SOME FINITE GROUPS WITH ZERO DEFICIENCY

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Abstract

We introduce further finite groups which can be presented with an equal number of generators and relations.

1. Introduction

Finite groups with zero deficiency include cyclic groups, certain metacyclic groups [5] and other classes of finite groups given in [2], [3] and [4]. In this paper we present a further class of two-generator, two-relation groups which we show are finite and hence introduce the smallest non-metacyclic p-group with zero deficiency. We also present a three-generator, three-relation finite group.

The groups presented are defined as;

$$G(\alpha,\beta,\gamma) = \{a, b \mid c^{-1}ac = a^{\alpha}, b^2 = a^{\beta}c^{\gamma}, c = a^{-1}b^{-1}ab\}, |\alpha| \neq 1, \gamma \geqq 0$$

and

$$G = \{a,b,c \mid b^{-1}ab = a^{-1}b^4, c^{-1}bc = b^{-1}c^4, a^{-1}ca = c^{-1}a^4\}.$$

2. Finiteness of G (α, β, γ)

The relations are:

(1) $$c^{-1}ac = a^{\alpha}, \qquad |\alpha| \neq 1,$$

(2) $$b^2 = a^{\beta}c^{\gamma}, \qquad \gamma \geqq 0,$$

(3) $$b^{-1}ab = ac.$$

Conjugation of (3) by b implies

$b^{-2}ab^2 = acb^{-1}cb$ whence (2) yields

$c^{-\gamma}ac^{\gamma} = acb^{-1}cb$ which together with (1) gives

(4) $$b^{-1}cb = c^{-1}a^{\alpha^{\gamma}-1}.$$

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Conjugation of (1) by b yields

$a^{1-\alpha^{\gamma}}ca^{\alpha} = (ac)^{\alpha}$ which with (1) gives, in conjunction with (3) and (4),

(5) $a^{\alpha-\alpha^{\gamma+1}+\alpha^{\gamma}} = c^{-1}(ac)^{\alpha}$, whereby (1) yields

(6) $c^{\alpha-1} = a^{\alpha^{\gamma}-\alpha^{\gamma+1}-\alpha^{2}-\cdots-\alpha^{\alpha}}$, for $\alpha > 1$, or

(7) $c^{\alpha-1} = a^{\alpha^{\gamma}-\alpha^{\gamma+1}+\alpha+\alpha^{2}+\cdots+\alpha^{1-\alpha}}$, for $\alpha < -1$, whence, from (1),

(8) $a^{\alpha|\alpha-1|-1} = 1$ and hence G is a finite group with order dividing

$$|2(\alpha-1)(\alpha^{|\alpha-1|}-1)|.$$

In the special case with $\alpha = -3$, $\beta = 4$ and $\gamma = 2$ then (7) gives $c^{-4} = a^{96}$ whence (1) implies

(9) $$a^{16} = 1,\ c^{4} = 1.$$

However conjugation of (2) by b gives

$$a^{4}c^{2} = a^{12}c^{-2} \text{ whence}$$

(10) $$a^{8} = 1.$$

Hence $G(-3, 4, 2)$ is a group of order dividing 64. In fact $G(-3, 4, 2)$ is group number 240 in [1] and since this group is the only non-metacyclic 2-group of order at most 64 with trivial Schur multiplicator then $G(-3, 4, 2)$ is the smallest non-metacyclic p-group with zero deficiency.

3. Finiteness of G

The relations are

(11) $b^{-1}ab = a^{-1}b^{4}$, $c^{-1}bc = b^{-1}c^{4}$, $a^{-1}ca = c^{-1}a^{4}$; whence

$b^{-1}ab^{2} = b^{-4}ab^{4}$ or

(12) $b^{-2}ab^{2} = a$. Similarly

(13) $c^{-2}bc^{2} = b$ and

(14) $a^{-2}ca^{2} = c$.

We use the identity

(15) $$[a, b, c^{a}][c, a, b^{c}][b, c, a^{b}] = 1$$

where $[x, y]$ denotes $x^{-1}y^{-1}xy$ and x^{y} denotes $y^{-1}xy$.

We have

(16) $$[a, b, c^{a}] = [a^{-2}b^{4}, c^{-1}a^{4}] = b^{-8}c^{16}$$

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which with the other equations equivalent to (16) together with (15) give, since the subgroup generated by a^2, b^2 and c^2 is abelian,

(17) $$a^8b^8c^8 = 1.$$

Since $[b^8c^8, b] = 1$ then $[a^8, b] = 1$ whence (11) yields $a^{16} = b^{32} = c^{64} = a^{128}$ whence

(18) $$a^{112} = 1.$$

(19) Similarly $b^{112} = c^{112} = 1$, whence G is a finite group with order dividing 7.2^{11}, since $b^{16} = c^{32}$.

References

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