \alpha' \leq \nu$ we have already constructed a closed set $F_\alpha \subset X$ and a family $\mathcal{G}_\alpha$ of open sets in X satisfying the following properties.

(i) $F_\alpha \subset F_\beta, \mathcal{G}_\alpha \subset \mathcal{G}_\beta$ for $\alpha < \beta < \alpha'$.
(ii) If $\alpha < \alpha'$ is a successor ordinal, then $x_\alpha \in F_\alpha$.
(iii) $\mathcal{G}_\alpha$ is a π-base in X for every $x \in F_\alpha$, $|\mathcal{G}_\alpha| \leq |\alpha| \cdot \mathfrak{m}$ and $\text{ord}(\mathcal{G}_\alpha) \leq \mathfrak{m}$ for any $\alpha < \alpha'$.
(iv) If $\alpha < \beta < \alpha'$, $U \in \mathcal{G}_\beta \setminus \mathcal{G}_\alpha$, then $F_\alpha \cap \overline{U} = \emptyset$.
(v) If $\alpha < \beta < \alpha'$, $U \in \bigwedge \mathcal{G}_\alpha$ and there exists $T \in \mathcal{T}$ such that $T \cap \overline{U} \neq \emptyset$ then $F_\beta \cap (T \cap \overline{U}) \neq \emptyset$.

F_0 is taken to be $\{x_0\}$ and $\mathcal{G}_0$ is a π-base for x_0 of cardinality up to $\mathfrak{m}$. We now construct the corresponding closed set $F_{\alpha'}$ and family $\mathcal{G}_{\alpha'}$. Firstly we suppose that α' is a successor ordinal ($\alpha' = \beta + 1$) and consider the family

$$\mathcal{H} = \left\{ H : \text{there exist } U \in \bigwedge \mathcal{G}_\beta \text{ and } T \in \mathcal{T} \text{ such that } \emptyset \neq H = \overline{U} \cap T \subset X \setminus F_\beta \right\}.$$

For any $H \in \mathcal{H}$ we choose $y_H \in H$. If $x_{\alpha'} \in F_\beta$ we choose $Z = \{y_H : H \in \mathcal{H}\}$, and if $x_{\alpha'} \notin F_\beta$ then $Z = \{x_{\alpha'}\} \cup \{y_H : H \in \mathcal{H}\}$. It is clear that

$$|Z| \leq |H| \leq \sup\{|\mathcal{G}_\beta|, |\mathcal{T}|\} \leq \sup\{\mathrm{Nag}(X), |\beta|, \mathfrak{m}\} < \nu,$$

thus we may apply the induction hypothesis on the set Z to find a family $I(Z)$ satisfying conditions (a), (b) and (c) that are: $|I(Z)| \leq |Z| \cdot \mathfrak{m}$, $I(Z)$ is a π-base in X at every point x of Z and $\mathrm{ord}(I(Z)) \leq \mathfrak{m}$. By construction $F_\beta \cap Z = \emptyset$, so it is not restrictive to suppose that we can choose $I(Z)$ such that $\overline{U} \cap F_\beta = \emptyset$ for each $U \in I(Z)$. We consider the family $\mathcal{G}_{\alpha'} := \mathcal{G}_\beta \cup I(Z)$ and $F_{\alpha'} := F_\beta \cup \overline{Z}$. By the construction all properties (i)–(v) are satisfied.

If α' is a limit ordinal, $\alpha' \leq \nu$, we define the family $\mathcal{G}_{\alpha'} := \bigcup\{\mathcal{G}_\alpha : \alpha < \alpha'\}$ and the set $F_{\alpha'} := \overline{\bigcup\{F_\alpha : \alpha < \alpha'\}}$. Now properties (i), (iv)–(v) hold. Property (ii) holds vacuously. We have to prove (iii). We consider two cases.

(a) If $\mathrm{cof}(\alpha') \leq \mathfrak{m}$, then it follows from (i) that $\mathcal{G}_{\alpha'} = \bigcup\{\mathcal{G}_\alpha : \alpha \in M\}$ such that M is cofinal with α' and $\mathrm{ord}(\mathcal{G}_\alpha) \leq \mathfrak{m}$ for every $\alpha \in M$ so $\mathrm{ord}(\mathcal{G}_{\alpha'}) \leq \mathfrak{m}$.

(b) If $\mathrm{cof}(\alpha') > \mathfrak{m}$ then it follows from (i) and $t(X) \leq \mathfrak{m}$ that $F_{\alpha'} = \bigcup\{F_\alpha : \alpha < \alpha'\}$. We will prove that $\mathrm{ord}(F_{\alpha'}) \leq \mathfrak{m}$. Assume that there is a family $\mathcal{U} \subset \mathcal{G}_{\alpha'}$ such that $|\mathcal{U}| = \mathfrak{m}^+$ and $P = \bigcap \mathcal{U} \neq \emptyset$. Pick a set $C \in \mathcal{C}$ such that $C \cap P \neq \emptyset$. Then the family

$$\gamma_C = \{\overline{U} \cap C \cap F_{\alpha'} : U \in \mathcal{U}\}$$

satisfies the finite intersection property. Fix a finite family $\mathcal{V} \subset \mathcal{U}$ and consider $V = \bigcap \mathcal{V}$; then there exists $\alpha < \alpha'$ such that $\mathcal{V} \subset \mathcal{G}_\alpha$. Let $T \in \mathcal{T}$ be such that $C \subset T$; then $T \cap V \supset C \cap P \neq \emptyset$, so it follows from (v) that $F_{\alpha+1} \cap T \cap \overline{V} \neq \emptyset$ which implies that $(F_{\alpha'} \cap \overline{V}) \cap T \neq \emptyset$ for any $T \in \mathcal{T}$ with $C \subset T$, hence

$$F_{\alpha'} \cap \overline{V} \cap C \subset \bigcap\{\overline{U} \cap C \cap F_{\alpha'} : U \in \mathcal{V}\} \neq \emptyset.$$

Since the family γ_C consists of compact subsets of X, it follows that there is a point $x \in F_{\alpha'} \cap C \cap (\bigcap\{\overline{U} : U \in \mathcal{U}\})$. Now, there is $\alpha < \alpha'$ such that $x \in F_\alpha$. The family $\mathcal{U}' = \mathcal{U} \cap F_\alpha$ has cardinality at most $\mathfrak{m}$ so there exists $U \in \mathcal{U} \setminus \mathcal{U}' \subset \mathcal{G}_{\alpha'} \setminus \mathcal{G}_\alpha$ not empty. But, because of (iv), $\overline{U} \cap F_\alpha = \emptyset$, which is a contradiction with $x \in \overline{U} \cap F_\alpha$ for every $U \in \mathcal{U}$. Now $\mathrm{ord}(F_{\alpha'}) \leq \mathfrak{m}$ and we have completed the induction step for a limit ordinal $\alpha' \leq \nu$.

For all $\alpha \leq \nu$ we have constructed a family $\mathcal{G}_\alpha$ and a set F_α with conditions (i)–(v). Finally, by (ii), $S \subset F_\nu$ and the family $\mathcal{G}_\nu$ has properties (a), (b) and (c), thus P_ν holds. Now considering S a dense set of X we obtain the desired inequality of our theorem. $\square$

COROLLARY 4.3. *Let X be a topological space. Then*

$$\pi sw(X) \leq \sup\{\mathrm{Nag}(X), h\pi_\chi(X)\}. \tag{4.2}$$

PROOF. This follows from Theorem 4.2 using the facts that $\pi_\chi(X) \leq h\pi_\chi(X)$ and $t(X) \leq h\pi_\chi(X)$; see [4, Theorem 3.8]. $\square$

QUESTION 4.4. Can we replace the cardinal function $h\pi_\chi(X)$ by $hc\pi_\chi(X)$ in inequality (4.2)?

QUESTION 4.5. Does the inequality $p\pi_\chi(X) \le \sup\{t(X), \pi_\chi(X)\}$ hold in the class of Lindelöf Σ-spaces? In other words, is the theorem of Juhász and Szentmiklóssy stronger than that of Tkachuk [16, Theorem 3.1]?

The following question is more general.

QUESTION 4.6. Does the inequality $p\pi_\chi(X) \le \sup\{t(X), \pi_\chi(X), \text{Nag}(X)\}$ hold for X a topological space?

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