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THE SCHUR–ZASSENHAUS THEOREM IN LOCALLY FINITE GROUPS

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Abstract

Let $L = HK$ be a semidirect product of a normal locally finite π'-group H by a locally finite π-group K, where π is a set of primes. Suppose $C_K(H) = 1$ and L is Sylow π-sparse (which in the countable case just says that the Sylow π-subgroups of L are conjugate). This paper completes the characterization of those groups which can occur as K— this had previously been obtained under the assumption that L is locally soluble. The answer is the same—essentially that the groups occurring are those having a subgroup of finite index which is a subdirect product of so-called "pinched" groups.

This short note is an addendum to my paper Hartley (1975), with which familiarity will be assumed and from which any unexplained terminology is drawn. Its purpose is to point out that certain hypotheses of local solubility in one of the main results of that paper are superfluous. We improve our previous Theorem B to

THEOREM B. *Let G be a locally finite group and let $\pi = \pi(G)$. Then necessary and sufficient conditions that there exist a locally finite Sylow π-sparse group $L = HK$ such that $H \lhd L$, H is a π'-group, $C_K(H) = 1$ and $K \cong G$ are*

(i) *there exists a prime $q \notin \pi$,*

(ii) *G is almost subpropinched.*

PROOF. That (i) and (ii) imply the existence of such an L is seen exactly as in Hartley (1975).

Conversely, suppose that $L = HK$, where L is locally finite, $H \lhd L$, H is a π'-group, K is a π-group, $C_K(H) = 1$ and L is Sylow π-sparse. We have to show that (i) and (ii) hold for K; of these (i) is clear.

Let A be any abelian subgroup of K. Then HA is Sylow π-sparse, and by Hartley (1972) Lemma 3.5 for example, A has finite (Mal'cev special) rank. We may now apply a deep theorem of Šunkov (1971) to conclude that K has a locally soluble normal subgroup K_0 of finite index and finite rank. By Kargapolov (1959), K_0 is countable. Since HK_0 is Sylow π-sparse and (ii) of the theorem is unaffected by passing to subgroups of finite index, we may assume that K is countable and locally soluble. There exists a countable subgroup H_0 of H normalized by K and such that $C_K(H_0) = 1$. Hence we may assume that H, and therefore L also, is countable.

We can write $L = \bigcup_{i=1}^{\infty} F_i$, where

$$F_1 \leqq F_2 \leqq \cdots .$$

is a tower of finite subgroups of L. Let $E_i = F_i \cap H$. Then

$$(1) \qquad E_1 \leqq E_2 \leqq \cdots; \qquad \bigcup_{i=1}^{\infty} E_i = H.$$

Let p be any prime not in π, and let $\sigma = \pi \cup \{p\}$. We have that $E_i \lhd F_i$, F_i/E_i is a soluble σ-group, and since p is the only prime in σ dividing $|E_i|$, E_i has a nilpotent Hall σ-subgroup. By Hall (1956) Theorem D5, F_i satisfies the condition D_σ of that paper, and in particular every Sylow σ-subgroup of F_i is a Hall σ-subgroup. Therefore there exists a tower

$$Q_1 \leqq Q_2 \leqq \cdots .$$

such that Q_i is a Hall σ-subgroup of F_i for each i.

Let $Q = \bigcup_{i=1}^{\infty} Q_i$. Then Q is a Sylow σ-subgroup of L, and in fact, if $x \to \bar{x}$ is any homomorphism of L, then $\bar{Q}$ is a Sylow σ-subgroup of $\bar{L}$ (see Hartley (1971) Lemma 2.1). In particular, $L = HQ$. Since Q is countable and locally soluble, $H \cap Q$ is a p-group and $H/H \cap Q$ is a π-group, while $p \notin \pi$, we have $Q = (H \cap Q)K^*$ for some π-subgroup K^* of Q. Then K^* complements H in L, and as L is Sylow π-sparse, we have $K = K^{*h}$ for some $h \in H$. Let $H_p = (H \cap Q)^h$, $D_i = E_i^h$. Now

$$H \cap Q \cap E_i = Q \cap E_i = Q \cap F_i \cap E_i = Q_i \cap E_i,$$

which is a Sylow p-subgroup of E_i. Hence we have

(a) *K normalizes H_p,*

(b) *There exists a tower $D_1 \leqq D_2 \leqq \cdots$ (depending on p) of finite subgroups of H such that $\bigcup_{i=1}^{\infty} D_i = H$ and $H_p \cap D_i$ is a Sylow p-subgroup of D_i for each i.*

Now if $y \in L$, then $y^{h^{-1}} \in F_i$ for some i (where h is as above), and so $y^{h^{-1}}$ normalizes $E_i = F_i \cap H$. Therefore y normalizes D_i, and we also have

(c) *Every element of L normalizes all but finitely many of $D_1, D_2, \cdots$.*

We now form the direct product H^* of the groups H_p, as p ranges over all primes not in π. We have an obvious action of K on H^*, and form the split extension $L^* = H^*K$. Of course, H^* is locally nilpotent. We prove

LEMMA (i) *L^* is Sylow π-sparse.*

(ii) $C_K(H^*) = 1$.

PROOF. (i) Let B be any subgroup of K. Since L is Sylow π-sparse and countable, Hartley (1971) Lemma 4.3 shows that $C_H(B) = C_H(F)$ for some finite subgroup F of B. Then $C_{H_p}(B) = C_{H_p}(F)$ for all $p \notin \pi$, and hence $C_{H^*}(B) = C_{H^*}(F)$. By Hartley (1971) Lemma 4.3 again, L^* is Sylow π-sparse.

(ii) Let $x \in C_K(H^*)$, let Y be any finite $\langle x \rangle$-invariant subgroup of H, and let p be a prime divisor of $|Y|$. We have $Y \leqq D_i$ for some i, where D_i comes from the tower given by (b) corresponding to H_p, and (c) allows us to suppose that D_i is $\langle x \rangle$-invariant. Consider now $D_i\langle x \rangle$. We have that $(|D_i|, |\langle x \rangle|) = 1$, and by (b) and the fact that x centralizes H^*, $C = C_{D_i}(x)$ contains a Sylow p-subgroup of D_i. Now by well known results on coprime automorphism groups (see Gorenstein (1968), Theorem 6.2.2)), C contains every $\langle x \rangle$-invariant p-subgroup of D_i, and x leaves some Sylow p-subgroup of Y invariant. Therefore x centralizes a Sylow p-subgroup of Y. Since this holds for all p dividing $|Y|$, we find that x centralizes Y. Hence x centralizes H, and so by hypothesis, $x = 1$.

Returning to the proof of Theorem B, we see that the lemma allows us to consider H^* instead of H and so assume that L is locally soluble. But then we complete the proof by applying Theorem B of Hartley (1975).

It is less clear whether the requirement of local solubility in Hartley (1975) Theorem A is superfluous.

In conclusion, we draw attention to the Theorem B of Rae (1972), which gives information about the structure of H in the situation of Theorem B.

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