1
Introduction

This book introduces and studies C^∞-*algebraic geometry with corners*. It extends the existing theory of C^∞-algebraic geometry [23, 42, 49, 52, 75], a version of Algebraic Geometry that generalizes smooth manifolds to a huge class of singular spaces. C^∞-algebraic geometry with corners is a generalization based on manifolds with corners rather than on manifolds. It is related to log geometry [78], as log structures are a kind of boundary in Algebraic Geometry.

The motivation for this book is two-fold. Firstly, it presents an introduction to C^∞-algebraic geometry with corners, along with many foundational results, in one self-contained volume. Secondly, it provides the foundations needed to extend current work on Derived Differential Geometry [6, 7, 8, 10, 11, 12, 13, 43, 44, 62, 88, 89, 87] to include derived C^∞-schemes and C^∞-stacks with corners, and hence derived manifolds and derived orbifolds with corners.

Derived orbifolds with corners have important applications in Symplectic Geometry, for example, in the most general versions of Lagrangian Floer cohomology [28, 29], Fukaya categories [3, 83], and Symplectic Field Theory [24]. In these areas one studies moduli spaces $\bar{\mathcal{M}}$ of J-holomorphic curves, which should be derived orbifolds with corners. (Related structures are the 'Kuranishi spaces with corners' of Fukaya–Oh–Ohta–Ono [29, 30] and 'polyfolds with corners' of Hofer–Wysocki–Zehnder [37, 38], but derived orbifolds with corners have better functoriality and are, we think, more beautiful.)

C^∞-*algebraic geometry* was originally suggested by William Lawvere in the late 1960's [58], and our primary reference is the second author's monograph [49]. In C^∞-algebraic geometry, rings are replaced with C^∞-*rings*, such as the space $C^\infty(X)$ of smooth functions $c : X \to \mathbb{R}$ of a manifold X. C^∞-rings are $\mathbb{R}$-algebras but with a far richer structure – for each smooth function $f : \mathbb{R}^n \to \mathbb{R}$ there is an n-fold operation $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ acting by $\Phi_f : c_1, \ldots, c_n \mapsto f(c_1, \ldots, c_n)$, satisfying many natural identities.

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The basic objects of C^∞-algebraic geometry are C^∞*-schemes*, which form a category $\mathbf{C^\infty Sch}$. They are special examples of (*local*) C^∞*-ringed spaces* $\underline{X} = (X, \mathcal{O}_X)$, a topological space X with a sheaf of C^∞-rings $\mathcal{O}_X$. These form categories $\mathbf{LC^\infty RS} \subset \mathbf{C^\infty RS}$. As in Algebraic Geometry, there is a *spectrum functor* $\operatorname{Spec} : \mathbf{C^\infty Rings}^{\rm op} \to \mathbf{LC^\infty RS}$, and a C^∞-scheme is a local C^∞-ringed space $\underline{X}$ covered by open subspaces $\underline{U} \subseteq \underline{X}$ with $\underline{U} \cong \operatorname{Spec} \mathfrak{C}$ for some $\mathfrak{C} \in \mathbf{C^\infty Rings}$. Any smooth manifold X determines a C^∞-scheme $\underline{X} = \operatorname{Spec} C^\infty(X)$, which is X with its sheaf of smooth functions $\mathcal{O}_X$, giving an embedding $\mathbf{Man} \subset \mathbf{C^\infty Sch}$ as a full subcategory.

C^∞-schemes are far more general than manifolds, and $\mathbf{C^\infty Sch}$ contains many singular or infinite-dimensional objects. In addition, the category of C^∞-schemes addresses several shortcomings of the category of smooth manifolds: it is Cartesian closed, and has all finite limits and directed colimits, unlike the category of manifolds. Thus all fibre products (not just transverse ones) exist in $\mathbf{C^\infty Sch}$, and many spaces constructed from fibre products of manifolds can be studied. These good properties are some of the reasons C^∞-algebraic geometry has been applied in the foundations of Synthetic Differential Geometry [23, 52, 73, 74, 75], and Derived Differential Geometry.

Manifolds with corners X are a generalization of manifolds locally modelled on $[0,\infty)^k \times \mathbb{R}^{m-k}$ rather than on $\mathbb{R}^m$. They occur in many places in Differential Geometry. There are several candidates for morphisms of manifolds with corners, but for reasons explained later we choose the 'b-maps' of Melrose [69, 70, 71, 72], which we just call 'smooth maps', to be the morphisms in the category $\mathbf{Man^c}$ of manifolds with corners.

A manifold with corners X of dimension n has a *boundary* ∂X, a manifold with corners of dimension $n-1$, with a (not necessarily injective) inclusion map $\Pi_X : \partial X \to X$. The *interior* is $X^\circ = X \setminus \Pi_X(\partial X)$, an ordinary n-manifold. A smooth map $f : X \to Y$ is called *interior* if $f(X^\circ) \subset Y^\circ$. We write $\mathbf{Man^c_{in}} \subset \mathbf{Man^c}$ for the subcategory with morphisms interior maps.

The symmetric group S_k acts freely on $\partial^k X$, and the quotient $\partial^k X/S_k$ is the k*-corners* $C_k(X)$, a manifold with corners of dimension $n-k$, with $C_0(X) = X$ and $C_1(X) = \partial X$. For example, the square $[0,1]^2$ has

$$C_0\big([0,1]^2\big) = [0,1]^2, \qquad C_2\big([0,1]^2\big) = \{0,1\}^2,$$
$$C_1\big([0,1]^2\big) = \partial\big([0,1]^2\big) = \big(\{0,1\} \times [0,1]\big) \amalg \big([0,1] \times \{0,1\}\big).$$

The *corners* of X is $C(X) = \coprod_{k=0}^n C_k(X)$, a *manifold with corners of mixed dimension*, which form a category $\mathbf{\check{M}an^c}$. If $f : X \to Y$ is a smooth map of manifolds with corners (possibly of mixed dimension), we can define a natural interior map $C(f) : C(X) \to C(Y)$, which need not map $C_k(X) \to$

$C_k(Y)$. This defines a *corner functor* $C : \check{\mathbf{Man}}^{\mathbf{c}} \to \check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{in}}$, which is right adjoint to the inclusion $\mathrm{inc} : \check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{in}} \hookrightarrow \check{\mathbf{Man}}^{\mathbf{c}}$. This means that not only do boundaries ∂X and corners $C_k(X)$ combine into a functorial object $C(X)$, but they are canonically determined just by the two notions of smooth map and interior map of manifolds with corners.

Manifolds with g-corners $\mathbf{Man}^{\mathbf{gc}}$, as in [46] and §3.3, are a generalization of manifolds with corners whose local models X_P are more general than $[0,\infty)^k \times \mathbb{R}^{m-k}$, and depend on a weakly toric monoid P. They have nicer properties than manifolds with corners in some ways – for example, in the existence of b-transverse fibre products – and come up in some applications; for example, moduli spaces of 'quilts' in Symplectic Geometry [67, 68] may be manifolds with g-corners, but not manifolds with corners.

To extend C^∞-algebraic geometry to include manifolds with (g-)corners, we introduce the notion of a *C^∞-ring with corners* $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$, which is a pair consisting of a C^∞-ring $\mathfrak{C}$ and a commutative monoid $\mathfrak{C}_{\mathrm{ex}}$ with many intertwining relationships. If X is a manifold with corners, the corresponding C^∞-ring with corners is $\boldsymbol{C}^\infty(X) = (C^\infty(X), \mathrm{Ex}(X))$, where $C^\infty(X)$ is the C^∞-ring of smooth maps $X \to \mathbb{R}$, and $\mathrm{Ex}(X)$ is the monoid of smooth ('exterior') maps $X \to [0,\infty)$, with the monoid operation being multiplication. The monoid holds information about the corners structure of X.

The definition of C^∞-scheme with corners then follows those of schemes in Algebraic Geometry [36, §II], or C^∞-schemes. We introduce categories $\mathbf{LC^\infty RS^c} \subset \mathbf{C^\infty RS^c}$ of (*local*) *C^∞-ringed spaces with corners* $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$, which are topological spaces X with sheaves of C^∞-ringed spaces with corners $\boldsymbol{\mathcal{O}}_X$. We construct a *spectrum functor* $\mathrm{Spec^c} : (\mathbf{C^\infty Rings^c})^{\mathrm{op}} \to \mathbf{LC^\infty RS^c}$, which is left adjoint to the global sections functor $\Gamma^{\mathrm{c}} : \mathbf{LC^\infty RS^c} \to (\mathbf{C^\infty Rings^c})^{\mathrm{op}}$. A *$C^\infty$-scheme with corners* $\boldsymbol{X}$ is then a local C^∞-ringed space with corners which can be covered by open $\boldsymbol{U} \subseteq \boldsymbol{X}$ with $\boldsymbol{U} \cong \mathrm{Spec^c}\,\boldsymbol{\mathfrak{C}}$ for some C^∞-ring with corners $\boldsymbol{\mathfrak{C}}$.

We define many interesting subcategories of $\mathbf{C^\infty Sch^c}$, as in Figure 1.1, which reproduces a diagram (Figure 5.1) of subcategories of the category of C^∞-schemes with corners $\mathbf{C^\infty Sch^c}$ from §5.6. For example, *firm* C^∞-schemes with corners $\mathbf{C^\infty Sch^c_{fi}} \subset \mathbf{C^\infty Sch^c}$ have only finitely many boundary faces at each point. We define a subcategory $\mathbf{C^\infty Sch^c_{in}} \subset \mathbf{C^\infty Sch^c}$ of *interior* C^∞-schemes with corners, with *interior morphisms* corresponding to interior maps of manifolds with (g-)corners. We study categorical properties of $\mathbf{C^\infty Sch^c}$ and its subcategories. For example, fibre products and all finite limits exist in $\mathbf{C^\infty Sch^c_{fi}}$, and in most of the other subcategories in Figure 1.1.

We construct a *corner functor* $C : \mathbf{C^\infty Sch^c} \to \mathbf{C^\infty Sch^c_{in}}$, which is right adjoint to $\mathrm{inc} : \mathbf{C^\infty Sch^c_{in}} \hookrightarrow \mathbf{C^\infty Sch^c}$. If $\boldsymbol{X}$ is a firm C^∞-scheme with

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$$
\begin{tikzcd}
& \mathbf{Man^c_{in}} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathbf{Man^{gc}_{in}} \arrow[dl] \arrow[dd] & & & & & & & \\
\mathbf{Man^c} \arrow[rr] \arrow[dd] & & \mathbf{Man^{gc}} \arrow[dd] & & & & & & \mathbf{C^\infty Sch^c_{fg}} \arrow[rr] & & \mathbf{C^\infty Sch^c_{fp}} \arrow[dd] \\
& \mathbf{C^\infty Sch^c_{si}} \arrow[rr] \arrow[dl] & & \mathbf{C^\infty Sch^c_{to}} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathbf{C^\infty Sch^c_{fi,tf}} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathbf{C^\infty Sch^c_{fi,\mathbb{Z}}} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathbf{C^\infty Sch^c_{fi,in}} \arrow[dl] \arrow[dd] & \\
\mathbf{C^\infty Sch^c_{si,ex}} \arrow[rr] & & \mathbf{C^\infty Sch^c_{to,ex}} \arrow[rr] \arrow[dd] & & \mathbf{C^\infty Sch^c_{fi,tf,ex}} \arrow[rr] \arrow[dd] & & \mathbf{C^\infty Sch^c_{fi,\mathbb{Z},ex}} \arrow[rr] \arrow[dd] & & \mathbf{C^\infty Sch^c_{fi,in,ex}} \arrow[rr] \arrow[dd] & & \mathbf{C^\infty Sch^c_{fi}} \arrow[dd] \\
& & & \mathbf{C^\infty Sch^c_{sa}} \arrow[rr] \arrow[dl] & & \mathbf{C^\infty Sch^c_{tf}} \arrow[rr] \arrow[dl] & & \mathbf{C^\infty Sch^c_{\mathbb{Z}}} \arrow[rr] \arrow[dl] & & \mathbf{C^\infty Sch^c_{in}} \arrow[dl] & \\
& & \mathbf{C^\infty Sch^c_{sa,ex}} \arrow[rr] & & \mathbf{C^\infty Sch^c_{tf,ex}} \arrow[rr] & & \mathbf{C^\infty Sch^c_{\mathbb{Z},ex}} \arrow[rr] & & \mathbf{C^\infty Sch^c_{in,ex}} \arrow[rr] & & \mathbf{C^\infty Sch^c}.
\end{tikzcd}
$$

Figure 1.1 Subcategories of C^∞-schemes with corners $\mathbf{C^\infty Sch^c}$

corners we may write $C(\boldsymbol{X}) = \coprod_{k=0}^{\infty} C_k(\boldsymbol{X})$, where $C_k(\boldsymbol{X})$ is the *k-corners* of $\boldsymbol{X}$, with $C_0(\boldsymbol{X}) \cong \boldsymbol{X}$, and we define the *boundary* $\partial\boldsymbol{X} = C_1(\boldsymbol{X})$.

There are full and faithful inclusion functors $\mathbf{Man^c}, \mathbf{Man^{gc}} \hookrightarrow \mathbf{C^\infty Sch^c_{fi}}$ and $\mathbf{Man^c_{in}}, \mathbf{Man^{gc}_{in}} \hookrightarrow \mathbf{C^\infty Sch^c_{fi,in}}$, which commute with corner functors, boundaries ∂X, and k-corners $C_k(X)$, and preserve (b-)transverse fibre products. Thus, manifolds with (g-)corners can be regarded as special examples of C^∞-schemes with corners, and C^∞-schemes with corners generalize manifolds with (g-)corners.

The geometry of 'things with corners' turns out to be surprisingly interesting and complicated, and has received relatively little attention – for example, our observation that corners arise as right adjoints to inclusions like $\mathbf{Man^c_{in}} \hookrightarrow \mathbf{Man^c}$ appears to be new. We hope our book will inspire further research.

How this book came to be written

Beginning from foundational work of Lurie [62, §4.5] and Spivak [87], the second author has since 2009 been working on a theory of Derived Differential Geometry [43, 44], which is the study of *derived manifolds* and *derived orbifolds*. Here 'derived' is in the sense of Derived Algebraic Geometry [61, 62, 90, 91, 92]. This was aimed at applications in Symplectic Geometry, and is further explained in §2.9 and §8.4, though we summarize below.

Fukaya, Oh, Ohta and Ono [28, 29, 30, 31] were developing theories of Gromov–Witten invariants, Lagrangian Floer cohomology, and Fukaya categories, involving 'Kuranishi spaces', a geometric structure they put on moduli spaces of J-holomorphic curves. At the time there were problems with the definition and theory of Kuranishi spaces. The second author's view was that

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Kuranishi spaces are actually derived orbifolds, and ideas from derived geometry were needed to understand them. Following [62, 87], he defined derived manifolds and derived orbifolds as special kinds of derived C^∞-schemes and derived C^∞-stacks.

For more advanced applications in Symplectic Geometry [28, 29], it was essential to have a theory of derived orbifolds *with corners*. To do this properly in the world of C^∞-algebraic geometry, it was necessary to go right back to the beginning, and introduce notions of C^∞-ring with corners and C^∞-scheme with corners, before defining derived C^∞-schemes and C^∞-stacks with corners.

The first attempt at this was the 2014 MSc thesis of Elana Kalashnikov [51], supervised by the second author, which defined notions of C^∞-rings with corners and C^∞-schemes with corners. Building on [51], the first author's 2019 PhD thesis [25] went into the subject in much more detail, again supervised by the second author.

One of the goals of the PhD was to find the best foundations for C^∞-algebraic geometry with corners, and its possible future applications, and the authors explored many options before settling on the definitions given here. This book is a rewritten and expanded version of [25]. It is designed in particular to provide the foundations of theories of 'derived C^∞-schemes with corners' and 'derived manifolds and orbifolds with corners' in the final version of the second author's book [44].

Why we chose these definitions

In Chapters 3 and 4 we have chosen particular definitions of the notions of smooth map of manifolds with corners, and of C^∞-ring with corners, which determine the course of the rest of the theory. The reader may wonder whether these choices could have been otherwise, leading to a different way of algebraizing 'things with corners', and a different theory of C^∞-algebraic geometry with corners. Our answer to this comes in two parts.

For the first part, knowing C^∞-algebraic geometry, the rough starting point for a theory of C^∞-algebraic geometry with corners is obvious. As in §2.2, a *categorical C^∞-ring* is a product-preserving functor $F : \mathbf{Euc} \to \mathbf{Sets}$, where $\mathbf{Euc} \subset \mathbf{Man}$ is the full subcategory of Euclidean spaces $\mathbb{R}^n$, and a C^∞-ring $\mathfrak{C}$ is an equivalent way of repackaging the data in F, with $\mathfrak{C} = F(\mathbb{R})$, and many operations on $\mathfrak{C}$. Here $\mathbf{Euc}$ is an Algebraic Theory [1], and a (categorical) C^∞-ring is an algebra over this Algebraic Theory.

Thus it is clear that a *categorical C^∞-ring with corners* should be a product-preserving functor $F : \mathbf{Euc^c} \to \mathbf{Sets}$, where $\mathbf{Euc^c} \subset \mathbf{Man^c}$ is the full

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subcategory of Euclidean corner spaces $\mathbb{R}^m_k = [0,\infty)^k \times \mathbb{R}^{m-k}$. Then a C^∞-*ring with corners* $\mathfrak{C} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ should be an equivalent way of repackaging F, with $\mathfrak{C} = F(\mathbb{R})$ and $\mathfrak{C}_{\rm ex} = F([0,\infty))$, and many operations on $\mathfrak{C}, \mathfrak{C}_{\rm ex}$.

There are possible choices in two places in this definition. Firstly, how should we define morphisms in $\mathbf{Man^c}$ and $\mathbf{Euc^c}$? (As we will see in §3.1, there are many different classes of 'smooth maps' between manifolds with corners.) And secondly, do we want to impose any additional conditions on the functors F?

For the first, in order for 'product-preserving' to make sense, the class of morphisms of manifolds with corners X must be closed under products and direct products. Also, for good properties of boundaries ∂X we want the inclusion $\Pi_X : \partial X \to X$ to be a morphism in $\mathbf{Man^c}$ (this excludes interior morphisms, for example). With these restrictions, there are only really three sensible choices for the morphisms in $\mathbf{Man^c}$, which are called 'weakly smooth', 'smooth' (or 'b-maps'), and 'strongly smooth' in §3.1, with

$$\{\text{weakly smooth}\} \subset \{\text{smooth}\} \subset \{\text{strongly smooth}\}.$$

We rejected 'weakly smooth' maps, as they would cause C^∞-schemes with corners not to have well-behaved notions of boundary and corners. We found that smooth maps led to a richer and more interesting theory than strongly smooth maps, and the latter did not work for all the applications we had in mind (e.g., it does not correctly model manifolds with g-corners in §3.3). Also the strongly smooth theory is strictly contained in the smooth theory. So we decided on smooth maps as the morphisms in $\mathbf{Man^c}$ and $\mathbf{Euc^c}$.

For the second, we impose an extra condition for $\mathfrak{C} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ coming from a product-preserving $F : \mathbf{Euc^c} \to \mathbf{Sets}$ to be a C^∞-ring with corners, that invertible functions in $\mathfrak{C}_{\rm ex}$ should have logs in $\mathfrak{C}$. To motivate this, note that if X is a manifold with corners and $f : X \to [0,\infty)$ is smooth with inverse $1/f$ then f maps $X \to (0,\infty)$, and so $\log f : X \to \mathbb{R}$ is well defined and smooth.

The second part of our answer is that we have defined not one theory of C^∞-rings and schemes with corners, but many, as in Figure 1.1; the overarching definitions of C^∞-rings and schemes with corners are the most general, which contain all the rest. General C^∞-schemes with corners $\boldsymbol{X} \in \mathbf{C^\infty Sch^c}$ may have 'corner behaviour' which is complicated and pathological, for example, $\boldsymbol{X} = [0,\infty)^{\mathbb{N}}$ has infinitely many boundary faces at a point. The subcategories of $\mathbf{C^\infty Sch^c}$ in Figure 1.1 have 'corner behaviour' which is progressively nicer, and more like ordinary manifolds with corners, as we impose more conditions.

The smallest categories $\mathbf{C^\infty Sch^c_{si}}, \mathbf{C^\infty Sch^c_{si,ex}}$, with 'simple corners', behave exactly like manifolds with corners, and $\mathbf{C^\infty Sch^c_{to}}, \mathbf{C^\infty Sch^c_{to,ex}}$, with 'toric corners', behave exactly like manifolds with g-corners. Different subcategories may be appropriate for different applications.

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In the rest of this introduction we summarize the contents of the chapters.

Chapter 2: C^∞-rings and C^∞-schemes

We begin with background on category theory, and on C^∞-rings and C^∞-schemes, mostly following the second author [49]. A *categorical C^∞-ring* is a product-preserving functor $F : \mathbf{Euc} \to \mathbf{Sets}$, where $\mathbf{Euc} \subset \mathbf{Man}$ is the full subcategory of the category of smooth manifolds $\mathbf{Man}$ with objects the Euclidean spaces $\mathbb{R}^n$ for $n \geqslant 0$. These form a category $\mathbf{CC^\infty Rings}$, with morphisms natural transformations, which is equivalent to the category $\mathbf{C^\infty Rings}$ of C^∞-rings by mapping $F \mapsto \mathfrak{C} = F(\mathbb{R})$.

Here a *C^∞-ring* is data $\bigl(\mathfrak{C}, (\Phi_f)_{f:\mathbb{R}^n\to\mathbb{R}\ C^\infty}\bigr)$, where $\mathfrak{C}$ is a set and $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ a map for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, satisfying some axioms. Usually we write this as $\mathfrak{C}$, leaving the C^∞-operations Φ_f implicit. The motivating examples for X a smooth manifold are $\mathfrak{C} = C^\infty(X)$, the set of smooth functions $c : X \to \mathbb{R}$, with operations $\Phi_f(c_1, \ldots, c_n)(x) = f(c_1(x), \ldots, c_n(x))$. We discuss *$\mathfrak{C}$-modules*, which are just modules over $\mathfrak{C}$ considered as an $\mathbb{R}$-algebra, and the *cotangent module* $\Omega_{\mathfrak{C}}$, generalizing $\Gamma^\infty(T^*X)$, the vector space of smooth sections of T^*X, as a module over $C^\infty(X)$.

We then define the subcategory $\mathbf{C^\infty RS}$ of *C^∞-ringed spaces* $\underline{X} = (X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of C^∞-rings on X, and the subcategory $\mathbf{LC^\infty RS} \subset \mathbf{C^\infty RS}$ of *local C^∞-ringed spaces* for which the stalks $\mathcal{O}_{X,x}$ are local C^∞-rings. The global sections functor $\Gamma : \mathbf{LC^\infty RS} \to \mathbf{C^\infty Rings}^{\mathrm{op}}$ has a right adjoint $\mathrm{Spec} : \mathbf{C^\infty Rings}^{\mathrm{op}} \to \mathbf{LC^\infty RS}$, the *spectrum functor*. An object $\underline{X} \in \mathbf{LC^\infty RS}$ is called a *C^∞-scheme* if we may cover X with open $U \subseteq X$ with $(U, \mathcal{O}_X|_U) \cong \mathrm{Spec}\,\mathfrak{C}$ for $\mathfrak{C} \in \mathbf{C^\infty Rings}$. These form a full subcategory $\mathbf{C^\infty Sch} \subset \mathbf{LC^\infty RS}$.

There is a full embedding $\mathbf{Man} \hookrightarrow \mathbf{C^\infty Sch}$ mapping $X \mapsto \underline{X} = (X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is the sheaf of local smooth functions $c : X \to \mathbb{R}$.

In classical Algebraic Geometry $\Gamma \circ \mathrm{Spec} \cong \mathrm{Id} : \mathbf{Rings}^{\mathrm{op}} \to \mathbf{Rings}^{\mathrm{op}}$, so $\mathbf{Rings}^{\mathrm{op}}$ is equivalent to the category $\mathbf{ASch}$ of affine schemes. In C^∞-algebraic geometry it is not true that $\Gamma \circ \mathrm{Spec} \cong \mathrm{Id}$, but we do have $\mathrm{Spec} \circ \Gamma \circ \mathrm{Spec} \cong \mathrm{Spec}$. Define a C^∞-ring $\mathfrak{C}$ to be *complete* if $\mathfrak{C} \cong \Gamma \circ \mathrm{Spec}\,\mathfrak{D}$ for some $\mathfrak{D}$. Write $\mathbf{C^\infty Rings_{co}} \subset \mathbf{C^\infty Rings}$ for the full subcategory of complete C^∞-rings. Then $(\mathbf{C^\infty Rings_{co}})^{\mathrm{op}}$ is equivalent to the category $\mathbf{AC^\infty Sch}$ of affine C^∞-schemes.

We define *$\mathcal{O}_X$-modules* on a C^∞-scheme $\underline{X}$, and the *cotangent sheaf* $T^*\underline{X}$, which generalizes cotangent bundles of manifolds. We review applications of C^∞-algebraic geometry in Synthetic Differential Geometry and Derived Differential Geometry.

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Chapter 3: Manifolds with (g-)corners

We discuss *manifolds with corners*, locally modelled on $[0,\infty)^k \times \mathbb{R}^{m-k}$. The definition of manifold with corners X, Y is an obvious generalization of the definition of manifold. However, the definition of smooth map $f : X \to Y$ is *not* obvious: what conditions should we impose on f near the boundary and corners of X, Y? There are several non-equivalent definitions of morphisms of manifolds with corners in the literature. The one we choose is due to Melrose [71, §1.12], [55, §1], who calls them *b-maps*. This gives a category $\mathbf{Man^c}$.

A smooth map $f : X \to Y$ is called *interior* if $f(X^\circ) \subseteq Y^\circ$, where X° is the *interior* of X (that is, if X is locally modelled on $[0,\infty)^k \times \mathbb{R}^{m-k}$, then X° is locally modelled on $(0,\infty)^k \times \mathbb{R}^{m-k}$). Write $\mathbf{Man^c_{in}} \subset \mathbf{Man^c}$ for the subcategory with only interior morphisms.

A manifold with corners X has a *boundary* ∂X, a manifold with corners of dimension $\dim X - 1$, with a (non-injective) morphism $\Pi_X : \partial X \to X$. For example, if $X = [0,\infty)^2$, then ∂X is the disjoint union $(\{0\} \times [0,\infty)) \amalg ([0,\infty) \times \{0\})$ with the obvious map Π_X, so two points in ∂X lie over $(0,0) \in X$. There is a natural, free action of the symmetric group S_k on the kth boundary $\partial^k X$, permuting the boundary strata. The *k-corners* is $C_k(X) = \partial^k X / S_k$, of dimension $\dim X - k$. The *corners* of X is $C(X) = \coprod_{k=0}^{\dim X} C_k(X)$, an object in the category $\check{\mathbf{M}}\mathbf{an^c}$ of *manifolds with corners of mixed dimension*.

We call X a *manifold with faces* if $\Pi_X|_F : F \to X$ is injective for each connected component F of ∂X (called a *face*).

Now boundaries are not functorial: if $f : X \to Y$ is smooth, there is generally no natural morphism $\partial f : \partial X \to \partial Y$. However, there is a natural, interior morphism $C(f) : C(X) \to C(Y)$, giving the *corner functor* $C : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c_{in}}$ or $C : \check{\mathbf{M}}\mathbf{an^c} \to \check{\mathbf{M}}\mathbf{an^c_{in}}$. We explain that C is right adjoint to the inclusion $\mathrm{inc} : \check{\mathbf{M}}\mathbf{an^c_{in}} \hookrightarrow \check{\mathbf{M}}\mathbf{an^c}$. Thus, from a categorical point of view, boundaries and corners in $\mathbf{Man^c}$ are determined uniquely by the inclusion $\mathbf{Man^c_{in}} \hookrightarrow \mathbf{Man^c}$, that is, by the comparison between interior and smooth maps.

A manifold with corners X has two notions of (co)tangent bundle: the *tangent bundle* TX and its dual the *cotangent bundle* T^*X, and the *b-tangent bundle* ${}^bTX$ and its dual the *b-cotangent bundle* ${}^bT^*X$. Here TX is the obvious notion of tangent bundle. There is a morphism $I_X : {}^bTX \to TX$ which identifies $\Gamma^\infty({}^bTX)$ with the subspace of vector fields $v \in \Gamma^\infty(TX)$ which are tangent to every boundary stratum of X. Tangent bundles are functorial over smooth maps $f : X \to Y$ (that is, f lifts to $Tf : TX \to TY$ linear on the fibres). B-tangent bundles are functorial over interior maps $f : X \to Y$.

We also discuss the categories $\mathbf{Man^{gc}_{in}} \subset \mathbf{Man^{gc}}$ of *manifolds with gener-*

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alized corners, or *manifolds with g-corners*, introduced by the second author in [46]. These allow more-complicated local models than $[0,\infty)^k \times \mathbb{R}^{m-k}$. The local models X_P are parametrized by *weakly toric monoids* P, with $X_P = [0,\infty)^k \times \mathbb{R}^{m-k}$ when $P = \mathbb{N}^k \times \mathbb{Z}^{m-k}$. So we provide an introduction to the theory of monoids. (All monoids in this book are commutative.)

The simplest manifold with g-corners which is not a manifold with corners is $X = \{(w,x,y,z) \in [0,\infty)^4 : wx = yz\}$. Most of the theory above extends to manifolds with g-corners, except that tangent bundles TX are not well-behaved, and we no longer have $C_k(X) \cong \partial^k X/S_k$.

We call morphisms $g : X \to Z$, $h : Y \to Z$ in $\mathbf{Man^c_{in}}$ or $\mathbf{Man^{gc}_{in}}$ *b-transverse* if ${}^bT_xg \oplus {}^bT_yh : {}^bT_xX \oplus {}^bT_yY \to {}^bT_zZ$ is surjective for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z$. Manifolds with g-corners have the nice property that all b-transverse fibre products exist in $\mathbf{Man^{gc}_{in}}$, whereas fibre products only exist in $\mathbf{Man^c_{in}}, \mathbf{Man^c}$ under rather restrictive conditions. We can think of $\mathbf{Man^{gc}_{in}}$ as a kind of closure of $\mathbf{Man^c_{in}}$ under b-transverse fibre products.

We discuss applications of manifolds with (g-)corners in the literature, including in Topological Quantum Field Theories, analysis of partial differential equations, and moduli problems in Morse homology and Floer theories whose moduli spaces are manifolds with (g-)corners.

Chapter 4: (Pre) C^∞-rings with corners

We introduce *C^∞-rings with corners*. To decide on the correct definition, the obvious starting point is categorical C^∞-rings, that is, product-preserving functors $F : \mathbf{Euc} \to \mathbf{Sets}$. We define a *categorical pre C^∞-ring with corners* to be a product-preserving functor $F : \mathbf{Euc^c} \to \mathbf{Sets}$, where $\mathbf{Euc^c} \subset \mathbf{Man^c}$ is the full subcategory with objects $[0,\infty)^m \times \mathbb{R}^n$ for $m,n \geqslant 0$. These form a category $\mathbf{CPC^\infty Rings^c}$, with morphisms natural transformations.

Then we define the category $\mathbf{PC^\infty Rings^c}$ of *pre C^∞-rings with corners*, with objects $\big(\mathfrak{C}, \mathfrak{C}_{\rm ex}, (\Phi_f)_{f:\mathbb{R}^m\times[0,\infty)^n\to\mathbb{R}\ C^\infty}, (\Psi_g)_{g:\mathbb{R}^m\times[0,\infty)^n\to[0,\infty)\ C^\infty}\big)$ for $\mathfrak{C}, \mathfrak{C}_{\rm ex}$ sets and $\Phi_f : \mathfrak{C}^m \times \mathfrak{C}_{\rm ex}^n \to \mathfrak{C}$, $\Psi_g : \mathfrak{C}^m \times \mathfrak{C}_{\rm ex}^n \to \mathfrak{C}_{\rm ex}$ maps, satisfying some axioms. Usually we write this as $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$, leaving the C^∞-operations Φ_f, Ψ_g implicit. There is an equivalence $\mathbf{CPC^\infty Rings^c} \to \mathbf{PC^\infty Rings^c}$ mapping $F \mapsto (\mathfrak{C}, \mathfrak{C}_{\rm ex}) = (F(\mathbb{R}), F([0,\infty)))$. The motivating example, for X a manifold with (g-)corners, is $(\mathfrak{C}, \mathfrak{C}_{\rm ex}) = (C^\infty(X), \mathrm{Ex}(X))$, where $C^\infty(X)$ is the set of smooth maps $c : X \to \mathbb{R}$, and $\mathrm{Ex}(X)$ the set of *exterior* (i.e. smooth, but not necessarily interior) maps $c' : X \to [0,\infty)$.

We define the category of *C^∞-rings with corners* $\mathbf{C^\infty Rings^c}$ to be the full subcategory of $(\mathfrak{C}, \mathfrak{C}_{\rm ex})$ in $\mathbf{PC^\infty Rings^c}$ satisfying an extra condition, that if $c' \in \mathfrak{C}_{\rm ex}$ is invertible in $\mathfrak{C}_{\rm ex}$ then $c' = \Psi_{\exp}(c)$ for some $c \in \mathfrak{C}$. In terms of

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spaces X, this says that if $c' : X \to [0,\infty)$ is smooth and invertible (hence positive) then $c' = \exp c$ for some smooth $c = \log c' : X \to \mathbb{R}$.

Now we could also have started with $\mathbf{Man^c_{in}}$. So we write $\mathbf{Euc^c_{in}} \subset \mathbf{Man^c_{in}}$ for the full subcategory with objects $[0,\infty)^m \times \mathbb{R}^n$, and define the category $\mathbf{CPC^\infty Rings^c_{in}}$ of *categorical interior pre C^∞-rings with corners* to be product-preserving functors $F : \mathbf{Euc^c_{in}} \to \mathbf{Sets}$, with morphisms natural transformations. We define a subcategory $\mathbf{PC^\infty Rings^c_{in}} \subset \mathbf{PC^\infty Rings^c}$ of *interior pre C^∞-rings with corners*, and an equivalence $\mathbf{CPC^\infty Rings^c_{in}} \to \mathbf{PC^\infty Rings^c_{in}}$ mapping $F \mapsto (\mathfrak{C}, \mathfrak{C}_{\rm ex}) = (F(\mathbb{R}), F([0,\infty)) \amalg \{0\})$. The category $\mathbf{C^\infty Rings^c_{in}}$ of *interior C^∞-rings with corners* is the intersection $\mathbf{PC^\infty Rings^c_{in}} \cap \mathbf{C^\infty Rings^c}$ in $\mathbf{PC^\infty Rings^c}$. The motivating example, for X a manifold with (g-)corners, is $(\mathfrak{C}, \mathfrak{C}_{\rm ex}) = (C^\infty(X), \mathrm{In}(X) \amalg \{0\})$, for $\mathrm{In}(X)$ the interior maps $c' : X \to [0,\infty)$.

Although a C^∞-ring with corners $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ has a huge number of C^∞-operations Φ_f, Ψ_g, it is helpful much of the time to focus on a small subset of these, giving $\mathfrak{C}, \mathfrak{C}_{\rm ex}$ smaller, more-manageable structures. In particular, we often think of $\mathfrak{C}$ as an $\mathbb{R}$-*algebra*, with addition and multiplication given by $\Phi_{f_+}, \Phi_{f_\cdot} : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ for $f_+, f_\cdot : \mathbb{R}^2 \to \mathbb{R}$ given by $f_+(x,y) = x+y$, $f_\cdot(x,y) = xy$. And we think of $\mathfrak{C}_{\rm ex}$ as a *monoid*, with multiplication given by $\Psi_{g_\cdot} : \mathfrak{C}_{\rm ex} \times \mathfrak{C}_{\rm ex} \to \mathfrak{C}_{\rm ex}$ for $g_\cdot : [0,\infty)^2 \to [0,\infty)$ given by $g_\cdot(x,y) = xy$. Note that monoids also control the corner structure of manifolds with g-corners, in the same way.

We define various important subcategories of C^∞-rings with corners by imposing conditions on $\mathfrak{C}_{\rm ex}$ as a monoid. For example, a C^∞-ring with corners $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ is *interior* if $\mathfrak{C}_{\rm ex} = \mathfrak{C}_{\rm in} \amalg \{0\}$ where $\mathfrak{C}_{\rm in}$ is a submonoid of $\mathfrak{C}_{\rm ex}$, that is, $\mathfrak{C}_{\rm ex}$ has no zero divisors. In terms of spaces X, we interpret $\mathfrak{C}_{\rm in}$ as the monoid $\mathrm{In}(X)$ of interior maps $c' : X \to [0,\infty)$. If $\boldsymbol{\mathfrak{C}}, \boldsymbol{\mathfrak{D}}$ are interior, a morphism $\boldsymbol{\phi} = (\phi, \phi_{\rm ex}) : \boldsymbol{\mathfrak{C}} \to \boldsymbol{\mathfrak{D}}$ is *interior* if $\phi_{\rm ex} : \mathfrak{C}_{\rm ex} \to \mathfrak{D}_{\rm ex}$ maps $\mathfrak{C}_{\rm in} \to \mathfrak{D}_{\rm in}$. The subcategory $\mathbf{C^\infty Sch^c_{fi}} \subset \mathbf{C^\infty Sch^c}$ of *firm* C^∞-rings with corners are those whose sharpening $\mathfrak{C}^\sharp_{\rm ex}$ is finitely generated.

Chapter 5: C^∞-schemes with corners

We define the category $\mathbf{C^\infty RS^c}$ of *C^∞-ringed spaces with corners* $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X)$, where X is a topological space and $\boldsymbol{\mathcal{O}}_X = (\mathcal{O}_X, \mathcal{O}_X^{\rm ex})$ is a sheaf of C^∞-rings with corners on X, and the subcategory $\mathbf{LC^\infty RS^c} \subset \mathbf{C^\infty RS^c}$ of *local C^∞-ringed spaces with corners* for which the stalks $\boldsymbol{\mathcal{O}}_{X,x}$ for $x \in X$ are local C^∞-rings with corners. The global sections functor $\Gamma^{\rm c} : \mathbf{LC^\infty RS^c} \to (\mathbf{C^\infty Rings^c})^{\rm op}$ has a right adjoint $\mathrm{Spec}^{\rm c} : (\mathbf{C^\infty Rings^c})^{\rm op} \to \mathbf{LC^\infty RS^c}$, the *spectrum functor*. An object $\boldsymbol{X}$ in $\mathbf{LC^\infty RS^c}$ is called a *C^∞-scheme with*

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corners if we may cover X with open $U \subseteq X$ with $(U, \boldsymbol{\mathcal{O}}_X|_U) \cong \mathrm{Spec^c}\, \mathfrak{C}$ for $\mathfrak{C}$ in $\mathbf{C^\infty Rings^c}$. These form a full subcategory $\mathbf{C^\infty Sch^c} \subset \mathbf{LC^\infty RS^c}$.

One might expect that to define corresponding subcategories $\mathbf{C^\infty RS^c_{in}} \subset \mathbf{C^\infty RS^c}, \mathbf{LC^\infty RS^c_{in}} \subset \mathbf{LC^\infty RS^c}$ of *interior* (local) C^∞-ringed spaces with corners, we should just replace $\mathbf{C^\infty Rings^c}$ by $\mathbf{C^\infty Rings^c_{in}} \subset \mathbf{C^\infty Rings^c}$. However, as $\mathrm{inc} : \mathbf{C^\infty Rings^c_{in}} \hookrightarrow \mathbf{C^\infty Rings^c}$ does not preserve limits, a sheaf of interior C^∞-rings with corners is only a presheaf of C^∞-rings with corners. So we define $\boldsymbol{X} = (X, \boldsymbol{\mathcal{O}}_X) \in \mathbf{C^\infty RS^c}$ to be *interior* if $\boldsymbol{\mathcal{O}}_X$ is the sheafification, as a sheaf valued in $\mathbf{C^\infty Rings^c}$, of a sheaf valued in $\mathbf{C^\infty Rings^c_{in}}$. Then the stalks $\boldsymbol{\mathcal{O}}_{X,x}$ of $\boldsymbol{\mathcal{O}}_X$ lie in $\mathbf{C^\infty Rings^c_{in}}$, so $\mathcal{O}^{\mathrm{ex}}_{X,x} = \mathcal{O}^{\mathrm{in}}_{X,x} \amalg \{0\}$.

We define $\Gamma^{\mathrm{c}}_{\mathrm{in}} : \mathbf{LC^\infty RS^c_{in}} \to (\mathbf{C^\infty Rings^c_{in}})^{\mathrm{op}}$ by $\Gamma^{\mathrm{c}}_{\mathrm{in}}(\boldsymbol{X}) = (\Gamma(\mathcal{O}_X), \Gamma(\mathcal{O}^{\mathrm{in}}_X) \amalg \{0\})$, where $\Gamma(\mathcal{O}^{\mathrm{in}}_X) \subset \Gamma(\mathcal{O}^{\mathrm{ex}}_X)$ is the subset of sections s with $s|_x \in \mathcal{O}^{\mathrm{in}}_{X,x} \subset \mathcal{O}^{\mathrm{ex}}_{X,x}$ for each $x \in X$. Then $\Gamma^{\mathrm{c}}_{\mathrm{in}}$ has a right adjoint $\mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}} : (\mathbf{C^\infty Rings^c_{in}})^{\mathrm{op}} \to \mathbf{LC^\infty RS^c_{in}}$, where $\mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}} = \mathrm{Spec^c}|_{(\mathbf{C^\infty Rings^c_{in}})^{\mathrm{op}}}$. An object $\boldsymbol{X} \in \mathbf{LC^\infty RS^c_{in}}$ is called an *interior C^∞-scheme with corners* if we may cover X with open $U \subseteq X$ with $(U, \boldsymbol{\mathcal{O}}_X|_U) \cong \mathrm{Spec}^{\mathrm{c}}_{\mathrm{in}}\, \mathfrak{C}$ for $\mathfrak{C} \in \mathbf{C^\infty Rings^c_{in}}$. These form a full subcategory $\mathbf{C^\infty Sch^c_{in}} \subset \mathbf{LC^\infty RS^c_{in}}$, with $\mathbf{C^\infty Sch^c_{in}} \subset \mathbf{C^\infty Sch^c}$.

We define full embeddings $\mathbf{Man^{gc}} \hookrightarrow \mathbf{C^\infty Sch^c}$ or $\mathbf{Man^{gc}_{in}} \hookrightarrow \mathbf{C^\infty Sch^c_{in}}$ mapping $X \mapsto (X, (\mathcal{O}_X, \mathcal{O}^{\mathrm{ex}}_X))$ or $X \mapsto (X, (\mathcal{O}_X, \mathcal{O}^{\mathrm{in}}_X \amalg \{0\}))$, where $\mathcal{O}_X$, $\mathcal{O}^{\mathrm{ex}}_X, \mathcal{O}^{\mathrm{in}}_X$ are the sheaves of local smooth functions $X \to \mathbb{R}$, exterior functions $X \to [0,\infty)$, and interior functions $X \to [0,\infty)$, respectively. Thus, manifolds with (g-)corners may be regarded as special examples of C^∞-schemes with corners. Manifolds with (g-)faces map to *affine* C^∞-schemes with corners.

For C^∞-schemes, $\Gamma \circ \mathrm{Spec} \not\cong \mathrm{Id}$ but $\mathrm{Spec} \circ \Gamma \circ \mathrm{Spec} \cong \mathrm{Spec}$. Thus we can define a full subcategory $\mathbf{C^\infty Rings_{co}} \subset \mathbf{C^\infty Rings}$ of *complete* C^∞-rings, with $(\mathbf{C^\infty Rings_{co}})^{\mathrm{op}}$ equivalent to the category $\mathbf{AC^\infty Sch}$ of affine C^∞-schemes.

For C^∞-schemes with corners, the situation is worse: we have both $\Gamma^{\mathrm{c}} \circ \mathrm{Spec^c} \not\cong \mathrm{Id}$ and $\mathrm{Spec^c} \circ \Gamma^{\mathrm{c}} \circ \mathrm{Spec^c} \not\cong \mathrm{Spec^c}$. To see why, note that if X is a manifold with corners then smooth functions $c : X \to \mathbb{R}$ are essentially local objects – they can be glued using partitions of unity.

However, smooth functions $c' : X \to [0,\infty)$ have some *strictly global* behaviour: there is a locally constant function $\mu_{c'} : \partial X \to \mathbb{N} \amalg \{\infty\}$ giving the order of vanishing of c' along the boundary ∂X. So the behaviour of c' near distant points x, y in X is linked, if x, y lie in the image of the same connected component of ∂X. This means that smooth functions $c' : X \to [0,\infty)$ cannot always be glued using partitions of unity, and localizing a C^∞-ring with corners

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$\mathfrak{C} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$ at an $\mathbb{R}$-point $x : \mathfrak{C} \to \mathbb{R}$, as one does to define $\operatorname{Spec}^{\rm c}$, does not see only local behaviour around x.

Since $\operatorname{Spec}^{\rm c} \circ \Gamma^{\rm c} \circ \operatorname{Spec}^{\rm c} \not\cong \operatorname{Spec}^{\rm c}$, we cannot define a subcategory of 'complete' C^∞-rings with corners equivalent to affine C^∞-schemes with corners. As a partial substitute, we define *semi-complete* C^∞-rings with corners $\mathfrak{C} = (\mathfrak{C}, \mathfrak{C}_{\rm ex})$, such that $\Gamma^{\rm c} \circ \operatorname{Spec}^{\rm c}$ is an isomorphism on $\mathfrak{C}$ and injective on $\mathfrak{C}_{\rm ex}$.

If $\boldsymbol{X}, \boldsymbol{Y} \in \mathbf{C^\infty Sch^c}$, a morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{C^\infty Sch^c}$ is a morphism in $\mathbf{LC^\infty RS^c}$. Although locally we can write $\boldsymbol{X} \cong \operatorname{Spec}^{\rm c} \boldsymbol{\mathfrak{C}}$, $\boldsymbol{X} \cong \operatorname{Spec}^{\rm c} \boldsymbol{\mathfrak{D}}$, because of the lack of a good notion of completeness, we do *not* know that locally we can write $\boldsymbol{f} = \operatorname{Spec}^{\rm c} \boldsymbol{\phi}$ for some $\boldsymbol{\phi} : \boldsymbol{\mathfrak{D}} \to \boldsymbol{\mathfrak{C}}$ in $\mathbf{C^\infty Rings^c}$. One problem this causes is that $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ are morphisms in $\mathbf{C^\infty Sch^c}$, we do not know that the fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ in $\mathbf{LC^\infty RS^c}$ (which always exists) lies in $\mathbf{C^\infty Sch^c}$, if $\boldsymbol{g}, \boldsymbol{h}$ are not locally of the form $\operatorname{Spec}^{\rm c} \boldsymbol{\phi}$. So we do not know that all fibre products exist in $\mathbf{C^\infty Sch^c}$.

To get around this, we introduce the full subcategory $\mathbf{C^\infty Sch^c_{fi}} \subset \mathbf{C^\infty Sch^c}$ of *firm* C^∞-schemes with corners $\boldsymbol{X}$, which are locally of the form $\operatorname{Spec}^{\rm c} \boldsymbol{\mathfrak{C}}$ for $\boldsymbol{\mathfrak{C}}$ a firm C^∞-ring with corners. Morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{C^\infty Sch^c_{fi}}$ are always locally of the form $\operatorname{Spec}^{\rm c} \boldsymbol{\phi}$, so we can prove that $\mathbf{C^\infty Sch^c_{fi}}$ is closed under fibre products in $\mathbf{LC^\infty RS^c}$, and thus all fibre products exist in $\mathbf{C^\infty Sch^c_{fi}}$.

In general, $\mathbf{C^\infty Sch^c}$ contains a huge variety of objects, many of which are very singular and pathological, and do not fit with our intuitions about manifolds with corners. So it can be helpful to restrict to smaller subcategories of better-behaved objects in $\mathbf{C^\infty Sch^c}$, such as $\mathbf{C^\infty Sch^c_{fi}}$. For example, for $\boldsymbol{X}$ to be firm means that locally $\boldsymbol{X}$ has only finitely many boundary strata, which seems likely to hold in almost every interesting application.

We define many subcategories of $\mathbf{C^\infty Sch^c_{in}}, \mathbf{C^\infty Sch^c}$, and prove results such as existence of reflection functors between them, and existence of fibre products and finite limits in them. Two particularly interesting and well-behaved examples are the full subcategories $\mathbf{C^\infty Sch^c_{to}} \subset \mathbf{C^\infty Sch^c_{in}}$, $\mathbf{C^\infty Sch^c_{to,ex}} \subset \mathbf{C^\infty Sch^c}$ of *toric* C^∞-schemes with corners, whose corner structure is controlled by toric monoids in the same way that manifolds with g-corners are.

Chapter 6: Boundaries, corners, and the corner functor

One of the most important properties of manifolds with corners X is the existence of boundaries ∂X, and clearly we want to generalize this to C^∞-schemes with corners. Our starting point, as in Chapter 3, is that the boundary $\partial X =$

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$C_1(X)$ is part of the corners $C(X) = \coprod_{k=0}^{\dim X} C_k(X)$, and the corner functor $C : \check{\mathbf{Man}}^{\mathbf{c}} \to \check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{in}}$ is right adjoint to the inclusion $\mathrm{inc} : \check{\mathbf{Man}}^{\mathbf{c}}_{\mathbf{in}} \hookrightarrow \check{\mathbf{Man}}^{\mathbf{c}}$.

We construct a right adjoint *corner functor* $C : \mathbf{LC^\infty RS^c} \to \mathbf{LC^\infty RS^c_{in}}$ to the inclusion $\mathrm{inc} : \mathbf{LC^\infty RS^c_{in}} \hookrightarrow \mathbf{LC^\infty RS^c}$. We prove that the restriction to $\mathbf{C^\infty Sch^c}$ maps to $\mathbf{C^\infty Sch^c_{in}}$, giving $C : \mathbf{C^\infty Sch^c} \to \mathbf{C^\infty Sch^c_{in}}$ right adjoint to $\mathrm{inc} : \mathbf{C^\infty Sch^c_{in}} \hookrightarrow \mathbf{C^\infty Sch^c}$. For $\boldsymbol{X}$ in $\mathbf{LC^\infty RS^c}$, points in $C(X)$ are pairs (x, P) for $x \in X$ and $P \subset \mathcal{O}^{\mathrm{ex}}_{X,x}$ a prime ideal in the monoid $\mathcal{O}^{\mathrm{ex}}_{X,x}$ of the stalk $\boldsymbol{\mathcal{O}}_{X,x}$ of $\boldsymbol{\mathcal{O}}_X$ at x. This is an analogue, for $X \in \mathbf{Man^c}$, of a point in $C(X)$ being (x, γ) for $x \in X$ and γ a local corner component of X at x.

The corner functors for $\mathbf{Man^c}, \mathbf{Man^{gc}}$ and $\mathbf{C^\infty Sch^c}$ commute with the embeddings $\mathbf{Man^c}, \mathbf{Man^{gc}} \hookrightarrow \mathbf{C^\infty Sch^c}$.

To get an analogue of the decomposition $C(X) = \coprod_{k \geqslant 0} C_k(X)$ for C^∞-schemes with corners, we restrict to the subcategories $\mathbf{C^\infty Sch^c_{fi}} \subset \mathbf{C^\infty Sch^c}$ and $\mathbf{C^\infty Sch^c_{fi,in}} \subset \mathbf{C^\infty Sch^c_{in}}$ of *firm* C^∞-schemes with corners, where $C : \mathbf{C^\infty Sch^c_{fi}} \to \mathbf{C^\infty Sch^c_{fi,in}}$. For $\boldsymbol{X}$ firm there is a locally constant sheaf $\check{M}^{\mathrm{ex}}_{C(X)}$ of finitely generated monoids on $C(\boldsymbol{X})$, with $\check{M}^{\mathrm{ex}}_{C(X)}|_{(x,P)} = \mathcal{O}^{\mathrm{ex}}_{X,x}/[c' = 1$ if $c' \in \mathcal{O}^{\mathrm{ex}}_{X,x} \setminus P]$. We define a decomposition $C(\boldsymbol{X}) = \coprod_{k \geqslant 0} C_k(\boldsymbol{X})$ with $C_k(\boldsymbol{X})$ open and closed in $C(\boldsymbol{X})$, by saying that $(x, P) \in C_k(\boldsymbol{X})$ if the maximum length of a chain of prime ideals in $\check{M}^{\mathrm{ex}}_{C(X)}|_{(x,P)}$ is $k+1$. This recovers the usual decomposition $C(X) = \coprod_{k \geqslant 0} C_k(X)$ if X is a manifold with (g-)corners. We define the *boundary* $\partial \boldsymbol{X} = C_1(\boldsymbol{X})$.

For toric C^∞-schemes with corners C maps $\mathbf{C^\infty Sch^c_{to,ex}} \to \mathbf{C^\infty Sch^c_{to}}$, and C preserves fibre products, and all fibre products exist in $\mathbf{C^\infty Sch^c_{to}}$. We use this to give criteria for when fibre products exist in $\mathbf{C^\infty Sch^c_{to,ex}}$. This is an analogue of results in [46] giving criteria for when fibre products exist in $\mathbf{Man^{gc}}$, given that b-transverse fibre products exist in $\mathbf{Man^{gc}_{in}}$.

Chapter 7: Modules, and sheaves of modules

If $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ is a C^∞-ring with corners, we define a $\boldsymbol{\mathfrak{C}}$-*module* to be a module over $\mathfrak{C}$ as an $\mathbb{R}$-algebra. Similarly, if $\boldsymbol{X} = (X, (\mathcal{O}_X, \mathcal{O}^{\mathrm{ex}}_X))$ is a C^∞-scheme with corners, we consider $\mathcal{O}_X$-*modules* on X, which are just modules on the underlying C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$. So the theory of modules over C^∞-rings with corners and C^∞-schemes with corners lifts immediately from modules over C^∞-rings and C^∞-schemes in [49, §5], with no additional theory required.

If X is a manifold with corners then, as in Chapter 3, we have two notions of cotangent bundle $T^*X, {}^bT^*X$, where ${}^bT^*X$ is functorial only under interior morphisms. Similarly, if $\boldsymbol{\mathfrak{C}} = (\mathfrak{C}, \mathfrak{C}_{\mathrm{ex}})$ is a C^∞-ring with corners, we have the

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cotangent module $\Omega_{\mathfrak{C}}$ of $\mathfrak{C}$ from [49, §5]. If $\boldsymbol{\mathfrak{C}}$ is interior we also define the *b-cotangent module* ${}^b\Omega_{\boldsymbol{\mathfrak{C}}}$, which uses the corner structure. If X is a manifold with (g-)faces and $\boldsymbol{\mathfrak{C}} = \boldsymbol{C}^\infty_{\rm in}(X)$ then $\Omega_{\mathfrak{C}} = \Gamma^\infty(T^*X)$ and ${}^b\Omega_{\boldsymbol{\mathfrak{C}}} = \Gamma^\infty({}^bT^*X)$. We show that b-cotangent modules are functorial under interior morphisms and have exact sequences for pushouts.

If $\boldsymbol{X}$ is a C^∞-scheme with corners we define the *cotangent sheaf* $T^*\boldsymbol{X}$, and if $\boldsymbol{X}$ is interior the *b-cotangent sheaf* ${}^bT^*\boldsymbol{X}$, by sheafifying the (b-)cotangent modules of $\boldsymbol{\mathcal{O}}_X(U)$ for open $U \subset X$. If $\boldsymbol{X} = F_{\mathbf{Man^c}}(X)$ for $X \in \mathbf{Man^c}$ these are the sheaves of sections of T^*X and ${}^bT^*X$. We show that Cartesian squares in subcategories such as $\mathbf{C^\infty Sch^c_{to}}, \mathbf{C^\infty Sch^c_{fi,in}} \subset \mathbf{C^\infty Sch^c_{in}}$ yield exact sequences of b-cotangent sheaves. On the corners $C(\boldsymbol{X})$ we construct an exact sequence relating ${}^bT^*C(\boldsymbol{X})$, $\boldsymbol{\Pi}^*_{\boldsymbol{X}}({}^bT^*\boldsymbol{X})$ and $\check{M}^{\rm ex}_{C(X)} \otimes_{\mathbb{N}} \mathcal{O}_{C(X)}$.

Chapter 8: Further generalizations and applications

Finally we propose four directions in which this book could be generalized and applied.

Synthetic Differential Geometry with corners Synthetic Differential Geometry is a subject in which one proves theorems about manifolds in Differential Geometry by reasoning using 'infinitesimals', as in Kock [52, 53]. C^∞-schemes are used to provide a 'model' for Synthetic Differential Geometry, and so show that the axioms of Synthetic Differential Geometry are consistent. In a similar way, one could develop a theory of 'Synthetic Differential Geometry with corners', for proving theorems about manifolds with corners using infinitesimals, and C^∞-schemes with corners could be used to show that it is consistent.

C^∞-stacks with corners In classical Algebraic Geometry, schemes are generalized to (Deligne–Mumford or Artin) stacks. The second author [49] extended the theory of C^∞-schemes to *C^∞-stacks*, including *Deligne–Mumford C^∞-stacks*. This corresponds to generalizing manifolds to orbifolds.

We discuss a theory of *C^∞-stacks with corners*, including *Deligne–Mumford C^∞-stacks with corners*. These generalize *orbifolds with (g-)corners*. Much of the theory follows from [49] with only cosmetic changes.

C^∞-rings and C^∞-schemes with a-corners Our theory starts with the categories $\mathbf{Man^c_{in}} \subset \mathbf{Man^c}$ of manifolds with corners defined in Chapter 3. The second author [47] also defined categories $\mathbf{Man^{ac}_{in}} \subset \mathbf{Man^{ac}}$ of *manifolds with analytic corners*, or *manifolds with a-corners*. Even the simplest objects $[\![0,\infty)$ in $\mathbf{Man^{ac}}$ and $[0,\infty)$ in $\mathbf{Man^c}$ have different smooth structures. There

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is also a category $\mathbf{Man^{c,ac}}$ of *manifolds with corners and a-corners* containing both $\mathbf{Man^c}$ and $\mathbf{Man^{ac}}$.

Manifolds with a-corners have applications to partial differential equations with boundary conditions of asymptotic type, and to moduli spaces with boundary and corners, such as moduli spaces of Morse flow lines, in which (we argue) manifolds with a-corners give the correct smooth structure.

This entire book could be rewritten over $\mathbf{Man^{ac}_{in}} \subset \mathbf{Man^{ac}}$ or $\mathbf{Man^{c,ac}_{in}} \subset \mathbf{Man^{c,ac}}$ rather than $\mathbf{Man^c_{in}} \subset \mathbf{Man^c}$. We explain the first steps in this.

Derived manifolds and derived orbifolds with corners Classical Algebraic Geometry has been generalized to Derived Algebraic Geometry, which is now a major area of mathematics. As in §2.9, classical Differential Geometry can be generalized to *Derived Differential Geometry*, the study of *derived manifolds* and *derived orbifolds*, regarded as special examples of *derived C^∞-schemes* and *derived C^∞-stacks*. Some references are [6, 7, 8, 10, 11, 12, 13, 43, 44, 62, 87, 88, 89].

It is desirable to extend the subject to *derived manifolds with corners* and *derived orbifolds with corners*, regarded as special examples of *derived C^∞-schemes with corners* and *derived C^∞-stacks with corners*. This will be done by the second author in [44], with this book as its foundations (see also Steffens [88, 89]). Derived orbifolds with corners will have important applications in Symplectic Geometry, as (we argue) they are the correct way to make Fukaya–Ohta–Oh–Ono's 'Kuranishi spaces with corners' [28, 29, 30, 31] into well-behaved geometric spaces.

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