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SUCCESSIVE ITERATION AND POSITIVE SOLUTIONS FOR A p-LAPLACIAN MULTIPOINT BOUNDARY VALUE PROBLEM

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Abstract

In this paper, we study the existence of positive solutions for the one-dimensional p-Laplacian differential equation,

$$(\phi_p(u'(t)))' + q(t) f(t, u(t), u'(t)) = 0, \quad t \in (0, 1),$$

subject to the multipoint boundary condition

$$u'(0) = \sum_{i=1}^{n} \alpha_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i),$$

by applying a monotone iterative method.

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1. Introduction

In this paper, we will consider the positive solutions to the p-Laplacian boundary value problem

$$(\phi_p(u'(t)))' + q(t) f(t, u(t), u'(t)) = 0, \quad t \in (0, 1), \tag{1.1}$$

$$u'(0) = \sum_{i=1}^{n} \alpha_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i), \tag{1.2}$$

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where $\phi_p(s) = |s|^{p-2}s$ with $p > 1$, $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$, and α_i, β_i, f, q satisfy:

(H1) $0 \le \alpha_i, \beta_i < 1$ $(i = 1, 2, \ldots, n)$ satisfy $0 \le \sum_{i=1}^{n} \alpha_i, \sum_{i=1}^{n} \beta_i < 1$;

(H2) $f(t, x, y) \in C([0, 1] \times [0, +\infty) \times R \to [0, +\infty))$, $q(t)$ is a nonnegative continuous function defined on $(0, 1)$ and $q(t) \not\equiv 0$ on any subinterval of $(0, 1)$. In addition, $\int_0^1 q(t)\,dt < +\infty$.

Here, a positive solution of (1.1) and (1.2) means a solution u^* satisfying $u^*(t) > 0$, $0 < t < 1$.

The existence and multiplicity of positive solutions for linear and nonlinear multipoint boundary value problems have been widely studied by many authors (see [1–6, 8] and the references therein). However, there are not many papers which are concerned with the computational methods of these problems. Then the question arises: "*How can we find the solutions when their existence is known*?"

More recently, when under the assumption that f is allowed to depend only on t and u but not on u', Ma *et al.* [7] proved the existence of positive solutions of a multipoint p-Laplacian boundary value problem via a monotone iterative technique.

So, motivated by all the works above, we investigate here the iteration and existence of positive solutions for the multipoint boundary value problem with p-Laplacian (1.1) and (1.2), which will extend all the previous research. We do not require the existence of lower and upper solutions. By applying monotone iterative techniques, we construct some successive iterative schemes to approximate the solutions in this paper.

2. Preliminaries

In this section, we give the preliminaries and some definitions.

DEFINITION 2.1. Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that:

- $au + bv \in P$ for all $u, v \in P$ and all $a \ge 0, b \ge 0$; and
- $u, -u \in P$ implies $u = 0$.

DEFINITION 2.2. The map α is said to be concave on $[0, 1]$, if

$$\alpha(tu + (1 - t)v) \ge t\alpha(u) + (1 - t)\alpha(v)$$

for all $u, v \in [0, 1]$ and $t \in [0, 1]$.

Let E be the Banach space $C^1[0, 1]$ endowed with the norm

$$\|u\| := \max\Big\{\max_{0\le t\le 1} |u(t)|, \max_{0\le t\le 1} |u'(t)|\Big\}.$$

We denote $E_+ = C^1_+[0, 1] = \{u \in E \mid u(t) \ge 0, t \in [0, 1]\}$, and define the cone $P \subset E$ by

$$P = \{u \in E \mid u(t) \ge 0, u \text{ is concave and nonincreasing on } [0, 1]\}.$$

Throughout, it is assumed that (H1) and (H2) hold.

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LEMMA 2.1. *Suppose that* $y \in C^1[0, 1]$ *with* $(\phi_p(y'(t)))' \in L^1[0, 1]$ *satisfies*

$$-(\phi_p(y'(t)))' \geq 0, \quad t \in (0, 1),$$
$$y'(0) = \sum_{i=1}^{n} \alpha_i y'(\xi_i), \quad y(1) = \sum_{i=1}^{n} \beta_i y(\xi_i).$$

Then, $y(t)$ *is concave and* $y(t) \geq 0$*,* $y'(t) \leq 0$ *on* $[0, 1]$*, that is,* $y \in P$.

The proof is very easy since $0 \leq \sum_{i=1}^{n} \alpha_i, \sum_{i=1}^{n} \beta_i < 1$, so we omit it here.

For all $x \in C^1_+[0, 1]$, suppose that u is a solution of the problem (1.1) and (1.2). Then

$$u'(t) = \phi_p^{-1}\left(A_x - \int_0^t q(s) f(s, x(s), x'(s))\, ds\right),$$
$$u(t) = B_x - \int_t^1 \phi_p^{-1}\left(A_x - \int_0^s q(r) f(r, x(r), x'(r))\, dr\right) ds,$$

where A_x, B_x satisfy the boundary conditions, that is,

$$\phi_p^{-1}(A_x) = \sum_{i=1}^{n} \alpha_i \phi_p^{-1}\left(A_x - \int_0^{\xi_i} q(s) f(s, x(s), x'(s))\, ds\right),$$
$$B_x = \sum_{i=1}^{n} \beta_i \left[B_x - \int_{\xi_i}^1 \phi_p^{-1}\left(A_x - \int_0^s q(r) f(r, x(r), x'(r))\, dr\right) ds\right].$$

Hence,

$$u(t) = -\frac{\sum_{i=1}^{n} \beta_i \int_{\xi_i}^1 \phi_p^{-1}(A_x - \int_0^s q(r) f(r, x(r), x'(r))\, dr)\, ds}{1 - \sum_{i=1}^{n} \beta_i} - \int_t^1 \phi_p^{-1}\left(A_x - \int_0^s q(r) f(r, x(r), x'(r))\, dr\right) ds$$

where A_x satisfies (2.1).

LEMMA 2.2. *For all* $x \in C^1_+[0, 1]$*, there exists a unique* A_x *with*

$$A_x \in \left[-\frac{\phi_p(\sum_{i=1}^{n} \alpha_i)}{1 - \phi_p(\sum_{i=1}^{n} \alpha_i)} \int_0^1 q(s) f(s, x(s), x'(s))\, ds, 0\right]$$

satisfying (2.1).

The proof is similar to [7, proof of Lemma 2.2], so we omit it here.

For any $x \in C^1_+[0, 1]$, let A_x be the unique constant satisfying (2.1) corresponding to x. Then we have the following lemma.

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LEMMA 2.3. *$A_x : C^1_+[0, 1] \to R$ has the following properties:*

(a) *$A_x : C^1_+[0, 1] \to R$ is continuous about x;*
(b) *if $f(t, x, y)$ is nondecreasing about x and nonincreasing about y on $[0, 1] \times [0, +\infty) \times (-\infty, 0]$, then A_x is nonincreasing on P.*

The proof is similar to [7, proof of Lemma 2.3], so we omit it here.

3. Main results

For notational convenience, we denote

$$A = \frac{(1 - \sum_{i=1}^n \beta_i)\phi_p^{-1}(1 - \phi_p(\sum_{i=1}^n \alpha_i))}{(1 - \sum_{i=1}^n \beta_i \xi_i)\phi_p^{-1}(\int_0^1 q(s)\, ds)}.$$

We will prove the following existence results.

THEOREM 3.1. *Assume that* (H1) *and* (H2) *hold, and there exists $a > 0$ such that:*

(S1) *$f(t, x_1, y_1) \le f(t, x_2, y_2)$ for any $0 \le t \le 1, 0 \le x_1 \le x_2 \le a, -a \le y_2 \le y_1 \le 0$;*
(S2) *$\max_{0 \le t \le 1} f(t, a, -a) \le \phi_p(aA)$;*
(S3) *$f(t, 0, 0) \not\equiv 0$ for $0 \le t \le 1$.*

Then the boundary value problem (1.1) *and* (1.2) *has two positive nonincreasing, concave solutions w^* and v^* such that*

$$\begin{aligned} &0 < w^* \le a, \quad -a \le (w^*)' \le 0 \\ \text{and } &\lim_{n\to\infty} w_n = \lim_{n\to\infty} T^n w_0 = w^*, \quad \lim_{n\to\infty} (w_n)' = \lim_{n\to\infty} (T^n w_0)' = (w^*)', \\ \text{where } &w_0(t) = a - at\frac{1 - \sum_{i=1}^n \beta_i}{1 - \sum_{i=1}^n \beta_i \xi_i}, \quad 0 \le t \le 1, \end{aligned}$$

and

$$\begin{aligned} &0 < v^* \le a, \quad -a \le (v^*)' \le 0 \\ \text{and } &\lim_{n\to\infty} v_n = \lim_{n\to\infty} T^n v_0 = v^*, \quad \lim_{n\to\infty} (v_n)' = \lim_{n\to\infty} (T^n v_0)' = (v^*)', \\ \text{where } &v_0(t) = 0, \quad 0 \le t \le 1, \end{aligned}$$

where

$$\begin{aligned} (Tu)(t) = &-\frac{\sum_{i=1}^n \beta_i \int_{\xi_i}^1 \phi_p^{-1}(A_u - \int_0^s q(r) f(r, u(r), u'(r))\, dr)\, ds}{1 - \sum_{i=1}^n \beta_i} \\ &- \int_t^1 \phi_p^{-1}\left(A_u - \int_0^s q(r) f(r, u(r), u'(r))\, dr\right) ds. \end{aligned} \tag{3.1}$$

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The iterative schemes in Theorem 3.1 are

$$w_0(t) = a - at\frac{(1 - \sum_{i=1}^{n} \beta_i)}{(1 - \sum_{i=1}^{n} \beta_i \xi_i)}, \quad w_{n+1} = Tw_n = T^n w_0, \quad n = 0, 1, 2 \dots$$
$$\textit{and} \quad v_0(t) = 0, \quad v_{n+1} = Tv_n = T^n v_0, \quad n = 0, 1, 2 \dots.$$

They start off with a known linear function and the zero function, respectively.

PROOF. We define an operator $T : P \to E$ by (3.1). Then, from the definition of T, we deduce that, for each $u \in P$, there exists $Tu \in C^1[0, 1]$ which is nonnegative and satisfies (1.2). Moreover, by Lemma 2.1 we have that Tu is concave, that is, $Tu \in P$, and $(Tu)'(t) \le 0$ on $[0, 1]$. So, $T : P \to P$.

The continuity of T is obvious. Now, we prove that T is compact. Let $\Omega \subset P$ be a bounded set. It is easy to prove that $T(\Omega)$ is bounded and equicontinuous. Then the Arzelà-Ascoli theorem guarantees that $T\Omega$ is relatively compact, which means that T is compact. Then, $T : P \to P$ is completely continuous, and each fixed point of T in P is a solution of (1.1) and (1.2).

For any $u_i \in P (i = 1, 2)$ with $u_1 \le u_2$ and $u_1' \ge u_2'$, let $A_{u_i} (i = 1, 2)$ be two constants decided in (2.1) corresponding to $u_i \in P (i = 1, 2)$, then by (S1) and Lemma 2.3 we have $A_{u_1} \ge A_{u_2}$. From the definition of T, we can easily get $Tu_1 \le Tu_2$.

We denote

$$\bar{P}_a = \{u \in P \mid \|u\| \le a\}.$$

Then, in what follows, we first prove that $T : \bar{P}_a \to \bar{P}_a$. If $u \in \bar{P}_a$, then $\|u\| \le a$, and

$$0 \le u(t) \le u(0) = \max_{0\le t\le 1} |u(t)| \le \|u\| \le a,$$
$$-a \le -\|u\| \le -\max_{0\le t\le 1} |u'(t)| = u'(1) \le u'(t) \le 0.$$

So by (S1) and (S2)

$$0 \le f(t, u(t), u'(t)) \le f(t, a, -a) \le \max_{0\le t\le 1} f(t, a, -a) \le \phi_p(aA), \qquad \text{for } 0 \le t \le 1.$$

In fact,

$$\|Tu\| = \max\left\{\max_{0\le t\le 1} |(Tu)(t)|, \max_{0\le t\le 1} |(Tu)'(t)|\right\}$$
$$= \max\{(Tu)(0), -(Tu)'(1)\}.$$

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By (3.1) and Lemma 2.2,

$$\begin{aligned}
&(Tu)(0)\\
&= -\bigg[\sum_{i=1}^{n}\beta_i\int_{\xi_i}^{1}\phi_p^{-1}(A_u - \int_0^s q(r)f(r,u(r),u'(r))\,dr)\,ds\bigg]\bigg[1-\sum_{i=1}^{n}\beta_i\bigg]^{-1}\\
&\quad - \int_0^1 \phi_p^{-1}\bigg(A_u - \int_0^s q(r)f(r,u(r),u'(r))\,dr\bigg)\,ds\\
&\le \bigg[\sum_{i=1}^{n}\beta_i\int_{\xi_i}^{1}\phi_p^{-1}\bigg(\frac{\phi_p(\sum_{i=1}^n\alpha_i)}{1-\phi_p(\sum_{i=1}^n\alpha_i)}\int_0^1 q(s)f(s,u(s),u'(s))\,ds\\
&\quad + \int_0^1 q(r)f(r,u(r),u'(r))\,dr\bigg)\,ds\bigg]\bigg[1-\sum_{i=1}^{n}\beta_i\bigg]^{-1}\\
&\quad + \int_0^1 \phi_p^{-1}\bigg(\frac{\phi_p(\sum_{i=1}^n\alpha_i)}{1-\phi_p(\sum_{i=1}^n\alpha_i)}\int_0^1 q(s)f(s,u(s),u'(s))\,ds\\
&\quad + \int_0^1 q(r)f(r,u(r),u'(r))\,dr\bigg)\,ds\\
&\le \bigg[\bigg(1-\sum_{i=1}^{n}\beta_i\xi_i\bigg)\phi_p^{-1}\bigg(\int_0^1 q(s)f(s,u(s),u'(s))\,ds\bigg)\bigg]\\
&\quad \times\bigg[\bigg(1-\sum_{i=1}^{n}\beta_i\bigg)\phi_p^{-1}\bigg(1-\phi_p\bigg(\sum_{i=1}^{n}\alpha_i\bigg)\bigg)\bigg]^{-1}\\
&\le aA\bigg[\bigg(1-\sum_{i=1}^{n}\beta_i\xi_i\bigg)\phi_p^{-1}\bigg(\int_0^1 q(s)\,ds\bigg)\bigg]\\
&\quad \times\bigg[\bigg(1-\sum_{i=1}^{n}\beta_i\bigg)\phi_p^{-1}\bigg(1-\phi_p\bigg(\sum_{i=1}^{n}\alpha_i\bigg)\bigg)\bigg]^{-1}\\
&= a,
\end{aligned}$$

and

$$\begin{aligned}
-(Tu)'(1) &= \phi_p^{-1}\bigg(-A_u + \int_0^1 q(s)f(s,u(s),u'(s))\,ds\bigg)\\
&\le \phi_p^{-1}\bigg(\frac{\phi_p(\sum_{i=1}^n\alpha_i)}{1-\phi_p(\sum_{i=1}^n\alpha_i)}\int_0^1 q(s)f(s,u(s),u'(s))\,ds\\
&\quad + \int_0^1 q(s)f(s,u(s),u'(s))\,ds\bigg)\\
&\le aA\frac{\phi_p^{-1}(\int_0^1 q(s)\,ds)}{\phi_p^{-1}(1-\phi_p(\sum_{i=1}^n\alpha_i))} \le a.
\end{aligned}$$

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Thus, we obtain that

$$\|Tu\| = \max\{(Tu)(0), -(Tu)'(1)\} \le a.$$

Hence, we assert that $T : \bar{P}_a \to \bar{P}_a$.

Let

$$w_0(t) = a - at\frac{1-\sum_{i=1}^{n}\beta_i}{1-\sum_{i=1}^{n}\beta_i\xi_i}, \quad 0 \le t \le 1,$$

so then $w_0(t) \in \bar{P}_a$. Let $w_1 = Tw_0$, so then $w_1 \in \bar{P}_a$. We denote $w_{n+1} = Tw_n = T^n w_0$, $n = 0, 1, 2 \ldots$. Since $T : \bar{P}_a \to \bar{P}_a$, we have $w_n \in T\bar{P}_a \subseteq \bar{P}_a$, $n = 1, 2, \ldots$. Since T is completely continuous, we assert that $\{w_n\}_{n=1}^{\infty}$ is a sequentially compact set.

Since

$$\begin{aligned}
w_1(t) &= Tw_0(t) \\
&= -\bigg[\sum_{i=1}^{n}\beta_i\int_{\xi_i}^{1}\phi_p^{-1}\bigg(A_{w_0} - \int_0^s q(r)f(r, w_0(r), w_0'(r))\,dr\bigg)\,ds\bigg] \\
&\quad\times\bigg[1-\sum_{i=1}^{n}\beta_i\bigg]^{-1} - \int_t^1\phi_p^{-1}\bigg(A_{w_0} - \int_0^s q(r)f(r, w_0(r), w_0'(r))\,dr\bigg)\,ds \\
&\le \bigg[\sum_{i=1}^{n}\beta_i\int_{\xi_i}^{1}\phi_p^{-1}\bigg(\frac{\phi_p(\sum_{i=1}^{n}\alpha_i)}{1-\phi_p(\sum_{i=1}^{n}\alpha_i)}\int_0^1 q(s)f(s, w_0(s), w_0'(s))\,ds \\
&\quad + \int_0^1 q(r)f(r, w_0(r), w_0'(r))\,dr\bigg)\,ds\bigg]\bigg[1-\sum_{i=1}^{n}\beta_i\bigg]^{-1} \\
&\quad + \int_t^1\phi_p^{-1}\bigg(\frac{\phi_p(\sum_{i=1}^{n}\alpha_i)}{1-\phi_p(\sum_{i=1}^{n}\alpha_i)}\int_0^1 q(s)f(s, w_0(s), w_0'(s))\,ds \\
&\quad + \int_0^1 q(r)f(r, w_0(r), w_0'(r))\,dr\bigg)\,ds \\
&\le aA\frac{(1-\sum_{i=1}^{n}\beta_i\xi_i)\phi_p^{-1}(\int_0^1 q(s)\,ds)}{(1-\sum_{i=1}^{n}\beta_i)\phi_p^{-1}(1-\phi_p(\sum_{i=1}^{n}\alpha_i))} - tAa\frac{\phi_p^{-1}(\int_0^1 q(s)\,ds)}{\phi_p^{-1}(1-\phi_p(\sum_{i=1}^{n}\alpha_i))} \\
&= a - at\frac{1-\sum_{i=1}^{n}\beta_i}{1-\sum_{i=1}^{n}\beta_i\xi_i} = w_0(t), \quad 0 \le t \le 1,
\end{aligned}$$

and

$$\begin{aligned}
w_1'(t) &= (Tw_0)'(t) \\
&= \phi_p^{-1}\Big(A_{w_0} - \int_0^t q(s) f(s, w_0(s), w_0'(s))\, ds\Big) \\
&\geq \phi_p^{-1}\Big(-\frac{\phi_p(\sum_{i=1}^n \alpha_i)}{1-\phi_p(\sum_{i=1}^n \alpha_i)} \int_0^1 q(s) f(s, w_0(s), w_0'(s))\, ds \\
&\quad - \int_0^t q(s) f(s, w_0(s), w_0'(s))\, ds\Big) \\
&\geq \phi_p^{-1}\Big(-\frac{1}{1-\phi_p(\sum_{i=1}^n \alpha_i)} \int_0^1 q(s) f(s, w_0(s), w_0'(s))\, ds\Big) \\
&\geq -aA \frac{\phi_p^{-1}(\int_0^1 q(s)\, ds)}{\phi_p^{-1}(1-\phi_p(\sum_{i=1}^n \alpha_i))} \\
&= -a \frac{1-\sum_{i=1}^n \beta_i}{1-\sum_{i=1}^n \beta_i \xi_i} = w_0'(t), \quad 0 \leq t \leq 1,
\end{aligned}$$

so

$$\begin{aligned}
w_2(t) = Tw_1(t) \leq Tw_0(t) = w_1(t), \quad 0 \leq t \leq 1, \\
w_2'(t) = (Tw_1)'(t) \geq (Tw_0)'(t) = w_1'(t), \quad 0 \leq t \leq 1.
\end{aligned}$$

Hence, by the induction,

$$w_{n+1} \leq w_n, \quad w_{n+1}'(t) \geq w_n'(t), \quad 0 \leq t \leq 1,\ n = 0, 1, 2 \ldots.$$

Thus, there exists $w^* \in \bar{P}_a$ such that $w_n \to w^*$. Applying the continuity of T and $w_{n+1} = Tw_n$, we get $Tw^* = w^*$.

Let $v_0(t) = 0$, $0 \leq t \leq 1$, so then $v_0(t) \in \bar{P}_a$. Let $v_1 = Tv_0$, so then $v_1 \in \bar{P}_a$. We denote $v_{n+1} = Tv_n = T^n v_0$, $n = 0, 1, 2 \ldots.$ Since $T : \bar{P}_a \to \bar{P}_a$, we have $v_n \in T\bar{P}_a \subseteq \bar{P}_a$, $n = 1, 2, \ldots.$ Since T is completely continuous, we assert that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set.

Since $v_1 = Tv_0 = T0 \in \bar{P}_a$,

$$\begin{aligned}
v_1(t) = Tv_0(t) = (T0)(t) \geq 0, \quad 0 \leq t \leq 1, \\
v_1'(t) = (Tv_0)'(t) = (T0)'(t) \leq 0, \quad 0 \leq t \leq 1.
\end{aligned}$$

So,

$$\begin{aligned}
v_2(t) = Tv_1(t) \geq (T0)(t) = v_1(t), \quad 0 \leq t \leq 1, \\
v_2'(t) = (Tv_1)'(t) \leq (T0)'(t) = v_1'(t), \quad 0 \leq t \leq 1.
\end{aligned}$$

Therefore, it is similar to the earlier arguments and, by induction,

$$v_{n+1} \geq v_n, \quad v_{n+1}'(t) \leq v_n'(t), \quad 0 \leq t \leq 1,\ n = 0, 1, 2 \ldots.$$

Hence, there exists $v^* \in \bar{P}_a$ such that $v_n \to v^*$. Applying the continuity of T and $v_{n+1} = Tv_n$, we get $Tv^* = v^*$.

If $f(t, 0, 0) \not\equiv 0$, $0 \le t \le 1$, then the zero function is not the solution of (1.1) and (1.2). Thus, $\max_{0\le t\le 1} |v^*(t)| > 0$, and we have $v^* \ge \min\{t, 1-t\} \max_{0\le t\le 1} |v^*(t)| > 0$, $0 < t < 1$.

It is well known that each fixed point of T in P is a solution of (1.1) and (1.2). Hence, we assert that w^* and v^* are two positive nonincreasing, concave solutions of the problem (1.1) and (1.2).

The proof is completed. □

The following corollaries follow easily.

Corollary 3.1. *Assume that* (H1), (H2), (S1) *and* (S3) *hold, and there exists* $a > 0$ *such that:*

(C3.1) $\underline{\lim}_{\ell\to+\infty} \max_{0\le t\le 1}(f(t, \ell, -a)/\ell^{p-1}) \le \phi_p(A)$ *(in particular,* $\underline{\lim}_{\ell\to+\infty} \max_{0\le t\le 1}(f(t, \ell, -a)/\ell^{p-1}) = 0$*).*

Then the boundary value problem (1.1) *and* (1.2) *has two positive, concave solutions* w^* *and* v^**, and all other conclusions of Theorem* 3.1 *hold.*

Corollary 3.2. *Assume that* (H1), (H2) *and* (S3) *hold, and there exist* $0 < a_1 < a_2 < \cdots < a_n$ *such that:*

(C3.2.1) $f(t, x_1, y_1) \le f(t, x_2, y_2)$ *for any* $0 \le t \le 1$, $0 \le x_1 \le x_2 \le a_k$, $-a_k \le y_2 \le y_1 \le 0$, $k = 1, 2, \ldots, n$*;*

(C3.2.2) $\max_{0\le t\le 1} f(t, a_k, -a_k) \le \phi_p(a_k A)$, $k = 1, 2, \ldots, n$.

Then the boundary value problem (1.1) *and* (1.2) *has* $2n$ *positive, concave solutions* w_k^* *and* v_k^* *such that:*

$$
\begin{aligned}
&0 < w_k^* \le a_k, \quad -a_k < (w_k^*)' \le 0,\\
\textit{and}\quad &\lim_{n\to\infty} w_{k_n} = \lim_{n\to\infty} T^n w_{k_0} = w_k^*, \quad \lim_{n\to\infty} (w_{k_n})' = \lim_{n\to\infty} (T^n w_{k_0})' = (w_k^*)',\\
\textit{where}\quad &w_{k_0}(t) = a_k - a_k t \frac{1 - \sum_{i=1}^n \beta_i}{1 - \sum_{i=1}^n \beta_i \xi_i}, \quad 0 \le t \le 1,
\end{aligned}
$$

and

$$
\begin{aligned}
&0 < v_k^* \le a_k, \quad -a_k < (v_k^*)' \le 0,\\
\textit{and}\quad &\lim_{n\to\infty} v_{k_n} = \lim_{n\to\infty} T^n v_{k_0} = v_k^*, \quad \lim_{n\to\infty} (v_{k_n})' = \lim_{n\to\infty} (T^n v_{k_0})' = (v_k^*)',\\
\textit{where}\quad &v_{k_0}(t) = 0, \quad 0 \le t \le 1,
\end{aligned}
$$

with $(Tu)(t)$ *defined in the same way as in* (3.1).

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The iterative schemes in Corollary 3.2 are

$$w_{k_0}(t) = a_k - a_k t \frac{1 - \sum_{i=1}^{n} \beta_i}{1 - \sum_{i=1}^{n} \beta_i \xi_i}, \quad w_{k_{n+1}} = T w_{k_n}$$
$$= T^n w_{k_0}, \quad k = 1, 2 \ldots, n = 0, 1, 2 \ldots$$
$$\text{and} \quad v_{k_0}(t) = 0, \quad v_{k_{n+1}} = T v_{k_n} = T^n v_{k_0}, \quad k = 1, 2 \ldots, n = 0, 1, 2 \ldots.$$

They start off with known linear functions and zero functions respectively.

COROLLARY 3.3. *Assume that* (H1)*,* (H2)*,* (C3.2.1) *and* (S3) *hold, and there exist* $0 < a_1 < a_2 < \cdots < a_n$ *such that*

(C3.3) $\underline{\lim}_{\ell \to +\infty} \max_{0 \le t \le 1}(f(t, \ell, -a_k))/(\ell^{p-1}) \le \phi_p(A)$, $k = 1, 2, \ldots, n$ *(in particular,* $\underline{\lim}_{\ell \to +\infty} \max_{0 \le t \le 1}(f(t, \ell, -a_k)/\ell^{p-1}) = 0$, $k = 1, 2, \ldots, n$*).*

Then the boundary value problem (1.1) *and* (1.2) *has* $2n$ *positive, concave solutions* w_k^* *and* v_k^**, and all other conclusions of Corollary 3.2 hold.*

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