J. Aust. Math. Soc. **88** (2010), 339–352
doi:10.1017/S1446788710000224

THE SKITOVICH–DARMOIS THEOREM FOR LOCALLY COMPACT ABELIAN GROUPS

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(Received 2 December 2009; accepted 24 April 2010)

Communicated by G. A. Willis

Abstract

According to the Skitovich–Darmois theorem, the independence of two linear forms of n independent random variables implies that the random variables are Gaussian. We consider the case where independent random variables take values in a second countable locally compact abelian group X, and coefficients of the forms are topological automorphisms of X. We describe a wide class of groups X for which a group-theoretic analogue of the Skitovich–Darmois theorem holds true when $n = 2$.

2000 *Mathematics subject classification*: primary 60B15; secondary 62E10, 43A25.
Keywords and phrases: locally compact abelian group, Skitovich–Darmois theorem.

1. Introduction

By the classical Kac–Bernstein theorem, a Gaussian measure is characterized by the independence of the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ of independent random variables ξ_j. Let ξ_j be independent random variables and α_j and β_j be nonzero real numbers, where $j = 1, 2, \ldots, n$ and $n \geq 2$. Consider two linear forms given by $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ instead of the sum and the difference. The Skitovich–Darmois theorem asserts that, if the linear forms L_1 and L_2 are independent, then the random variables ξ_j are Gaussian [2, 19]. This theorem was generalized by Ghurye and Olkin [11] to the case where the ξ_j are independent vectors in the space $\mathbb{R}^m$ and the coefficients α_j and β_j are nonsingular matrices (see also [14, Ch. 3]). They proved that the independence of L_1 and L_2 implies that the random vectors ξ_j are Gaussian. In recent years, much attention has been devoted to generalizing the Skitovich–Darmois theorem to various algebraic structures such as locally compact abelian groups, Lie groups, and quantum groups (see [6], where one can find further references). The research stimulated by the Skitovich–Darmois

This research was supported in part by the Ukrainian–French grant PICS No 4769 (2009–2011).

theorem has been also continued in the classical case [1, 15]. We also note that group analogues of the Skitovich–Darmois theorem are closely connected with the 'positive definite functions of product type' introduced by Schmidt (see [8, 18]). This article is devoted to a generalization of the Skitovich–Darmois theorem on locally compact abelian groups.

Let X be a second countable locally compact abelian group, Y be its character group X^*, and (x, y) be the value of a character $y \in Y$ on an element $x \in X$. Denote by $M^1(X)$ the convolution semigroup of probability distributions on X. Denote by $\widehat{\mu}$ the characteristic function on Y of a distribution $\mu \in M^1(X)$:

$$\widehat{\mu}(y) = \int_X (x, y)\, d\mu(x) \quad \forall y \in Y.$$

A probability measure $\mu \in M^1(X)$ on the group X is called Gaussian (in the sense of Parthasarathy [16, Ch. 4.6]) if its characteristic function can be represented in the form

$$\widehat{\mu}(y) = (x, y) \exp\{-\varphi(y)\} \quad \forall y \in Y,$$

for some fixed $x \in X$, where φ is a continuous nonnegative function satisfying the equation

$$\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)] \quad \forall u, v \in Y.$$

Taking into account that we will deal here only with Gaussian measures in the sense of Parthasarathy, we will just call them Gaussian. Denote by $\Gamma(X)$ the set of Gaussian measures on X. We note that the support of a Gaussian measure is a shift of a connected subgroup of the group X. For this reason the class of Gaussian measures on a totally disconnected group coincides with the class of degenerate distributions. Denote by m_K the normalized Haar measure of a compact subgroup K of the group X, and by $I(X)$ the set of shifts of such measures. Let $\mathrm{Aut}(X)$ be the set of topological automorphisms of X. Suppose that $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$, where $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ and ξ_j are independent random variables taking values in X and with distributions μ_j (here $j = 1, 2$). We formulate the following general problem.

PROBLEM 1.1. Describe second countable locally compact abelian groups X with the property that, for any linear forms L_1 and L_2, the independence of L_1 and L_2 implies that each distribution μ_j is a convolution of a Gaussian measure and the normalized Haar measure of a compact subgroup of X, written $\mu_j \in \Gamma(X) * I(X)$. Equivalently, describe second countable locally compact abelian groups X for which the Skitovich–Darmois theorem holds true for two independent random variables.

Shifts by normalized Haar measures appear in a natural way in characterization problems on abelian groups; see, for instance [7, 13, 17]. It should be noted that if $\mu \in M^1(X)$ and $\mu = \gamma * m_K$, where $\gamma \in \Gamma(X)$, then μ is K-invariant and μ induces a Gaussian measure on the factor group X/K under the natural homomorphism $X \to X/K$. Problem 1.1 was first solved in the class of finite abelian groups [5], then in the class of second countable compact totally disconnected abelian groups [9] and discrete countable abelian groups [10].

For any natural number n, denote by $f_n : X \to X$ the homomorphism $f_n(x) = nx$, and write $X^{(n)}$ for its image $\mathrm{Im}\, f_n$ and $X_{(n)}$ for its kernel $\mathrm{Ker}\, f_n$. The following theorems hold.

THEOREM 1.2 [9]. *Let ξ_1 and ξ_2 be independent random variables taking values in a second countable compact totally disconnected abelian group X, with distributions μ_1 and μ_2. Let $\alpha_j, \beta_j \in \mathrm{Aut}(X)$. Then the independence of the two linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ implies that $\mu_1, \mu_2 \in I(X)$ if and only if for each prime p the factor group $X/X^{(p)}$ is finite.*

THEOREM 1.3 [10]. *Let ξ_1 and ξ_2 be independent random variables taking values in a discrete countable abelian group X, with distributions μ_1 and μ_2, and let α_j, $\beta_j \in \mathrm{Aut}(X)$. Then the independence of the two linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ implies that $\mu_1, \mu_2 \in I(X)$.*

The main result of this article is the following theorem.

THEOREM 1.4. *Assume that a locally compact abelian group X is of the form*

$$X = \mathbb{R}^m \times K \times D, \tag{1.1}$$

where $m \geq 0$; the group K is compact, totally disconnected and abelian, and the factor group $K/K^{(p)}$ is finite for each prime p; and D is a countable discrete abelian group. Let ξ_1 and ξ_2 be independent random variables taking values in X and with distributions μ_1 and μ_2. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{Aut}(X)$, and define the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$. Then the independence of L_1 and L_2 implies that $\mu_1, \mu_2 \in \Gamma(X) * I(X)$.

The proof of Theorem 1.4 is given in Section 2. In Section 3 we construct an example of a locally compact abelian group X that is not topologically isomorphic to a group of the form (1.1) and for which the Skitovich–Darmois theorem holds true for two independent random variables.

2. Proof of Theorem 1.4

To prove Theorem 1.4, we will use some structure and duality theory for locally compact abelian groups [12]. Let X be an arbitrary second countable locally compact abelian group. If H is a subgroup of its dual group Y, then we denote its annihilator $\{x \in X : (x, y) = 1 \text{ for all } y \in H\}$ by $A(X, H)$. We recall that a torsion abelian group G is called p-prime if the order of every element of G is a power of p. For any $\alpha \in \mathrm{Aut}(X)$ define the conjugate automorphism $\widetilde{\alpha} \in \mathrm{Aut}(Y)$ by the formula $(x, \widetilde{\alpha} y) = (\alpha x, y)$ for all $x \in X$ and $y \in Y$. Denote by I the identity automorphism of a group. If A and B are subsets of X, then denote by $A + B$ their arithmetic sum:

$$A + B = \{x \in X : x = x_1 + x_2,\ x_1 \in A,\ x_2 \in B\}.$$

Denote by $\mathcal{P}$ the set of prime numbers.

https://doi.org/10.1017/S1446788710000224 Published online by Cambridge University Press

Let $\mu \in M^1(X)$. Denote by $\sigma(\mu)$ the support of μ. It is useful to remark that if H is a closed subgroup of Y and $\widehat{\mu}(y) = 1$ for all $y \in H$, then the characteristic function $\widehat{\mu}$ is H-invariant, that is, $\widehat{\mu}(y + h) = \widehat{\mu}(y)$ for all $y \in Y$, $h \in H$, and $\sigma(\mu) \subseteq A(X, H)$. Let K be a compact subgroup of X. It should be noted that the characteristic function of the normalized Haar measure m_K is of the form

$$\widehat{m}_K(y) = \begin{cases} 1 & \text{if } y \in A(Y, K) \\ 0 & \text{if } y \notin A(Y, K). \end{cases} \tag{2.1}$$

We formulate as a lemma the following simple and well-known statement.

Lemma 2.1. *Let X be a second countable locally compact abelian group. If f is a characteristic function such that $|f(y)| = 1$ for all $y \in Y$, then $f(y) = (x, y)$ for some $x \in X$.*

For $\mu \in M^1(X)$, define the distribution $\bar{\mu} \in M^1(X)$ by the formula $\bar{\mu}(E) = \mu(-E)$ for all Borel subsets E of X. Observe that $\widehat{\bar{\mu}} = \overline{\widehat{\mu}}$.

We need some lemmas to prove Theorem 1.4. Let ξ be a random variable taking values in X and with distribution μ. Taking into account that the characteristic function of the distribution μ is an expectation, that is, $\widehat{\mu}(y) = \mathbf{E}[(\xi, y)]$, the following statement may be proved exactly as in the classical case.

Lemma 2.2. *Let X be a second countable locally compact abelian group. Let ξ_1 and ξ_2 be independent random variables taking values in X and with distributions μ_1 and μ_2. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{Aut}(X)$, and define the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$. Then the independence of L_1 and L_2 is equivalent to the characteristic functions $\widehat{\mu}_j$ satisfying the equation*

$$\widehat{\mu}_1(\widetilde{\alpha}_1 u + \widetilde{\beta}_1 v)\widehat{\mu}_2(\widetilde{\alpha}_2 u + \widetilde{\beta}_2 v) = \widehat{\mu}_1(\widetilde{\alpha}_1 u)\widehat{\mu}_2(\widetilde{\alpha}_2 u)\widehat{\mu}_1(\widetilde{\beta}_1 v)\widehat{\mu}_2(\widetilde{\beta}_2 v)$$

for all $u, v \in Y$.

Lemma 2.3 [3]. *Let Y be a second countable compact connected abelian group, and $a \in \mathrm{Aut}(Y)$. Then all solutions of the equation*

$$\widehat{\mu}_1(u + v)\widehat{\mu}_2(u + av) = \widehat{\mu}_1(u)\widehat{\mu}_2(u)\widehat{\mu}_1(v)\widehat{\mu}_2(av) \quad \forall u, v \in Y, \tag{2.2}$$

in the class of nonnegative characteristic functions of probability distributions, are identically equal to 1.

Lemma 2.4. *Assume that a locally compact abelian group H is of the form $F \times S$, where S is a second countable compact abelian group and F is a discrete countable torsion abelian group that satisfies condition (i): for all primes p the subgroup $\{d \in F : pd = 0\}$ is finite. Then for any element $h_0 \in H$ and any automorphism $a \in \mathrm{Aut}(H)$ there exists a compact subgroup Q of H, such that $h_0 \in Q$ and $a(Q) = Q$.*

PROOF. Consider a decomposition of the group F into the weak direct product of its p-prime subgroups:

$$\prod_{p\in\mathcal{P}}^{*} F_p.$$

Note that each element $d \in F$ can be represented in the form

$$d = \sum_{j=1}^{n} d_j,$$

where $d_j \in F_{p_j}$. Hence $p_j^{k_j} d_j = 0$ for some natural number k_j, when $j = 1, 2, \ldots, n$. We assume that k_j are minimal here. Put

$$B_j = \{c \in F_{p_j} : p_j^{k_j} c = 0\}.$$

Then

$$d \in B_1 \times \cdots \times B_n = B, \tag{2.3}$$

and it follows from (i) that the subgroup B is finite. It is obvious that, for any finite subset M of F, there exists a subgroup B of the form (2.3) containing M.

Denote elements of H by h or (d, k), where $d \in F$ and $k \in S$, and define the homomorphism $\pi : H \to F$ by $\pi(h) = \pi(d, k) = d$. Fix an automorphism $a \in \mathrm{Aut}(H)$, and consider the subgroup $S_a = \pi(a(S))$. Since the homomorphisms π and a are continuous and S is compact, S_a is also a compact subgroup. Taking into account the fact that F is a discrete group, we conclude that S_a is a finite group. Then

$$a(S) \subseteq S_a \times S. \tag{2.4}$$

Let $h_0 = (d_0, k_0) \in H$. Denote by B the subgroup of the form (2.3) containing the element d_0 and the subgroups S_a and $S_{a^{-1}}$. Then $h_0 \in B \times S$. We will verify that $a(B \times S) = B \times S$. Let $c \in B_j$. It is obvious that the elements c and $a(c)$ have equal orders, and these orders are p_j^l, where $l \le k_j$. All elements of the group H of order p_j^l, where $l \le k_j$, are contained in $B_j \times S$, and so $a(c) \in B_j \times S$. Hence $a(B_j) \subseteq B_j \times S$. This implies that

$$a(B) \subseteq B \times S. \tag{2.5}$$

Taking into account that $S_a \subseteq B$, it follows from (2.4) and (2.5) that

$$a(B \times S) \subseteq a(B) + a(S) \subseteq (B \times S) + (S_a \times S) = B \times S. \tag{2.6}$$

Reasoning similarly, we find that

$$a^{-1}(B \times S) \subseteq B \times S. \tag{2.7}$$

It follows from (2.6) and (2.7) that $a(B \times S) = B \times S$. Set $Q = B \times S$. Then Q is the required subgroup. □

REMARK. The statement of Lemma 2.4 is not true if we omit condition (i). Indeed, let G be an arbitrary finite abelian group and let $\Phi_i = G$, where $i = 0, \pm1, \pm2, \ldots$.

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Let F be the weak direct product

$$F = \prod_{i=-\infty}^{-1}{}^{*} \Phi_i.$$

Consider F with the discrete topology. Then F is a discrete countable torsion abelian group, and condition (i) fails for F. Let S be the direct product

$$S = \prod_{i=0}^{\infty} \Phi_i.$$

Consider S with the product topology. Obviously, S is a second countable compact abelian group. Put $H = F \times S$ and denote elements of the group H by $h = \{x_i\}_{i=-\infty}^{\infty}$, where $x_i \in \Phi_i$. Note that $x_i = 0$ if $i < n(h)$. Set $a\{x_i\}_{i=-\infty}^{\infty} = \{x_{i+1}\}_{i=-\infty}^{\infty}$. Then $a \in \mathrm{Aut}(H)$. It is clear that the orbit $\{a^n h\}_{n=0}^{\infty}$ of each nonzero element $h \in H$ is noncompact. Hence there does not exist a compact subgroup Q of H which is invariant with respect to a.

LEMMA 2.5 [10]. *Let Y be a second countable compact abelian group, and suppose that $a \in \mathrm{Aut}(Y)$. Then all solutions of Equation (2.2) in the class of nonnegative characteristic functions of probability distributions are of the form*

$$\widehat{\mu}_1(y) = \widehat{\mu}_2(y) = \begin{cases} 1 & \text{if } y \in E \\ 0 & \text{if } y \notin E, \end{cases} \tag{2.8}$$

where E is a subgroup of Y, and $a(E) = E$.

PROOF OF THEOREM 1.4. Let $\delta \in \mathrm{Aut}(X)$, and $\mu \in M^1(X)$. It is apparent that $\mu \in \Gamma(X) * I(X)$ if and only if $\delta\mu \in \Gamma(X) * I(X)$. Hence we can consider new independent random variables $\xi_1' = \alpha_1\xi_1$ and $\xi_2' = \alpha_2\xi_2$, and assume without loss of generality that $L_1 = \xi_1 + \xi_2$ and $L_2 = \delta_1\xi_1 + \delta_2\xi_2$, where $\delta_j \in \mathrm{Aut}(X)$. Moreover, since for each $\delta \in \mathrm{Aut}(X)$ the independence of the linear forms L_1 and L_2 is equivalent to the independence of the linear forms L_1 and δL_2, we may in addition assume that $L_2 = \xi_1 + \alpha\xi_2$, where $\alpha \in \mathrm{Aut}(X)$. Put $a = \widetilde{\alpha}$.

By Lemma 2.2 the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha\xi_2$ are independent precisely when the characteristic functions $\widehat{\mu}_j$ satisfy (2.2). Put $\nu_j = \mu_j * \bar{\mu}_j$. Then $\widehat{\nu}_j = |\widehat{\mu}_j|^2 \geq 0$, and the characteristic functions $\widehat{\nu}_j$ also satisfy Equation (2.2).

Let b_X be the subgroup of X consisting of all compact elements of X, and C_Y be the connected component of zero of the group $Y = X^*$. By the structure theorem for locally compact connected abelian groups, $C_Y = A \times B$, where $A \cong \mathbb{R}^m$ and B is a compact connected abelian group. It is easy to see that the restriction of any automorphism $d \in \mathrm{Aut}(Y)$ to the subgroup B is a topological automorphism of the subgroup B. Hence we can consider the restriction of Equation (2.2) for the characteristic functions $\widehat{\nu}_j$ to the subgroup B. Since B is a second countable compact connected abelian group, by Lemma 2.3 all characteristic functions $\widehat{\nu}_j$ are equal to 1 on B. Therefore $\sigma(\nu_j) \subseteq A(X, B) = \mathbb{R}^m \times b_X$. Since μ_j is a divisor of ν_j, the distributions μ_j can be replaced by their shifts μ'_j in such a way that all the supports $\sigma(\mu'_j) \subseteq \mathbb{R}^m \times b_X$. Since the restriction of any automorphism $d \in \mathrm{Aut}(Y)$ to the subgroup B is an automorphism of the subgroup B, and $A(X, B) = \mathbb{R}^m \times b_X$, the restriction of any automorphism $\delta \in \mathrm{Aut}(X)$ to the subgroup $\mathbb{R}^m \times b_X$ is an automorphism of the subgroup $\mathbb{R}^m \times b_X$. From what has been said, it follows that we can prove Theorem 1.4 assuming that D in (1.1) is a torsion group.

Put $G = K \times D$ and $H = G^*$. Then $Y \cong \mathbb{R}^m \times H$. To avoid introducing new notation, we will assume that $Y = \mathbb{R}^m \times H$. Denote elements of Y by y or (s, h), where $s \in \mathbb{R}^m$ and $h \in H$. Consider the character groups $F = K^*$ and $S = D^*$. Since K is a compact totally disconnected group, F is a discrete torsion group. Since D is a discrete torsion group, S is a compact totally disconnected group. Hence the group $H \cong F \times S$ is totally disconnected and consists of compact elements. Since H is a totally disconnected group, $\mathbb{R}^m$ is the connected component of zero of the group Y. Hence the restriction of any automorphism $d \in \mathrm{Aut}(Y)$ to the subgroup $\mathbb{R}^m$ is a topological automorphism of the subgroup $\mathbb{R}^m$. Denote this restriction by $d_{\mathbb{R}^m}$. Since H consists of all compact elements of the group Y, the restriction of any automorphism $d \in \mathrm{Aut}(Y)$ to the subgroup H is also a topological automorphism of the subgroup H. Denote this restriction by d_H. Hence each automorphism $d \in \mathrm{Aut}(Y)$ can be written in the form $d(s, h) = (d_{\mathbb{R}^m} s, d_H h)$.

By Lemma 2.2, the independence of the linear forms L_1 and L_2 is equivalent to the fact that the characteristic functions $\widehat{\mu}_j$ satisfy Equation (2.2), which in our notation may be written in the form

$$\begin{aligned} &\widehat{\mu}_1(s + s', h + h')\widehat{\mu}_2(s + a_{\mathbb{R}^m} s', h + a_H h') \\ &\quad = \widehat{\mu}_1(s, h)\widehat{\mu}_2(s, h)\widehat{\mu}_1(s', h')\widehat{\mu}_2(a_{\mathbb{R}^m} s', a_H h') \end{aligned} \tag{2.9}$$

for all $(s, h), (s', h') \in Y$. Putting $s = s' = 0$ in (2.9), we get the equation

$$\widehat{\mu}_1(0, h + h')\widehat{\mu}_2(0, h + a_H h') = \widehat{\mu}_1(0, h)\widehat{\mu}_2(0, h)\widehat{\mu}_1(0, h')\widehat{\mu}_2(0, a_H h') \tag{2.10}$$

for all $h, h' \in H$. We solve Equation (2.10). For this purpose we consider the distributions $\nu_j = \mu_j * \bar{\mu}_j$ and note that the restriction to H of the characteristic functions $\widehat{\nu}_j$ also satisfy Equation (2.10).

It is easy to see that a compact totally disconnected abelian group K has the property that for each prime p the factor group $K/K^{(p)}$ is finite if and only if its character group $F = K^*$ satisfies condition (i) of Lemma 2.4. It follows from (1.1) that the conditions of Lemma 2.4 are fulfilled for the group H. Let $h_0 \in H$. By Lemma 2.4, there exists a compact subgroup Q of H such that $h_0 \in Q$ and $a(Q) = Q$.

Consider the restriction of Equation (2.10) for the functions $\widehat{\nu}_j$ to Q. We may do so because $a_H(Q) = Q$. It follows from Lemma 2.5 that

$$\widehat{\nu}_1(0, h) = \widehat{\nu}_2(0, h) \quad \forall h \in Q,$$

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and the characteristic functions $\widehat{\nu}_j(0, \cdot)$ take only values 1 and 0 on Q. This implies that

$$\widehat{\nu}_1(0, h) = \widehat{\nu}_2(0, h) \quad \forall h \in H,$$

and the characteristic functions $\widehat{\nu}_j(0, \cdot)$ take only values 1 and 0 on H. Put

$$E = \{h \in H : \widehat{\nu}_1(0, h) = \widehat{\nu}_2(0, h) = 1\}.$$

We have

$$\widehat{\nu}_1(0, h) = \widehat{\nu}_2(0, h) = \begin{cases} 1 & \text{if } h \in E \\ 0 & \text{if } h \notin E. \end{cases} \tag{2.11}$$

Note that the set where a characteristic function is equal to 1 is a subgroup, hence E is an open subgroup of H. Put $M = A(G, E)$. Then M is a compact subgroup of G. It follows from the uniqueness theorem for characteristic functions, (2.1) and (2.11) that

$$\widehat{\nu}_1(0, h) = \widehat{\nu}_2(0, h) = \widehat{m}_M(h) \quad \forall h \in H. \tag{2.12}$$

Taking (2.1) and Lemma 2.1 into account, we can easily conclude from (2.12) that the characteristic functions $\widehat{\mu}_1(0, \cdot)$ and $\widehat{\mu}_2(0, \cdot)$ have the form

$$\widehat{\mu}_1(0, h) = \widehat{m}_M(h)(g_1, h), \quad \widehat{\mu}_2(0, h) = \widehat{m}_M(h)(g_2, h) \quad \forall h \in H, \tag{2.13}$$

where $g_1, g_2 \in G$. Note that $M^* \cong H/E$. Replacing the distributions μ_j with their shifts, we will assume that $g_1 = g_2 = 0$ in (2.13). Then

$$\widehat{\mu}_1(0, h) = \widehat{\mu}_2(0, h) = \begin{cases} 1 & \text{if } h \in E \\ 0 & \text{if } h \notin E. \end{cases} \tag{2.14}$$

From (2.14), the characteristic functions $\widehat{\mu}_1(\cdot, \cdot)$ and $\widehat{\mu}_2(\cdot, \cdot)$ are E-invariant. Lemmas 2.4 and 2.5 imply that $a_H(E) = E$, and we can pass from Equation (2.9) on the group Y to the induced equation on the factor group $Y/E \cong \mathbb{R}^m \times (H/E)$, putting $f([y]) = \widehat{\mu}_1(y)$, $g([y]) = \widehat{\mu}_2(y)$ and $\widehat{a}[y] = [ay]$, for all $y \in Y$. Note that the homomorphism $\widehat{a}$ on the factor group Y/E induced by the automorphism a is a topological automorphism of the factor group Y/E. Put $L = H/E$. Obviously, the restriction of the automorphism $\widehat{a} \in \mathrm{Aut}(Y/E)$ to L is a topological automorphism of L. Denote this restriction by $\widehat{a}_L$, and denote elements of the group $Y/E \cong \mathbb{R}^m \times L$ by (s, l), where $s \in \mathbb{R}^m$, and $l \in L$. In this notation, $\widehat{a}(s, l) = (a_{\mathbb{R}^m} s, \widehat{a}_L l)$. The equation induced by Equation (2.9) is of the form

$$f(s + s', l + l')g(s + a_{\mathbb{R}^m} s', l + \widehat{a}_L l') = f(s, l)g(s, l)f(s', l')g(a_{\mathbb{R}^m} s', \widehat{a}_L l') \tag{2.15}$$

for all $(s, l), (s', l') \in \mathbb{R}^m \times L$. It is easy to see that, for solutions of Equation (2.15), we have

$$\{l \in L : f(0, l) = 1\} = \{l \in L : g(0, l) = 1\} = \{0\}. \tag{2.16}$$

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Consider the subgroup

$$V = \{l \in L : \widehat{a}l = l\}.$$

It follows from (2.10) that the restrictions of the characteristic functions $\widehat{f}(0, \cdot)$ and $\widehat{g}(0, \cdot)$ to V satisfy the equation

$$f(0, l + l')g(0, l + l') = f(0, l)g(0, l)f(0, l')g(0, l') \quad \forall l, l' \in V. \tag{2.17}$$

Putting $l' = -l$ in (2.17) and considering (2.14), we deduce that $f(0, l) = g(0, l) = 1$ for all $l \in V$. Then (2.16) implies that $V = \{0\}$, that is,

$$\mathrm{Ker}(I - \widehat{a}_L) = \{0\}. \tag{2.18}$$

Since M is a compact subgroup of G, and G is totally disconnected, M is also a compact totally disconnected group. Hence L is a discrete torsion group. It is easy to see that M, as a compact subgroup of G, also satisfies the condition that for each prime p the factor group $M/M^{(p)}$ is finite. Hence the group L satisfies condition (i) of Lemma 2.4. It is not difficult to verify that each monomorphism of a discrete countable torsion abelian group satisfying condition (i) of Lemma 2.4 is an automorphism. So, it follows from (2.18) that $I - \widehat{a} \in \mathrm{Aut}(L)$.

Thus, for solutions of Equation (2.15), we have

$$f(0, l) = g(0, l) = \begin{cases} 1 & \text{if } l = 0 \\ 0 & \text{if } l \neq 0 \end{cases} \tag{2.19}$$

and

$$I - \widehat{a}_L \in \mathrm{Aut}(L). \tag{2.20}$$

Put $l = l' = 0$ in (2.15). Then by the Ghurye–Olkin theorem [11],

$$\begin{aligned} f(s, 0) &= \exp\{-\langle A_1 s, s\rangle + i\langle t_1, s\rangle\} \quad \forall s \in \mathbb{R}^m, \\ g(s, 0) &= \exp\{-\langle A_2 s, s\rangle + i\langle t_2, s\rangle\} \quad \forall s \in \mathbb{R}^m, \end{aligned} \tag{2.21}$$

where the A_j are symmetric positive semidefinite matrices, $t_j \in \mathbb{R}^m$, and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^m$.

Putting $s = 0$ and $l' = 0$ in (2.15), we see that

$$f(s', l)g(a_{\mathbb{R}^m} s', l) = f(0, l)g(0, l)f(s', 0)g(a_{\mathbb{R}^m} s', 0) \tag{2.22}$$

for all $s' \in \mathbb{R}^m$ and all $l \in L$. It follows from (2.19) that the right-hand side of Equation (2.22) vanishes when $l \neq 0$, so that, if $l \neq 0$, then

$$f(s', l)g(a_{\mathbb{R}^m} s', l) \equiv 0 \quad \forall s' \in \mathbb{R}^m.$$

Since $f(s, 0)$ and $g(s, 0)$ are entire functions of s, for any fixed $h \in H$ the functions $f(s, h)$ and $g(s, h)$ are also entire functions of s [4, Ch. 2]. Hence, when $l \neq 0$, either $f(s, l) \equiv 0$ or $g(s, l) \equiv 0$ for all $s \in \mathbb{R}^m$.

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Take an arbitrary nonzero element $l_0 \in L$, and find l and l' solving the system of equations

$$\begin{cases} l + l' = 0 \\ l + \widehat{a}_L l' = l_0. \end{cases}$$

It follows from (2.20) that this system has a unique solution. Substituting these l and l' into Equation (2.15) and taking (2.21) into account, we infer that $g(s, l_0) \equiv 0$ for all $s \in \mathbb{R}^m$. So we have obtained the representation

$$g(s, l) = \begin{cases} \exp\{-\langle A_2 s, s\rangle + i\langle t_2, s\rangle\} & \text{if } s \in \mathbb{R}^m \text{ and } l = 0 \\ 0 & \text{if } s \in \mathbb{R}^m \text{ and } l \neq 0. \end{cases} \tag{2.23}$$

Reasoning similarly, we obtain

$$f(s, l) = \begin{cases} \exp\{-\langle A_1 s, s\rangle + i\langle t_1, s\rangle\} & \text{if } s \in \mathbb{R}^m \text{ and } l = 0 \\ 0 & \text{if } s \in \mathbb{R}^m \text{ and } l \neq 0. \end{cases} \tag{2.24}$$

Returning from the functions f and g to the characteristic functions $\widehat{\mu}_j$, we conclude from (2.23) and (2.24) that $\mu_j \in \Gamma(X) * I(X)$. $\square$

REMARK. It is interesting to note that our proof of Theorem 1.4 uses the Ghurye–Olkin theorem [11] and Theorem 1.3 (Lemma 2.5), but does not use Theorem 1.2.

REMARK. Let Δ_p be the group of p-adic integers. The group Δ_p is compact and totally disconnected. It is not difficult to prove that a compact totally disconnected abelian group is such that the factor group $X/X^{(p)}$ is finite for each prime p if and only if X is topologically isomorphic to the direct product

$$\prod_{p \in \mathcal{P}} (\Delta_p^{n_p} \times G_p), \tag{2.25}$$

where each n_p is a nonnegative integer and G_p is a finite p-prime group, and possibly $G_p = \{0\}$.

According to Theorem 1.2, in the class of compact totally disconnected abelian groups X, the condition that the factor group $X/X^{(p)}$ is finite for each prime p is not only sufficient but also necessary for the Skitovich–Darmois theorem to be valid for two independent random variables.

It is interesting to remark in this connection that the groups of the form (1.1) are not the only groups for which the Skitovich–Darmois theorem is valid for two independent random variables. We will construct in Section 3 an example of a locally compact abelian group X which is not topologically isomorphic to a group of the form (1.1) and such that the independence of the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$, where $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, implies that $\mu_1, \mu_2 \in \Gamma(X) * I(X)$.

3. An example

In this section, we give an example of a locally compact abelian group X which is not topologically isomorphic to a group of the form (1.1) and for which the Skitovich–Darmois theorem holds for two independent random variables. We need some additional notation. Denote by $\mathbb{N}$ the set of natural numbers, and by p_n the nth prime number. Let p be a prime number, and define

$$\mathbb{Z}(p^\infty) = \{k/p^n : k = 0, 1, \ldots, p^n - 1, n = 0, 1, \ldots\}.$$

We define the operation in $\mathbb{Z}(p^\infty)$ as addition modulo 1, then $\mathbb{Z}(p^\infty)$ is an abelian group, which we endow with the discrete topology. Obviously, this group is isomorphic to the multiplicative group of p^nth roots of unity, where n goes through the nonnegative integers, considered with the discrete topology. Denote by $\mathbb{Z}(p)$ the subgroup $\{k/p : k = 0, 1, \ldots, p - 1\}$ of $\mathbb{Z}(p^\infty)$. Obviously, the group $\mathbb{Z}(p)$ is isomorphic to the multiplicative group of pth roots of unity. We note that the groups $\mathbb{Z}(p^\infty)$ and Δ_p are the character groups of one another.

Consider the group $\prod_{n=1}^\infty \mathbb{Z}(p_n^\infty)$ and let $Y = \{y = (y_1, y_2, \ldots, y_n, \ldots)\}$ be a subgroup of $\prod_{n=1}^\infty \mathbb{Z}(p_n^\infty)$ such that $y_n \in \mathbb{Z}(p_n)$ for all but finitely many numbers n. Put

$$L = \prod_{n=1}^\infty \mathbb{Z}(p_n).$$

Then L is a subgroup of Y. Consider the group L with the product topology. Obviously, L is a second countable compact abelian group. The subgroup L can be regarded as an open subgroup of Y. Then Y is transformed into a topological group. Note that Y is a second countable locally compact abelian group, which is nondiscrete, noncompact and totally disconnected. It is easy to see that

$$Y/L \cong \prod_{n=1}^{\infty}{}^{*}\, \mathbb{Z}(p_n^\infty).$$

Put $X = Y^*$ and verify that the group X is not topologically isomorphic to a group of the form (1.1). To this end, we note that $Y^{(n)} = Y$ for any $n \in \mathbb{N}$. This implies that $X_{(n)} = \{0\}$ for any $n \in \mathbb{N}$, that is, X is a torsion-free group.

Assume that the group X is topologically isomorphic to a group of the form (1.1). Then, obviously, $m = 0$. Moreover, because the group Y is nondiscrete and noncompact, the group X is also nondiscrete and noncompact. Hence $K \neq \{0\}$ and $D \neq \{0\}$ in (1.1). This implies that the group Y has a compact subgroup M as a direct factor, which is isomorphic to D^*. Since the group Y is totally disconnected, the subgroup M is also totally disconnected. It follows from this that D is a torsion group, in contrast to the fact that X is a torsion-free group. Hence the group X is not topologically isomorphic to a group of the form (1.1).

Let ξ_1 and ξ_2 be independent random variables taking values in X and with distributions μ_1 and μ_2. Let $\alpha_j, \beta_j \in \mathrm{Aut}(X)$. We verify that the independence of $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ implies that $\mu_1, \mu_2 \in I(X)$.

Reasoning as in the proof of Theorem 1.4, we can assume that $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha\xi_2$, where $\alpha \in \mathrm{Aut}(X)$ and $\widehat{\mu}_j \geq 0$ when $j = 1, 2$. Put $f = \widehat{\mu}_1$, $g = \widehat{\mu}_2$, and $a = \widetilde{\alpha}$. By Lemma 2.2, the independence of the linear forms L_1 and L_2 is equivalent to the characteristic functions $\widehat{\mu}_j$ satisfying (2.2), which in our notation takes the form

$$f(u+v)g(u+av) = f(u)g(u)f(v)g(av) \quad \forall u, v \in Y. \tag{3.1}$$

Let $d \in \mathrm{Aut}(Y)$ and $y = (y_1, y_2, \ldots, y_n, \ldots) \in L$.

Put $y^{(n)} = (y_1, y_2, \ldots, y_n, 0, 0, \ldots)$. Then $y^{(n)} \in L$ and $y^{(n)} \to y$. It follows from this that $dy^{(n)} \to dy$. Since, obviously, $dy^{(n)} \in L$, we conclude that $dy \in L$. Hence $d(L) \subseteq L$. This implies that the restriction of any automorphism $d \in \mathrm{Aut}(Y)$ to L is a topological automorphism of the subgroup L. In particular, the restriction of a to L is a topological automorphism of the subgroup L, and we can consider the restriction of Equation (3.1) to L. Put $G = L^*$. It is obvious that

$$G \cong \prod_{n=1}^{\infty}{}^{*}\, \mathbb{Z}(p_n).$$

Taking (2.1) into account, it follows from Lemma 2.3 that

$$f(y) = g(y) = m_K(y) \quad \forall y \in L,$$

where K is a finite subgroup of the group G. It is easy to see that K is of the form

$$K \cong \prod_{j=1}^{l} \mathbb{Z}(p_{n_j}).$$

Set $S = \{n_1, n_2, \ldots, n_l\}$, and $E = A(L, K)$. It is obvious that

$$E = \prod_{n \in \mathbb{N}/S} \mathbb{Z}(p_n).$$

Since $f(y) = g(y) = 1$ for all $y \in E$, the functions f and g are E-invariant. Moreover, $a(E) = E$ by Lemma 2.5. Therefore we can pass from Equation (3.1) on the group Y to the induced equation on the factor group Y/E, putting $\widehat{f}([y]) = f(y)$, $\widehat{g}([y]) = g(y)$ and $\widehat{a}[y] = ay$, for all $y \in Y$. Since $a(E) = E$, we have $\widehat{a} \in \mathrm{Aut}(Y/E)$. Thus, the functions $\widehat{f}([\cdot])$ and $\widehat{g}([\cdot])$ satisfy the equation

$$\widehat{f}([u]+[v])\widehat{g}([u]+\widehat{a}\,[v]) = \widehat{f}([u])\widehat{g}([u])\widehat{f}([v])\widehat{g}(\widehat{a}\,[v]) \tag{3.2}$$

for all $[u], [v] \in Y/E$. It is not difficult to check that

$$Y/E \cong \prod_{n=1}^{\infty}{}^{*}\, \mathbb{Z}(p_n^{\infty}).$$

https://doi.org/10.1017/S1446788710000224 Published online by Cambridge University Press

This implies that $Y/E \cong F^*$, where

$$F = \prod_{n=1}^{\infty} \Delta_{p_n}.$$

Since F is a compact totally disconnected abelian group such that for each prime p the factor group $F/F^{(p)}$ is finite, the required statement follows from (3.2), Lemma 2.2, and Theorem 1.2.

4. Comments

Let X_1 and X_2 be second countable locally compact abelian groups for which the Skitovich–Darmois theorem holds for two independent random variables (see Problem 1.1). A natural question arises: does the Skitovich–Darmois theorem hold for two independent random variables for the group $X = X_1 \times X_2$? Unfortunately, we do not know the answer to this question, but note that a positive answer would imply Theorem 1.4. In connection with this question, consider the Kac–Bernstein theorem instead of the Skitovich–Darmois theorem.

Assume that ξ_1 and ξ_2 are independent random variables taking values in a second countable locally compact abelian group X and with distributions μ_1 and μ_2. We will say that the Kac–Bernstein theorem holds on X if the independence of $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ implies that $\mu_1, \mu_2 \in \Gamma(X) * I(X)$. It is known (see [7, Theorem 7.10]) that the Kac–Bernstein theorem holds on a group X if and only if the connected component of zero of X contains no elements of order two. Therefore, if the Kac–Bernstein theorem holds on groups X_j when $j = 1, 2$, then it holds on the group $X = X_1 \times X_2$.

But if we assume in the Kac–Bernstein theorem that independent random variables ξ_1 and ξ_2 are identically distributed, then the Kac–Bernstein theorem holds on X if and only if the connected component of zero of X contains no more that one element of order two [7, Theorem 9.9]. In particular, the Kac–Bernstein theorem holds for identically distributed random variables on the circle group $\mathbb{T}$, but it fails on the two-dimensional torus $\mathbb{T}^2$.

Acknowledgement

We thank the referee for reading the manuscript carefully and offering useful remarks.

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