Eighth Meeting, Friday, June 14th, 1895.

JOHN M'COWAN, Esq., M.A., D.Sc., President, in the Chair.

A Summary of the Theory of the Refraction of their approximately Axial Pencils through a Series of Media bounded by coaxial Spherical Surfaces, with Applications to a Photographic Triplet, &c.

By PROFESSOR CHRYSTAL.

[*The Paper will be published in the next Volume.*]

On a Diophantine Equation.

By R. F. DAVIS, M.A.

In the consideration of Question 12612 appearing in the *Educational Times* for January of this year, proposed by the Rev. Dr. Haughton, F.R.S., of Trinity College, Dublin, the following Diophantine Equation suggests itself:

What values of x make $8x^3 - 8x + 16 = \square$?

Since it may be written $8x(x^2 - 1) + 16 = \square$ it is obvious that $x = 0, \pm 1$ are solutions. Also that $x = 2$ is a solution. Moreover $x = -\frac{3}{2}$ when substituted gives $-27 + 12 + 16 = 1$ and is therefore a solution,—marking approximately a limit to the negative root.

I. Put $8x^3 - 8x + 16 = (px^2 + x - 4)^2$; then after reduction and division by x^2, we have

$$p^2x^2 - 2x(4 - p) + 1 - 8p = 0 \quad \ldots \quad \ldots \quad \ldots \quad (A)$$

https://doi.org/10.1017/S0013091500031643 Published online by Cambridge University Press

It will be found that the roots of this equation are real and rational when $8p^3 - 8p + 16 = \square$ *which is the same Diophantine Equation as that with which we started.*

Hence the values of x obtained by experiment may be used for p in the equation (A) with the certainty of obtaining one or more fresh solutions.

Thus put $p = 0$ and we get $\quad - 8x + 1 = 0 \quad x = \frac{1}{8}$

„ „ $p = 1$ „ „ „ $x^2 - 6x + 7 = 0 \quad x = -1$ or 7

„ „ $p = 1$ „ „ „ $x^2 - 10x + 9 = 0 \quad x = +1$ or 9

„ „ $p = 2$ „ „ „ $4x^2 - 4x + 15 = 0 \quad x = \frac{5}{2}, \frac{3}{2},$

all depending on the fact that if one root of a quadratic equation be real and rational, so is the other root.

II. The equation (A) may be written

$$(px + 1)^2 = 8(x + p)$$
$$= 16a^2, \quad \text{say};$$

whence $\quad px + 1 = 4a, \quad$ and $\quad x + p = 2a^2.$

Thus
$$x(2a^2 - x) - 4a + 1 = 0$$
$$x^2 - 2a^2x + 4a - 1 = 0 \qquad \ldots \qquad \ldots \qquad (B)$$

and the roots of this equation are real and rational when

$$a^4 - 4a + 1 = \square.$$

Any value of x satisfying the original problem will, if substituted in (B), give two real and rational values of a. If one of these values of a be substituted in (B) and the equation then solved as regards x we get the original value of x and another value.

Thus we are led (somewhat blindly it is true) to an interminable series of solutions: such as

$$0,\ 1,\ -1,\ 2,\ 7,\ 9,\ 15,\ 496$$

$$\frac{5}{2},\ -\frac{3}{2},\ \frac{1}{8},\ \frac{26}{9},\ -\frac{7}{9},\ \frac{17}{25},\ -\frac{38}{25},\ \frac{39}{49},\ -\frac{55}{49},\ \frac{71}{81},\ \text{etc.}$$

https://doi.org/10.1017/S0013091500031643 Published online by Cambridge University Press