ON A THEOREM OF HUGHES AND THOMPSON

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We generalise the notion of proper Hughes subgroup in a group, asking that, for some prime p, the set-complement of the union of two proper subgroups consist solely of elements of order p. A structure theorem for non-p-groups of this type is proved.

1. INTRODUCTION

Let G be a group and p a prime. Define the Hughes subgroup $H_p(G)$ of G as that generated by the elements of G whose order is not p. In [3] Hughes asked if the index of $H_p(G)$ in G is necessarily 1, $|G|$ or p. The question received a positive answer for finite non-p-groups in the article [4] of Hughes and Thompson; and a negative answer for groups in general in Wall [8]. Khukhro [6] has shown that for finite groups the answer is yes 'almost always'. Hughes and Thompson [4] and Kegel [5] prove more: in a finite group G, $H_p(G)$ if proper, is nilpotent; their results may be formulated as follows.

THEOREM 1. (Hughes and Thompson [4]; Kegel [5]) *Let G be a finite group and p a prime. Suppose that H is a proper subgroup of G whose set-complement in G consists of elements of order p. Then H is nilpotent. If G is not a p-group then H is normal and of index p in G.*

A partial generalisation of this result is proved by Espuelas [2]. If G is a finite soluble group, p an odd prime, and H a proper subgroup of G whose complement consists of elements of order at most p^n, then H has nilpotent length at most n.

Another direction of generalisation is taken in [1]. There it is shown that a group G of exponent p^n, soluble of derived length d and in which $H_p(G)$ is proper, is nilpotent with class bounded by a function of p, n, d.

Yet another generalisation is proposed here. Suppose that a group G has subgroups A_i $(i \in I)$ the set-complement of whose union consists of elements of order p. What can we say about the structure of G? What hope is there of generalising Theorem 1?

We present results on this problem for the case when $|I| = 2$, and raise some questions. We show that in this case it is possible to characterise the groups concerned,

as it is in the Hughes-Thompson context, and in much the same way. The methods presented here do not seem to generalise; in a later joint article with Professors V. Fedri and L. Serena, completely different methods will be used to get results for finite groups in the general case. These results are, however, less complete in general than those presented here.

2. THE CLASSES $\mathcal{H}_n(p)$

Let p be a prime and n a positive integer. Define the class $\mathcal{H}_n(p)$ to consist of all finite groups G with subgroups A_i $(1 \leqslant i \leqslant n)$ whose set-union is not G, and the complement of whose set-union consists of elements of order p. It will be convenient to include the identity group in $\mathcal{H}_n(p)$. The groups with proper Hughes subgroup for exponent p comprise the class $\mathcal{H}_1(p)$. It is easy to construct groups in $\mathcal{H}_n(p)$ in general. For example, suppose that G_i $(1 \leqslant i \leqslant n)$ are groups in which $H_p(G_i) \neq G_i$ $(1 \leqslant i \leqslant n)$. Consider the direct product $G = \Pi(G_i : 1 \leqslant i \leqslant n)$, and its subgroups $A_i = \Pi(G_j : j \neq i) \times H_p(G_i)$ $(1 \leqslant i \leqslant n)$. Then $G \neq \cup(A_i : 1 \leqslant i \leqslant n)$, and every element of G outside this union has order p since it projects outside $H_p(G_i)$ for every i. In particular, for example, if S_3 denotes the symmetric group on three symbols, then $S_3 \times S_3$ is in $\mathcal{H}_2(2)$. However, since $S_3 \times S_3$ is generated by elements of order six, $S_3 \times S_3$ is not in $\mathcal{H}_1(2)$. By definition $\mathcal{H}_m(p) \subseteq \mathcal{H}_n(p)$ for $m \leqslant n$ and we may ask if the inclusion $\mathcal{H}_m(p) \subseteq \mathcal{H}_n(p)$ $(m < n)$ is always strict. This question is not further prosecuted in this article.

Another interesting question concerns the structure of groups in $\mathcal{H}_n(p)$. Do they necessarily have the structure of the example given above? This is not true in general. For example, S_3 lies in $\mathcal{H}_3(3)$. (Note also that S_3 lies in $\mathcal{H}_1(2)$, so the prime p chosen does matter.) A consequence of Theorem 1 is that the groups in $\mathcal{H}_1(p)$ are soluble. In general groups in $\mathcal{H}_n(p)$ are not soluble. For example, a group in which the Sylow p-subgroups are of order p and self-centralising is necessarily in $\mathcal{H}_n(p)$ for some n: the subgroups A_i could be chosen as the cyclic subgroups whose order is not p. An example is A_5, the alternating group on five symbols, and we can be more economical: every element of A_5 not of order five is contained in a copy of A_4 of which there are five in number. Hence A_5 is in $\mathcal{H}_5(5)$. Nevertheless we shall prove that in the case $n = 2$ the questions raised here do have positive answers.

We include in this section some elementary, useful results needed later. All groups considered in this paper are finite.

LEMMA 1. *Suppose that a group G has subgroups A_i $(1 \leqslant i \leqslant n)$ whose union is not G and the complement of whose union consists of elements of order p. Let $D = \cap(A_i : 1 \leqslant i \leqslant n)$. Then $core_G(D)$ is nilpotent.*

PROOF: If $N = core_G(D)$, and g is in $G \setminus \cup(A_i : 1 \leqslant i \leqslant n)$, then every element of

$N\langle g \rangle \setminus N$ is outside $\cup (A_i : 1 \leqslant i \leqslant n)$. Hence it has order p. Therefore, by Theorem 1, N is nilpotent.   □

LEMMA 2. *Let G be a group and $\mathcal{A}$ a union of subgroups of G. Suppose that $\mathcal{A} \neq G$ and that, for some prime p, every element of $G \setminus \mathcal{A}$ has order p. Then $C_G(x)$ is a p-group for all $x \in G \setminus \mathcal{A}$.*

PROOF: If y is a p'-element of $C_G(x)$ then xy is not of order p and therefore $xy \in \mathcal{A}$. Since $\mathcal{A}$ is a union of groups it follows that $x \in \mathcal{A}$, a contradiction. Hence $C_G(x)$ is a p-group.   □

A corollary of this is that nilpotent groups in $\mathcal{H}_n(p)$ are p-groups.

In section 4 we need the idea of a Frobenius group, one admitting a fixed-point-free automorphism of order p. Such a group is necessarily nilpotent, by a theorem of Thompson [7]. It is, moreover, a p'-group, since otherwise the automorphism normalises a Sylow p-subgroup of the group, therefore centralising some non-trivial element. By a Frobenius extension we mean a group $H\langle s \rangle$ where H is a group and s a fixed-point-free automorphism of it of prime order. The next lemma, though formulated in the language of the present context, is well known, and is included for completeness only.

LEMMA 3. *Let H be a nilpotent p'-group admitting an automorphism s of order p. Write $G = H\langle s \rangle$. Then G is in $\mathcal{H}_1(p)$ if and only if $\mathbf{C}_H(s) = 1$.*

PROOF: That the condition $C_H(s) = 1$ is necessary follows from Lemma 2.

Conversely suppose that $\mathbf{C}_H(s) = 1$. For all $h \in H$ we show that $(sh)^p = 1$. Now $(sh)^p = h^{s^{p-1}} h^{s^{p-2}} \ldots h^s h$. If H is Abelian this element is fixed by s, so $(sh)^p = 1$. If H is not Abelian we deduce, by induction on the class of H, that the element displayed above is in $\zeta_1(H)$. It is centralised by sh, and therefore by s, and is therefore 1. That is $(sh)^p = 1$ for all $h \in H$. Finally note that, for $i \in (1, 2, \ldots, p-1)$, $C_H(s^i) = C_H(s) = 1$ so, for all $h \in H$, $(s^i h)^p = 1$. Therefore $H_1(p) \subseteq H$ so $G \in \mathcal{H}_1(p)$.   □

3. INITIAL STRUCTURE THEOREM

In this short section we give a structure theorem for $\mathcal{H}_2(p)$ groups which is used in the next section to delineate their structure more precisely.

THEOREM 2. *Let G be a finite group which, for some prime p, is not a p-group. Suppose that G has proper subgroups A_1, A_2 and that the complement of their union contains only elements of order p. Then one of the following holds.*

(3.1)   One of $G \setminus A_1$, $G \setminus A_2$ consists of elements of order p.

(3.2) $O_{p'}(G) \subseteq A_1 \cap A_2$ and $G/O_{p'}(G)$ is a p-group.

PROOF: Choose a pair A, B of subgroups of G with the properties that $A \subseteq A_1$, $B \subseteq A_2$; that $G \setminus (A \cup B)$ consists of elements of order p; and that, subject to these

constraints, (A, B) is minimal in the component-wise ordering of pairs of subgroups of G. With this choice

(3.3) A, B normalise each other.

For, if b is in B then the complement of $A^b \cup B = (A \cup B)^b$ consists of elements of order p. Moreover $(A^b \cap A) \cup B = (A^b \cup B) \cap (A \cup B)$, so every element in the complement of the left-side has order p since it fails to lie in a term of the right-side. By minimality $A = A^b \cap A$ whence $A \subseteq A^b$ and therefore $A = A^b$. Since this holds for all b in B, B normalises A. Similarly A normalises B.

Suppose that (3.1) does not hold. That is, neither $G \setminus A_1$ nor $G \setminus A_2$ consists of elements of order p. The same is therefore true of $G \setminus A$ and $G \setminus B$. Hence, in particular, A, B are incomparable. Write $D = A \cap B$. Then, by (3.3), D is normal in AB. Moreover if x is not in AB then $x^p = 1$. Hence, by Theorem 1, AB is normal in G. Let a, b be elements of $A \setminus D$ and $B \setminus D$ respectively. Then ab lies outside $A \cup B$ and therefore has order p. Hence $1 = (ab)^p = a^p b^p \pmod{D}$ and so $a^{-p} = b^p \pmod{D}$, which means that

(3.4) A/D, B/D have exponent p.

That is, D is subnormal of p-power index in G. Hence all p'-elements of G must lie in D. Also every element of $D\langle ab \rangle \setminus D$ has order p, so D is nilpotent, by Theorem 1. It follows that $O_{p'}(G) = O_{p'}(D)$ and $G/O_{p'}(G)$ is a p-group. This completes the proof of Theorem 2.   ☐

The properties of a minimal pair (A, B) of subgroups introduced in this proof will be used again in the next section.

4. FINER STRUCTURE OF $\mathcal{H}_2(p)$-GROUPS

Theorem 1 reduces the classification of groups in $\mathcal{H}_1(p)$ to the following two problems: firstly, classify p-groups in $\mathcal{H}_1(p)$; secondly, classify Frobenius extensions. An $\mathcal{H}_1(p)$-group is a special type of subdirect product of a p-group in $\mathcal{H}_1(p)$ and a Frobenius extension.

THEOREM 3. *A finite non-p-group is in $\mathcal{H}_1(p)$ if and only if $G = PN$ where P is a p-group, N a p'-group normal in G, and where:*

(4.1) $C_N(P) = 1$;

(4.2) $H_p(P) \subseteq C_P(N)$;

(4.3) $|P : C_P(N)| = p$.

In particular a group in $\mathcal{H}_1(p)$ is a subdirect product of a p-group in $\mathcal{H}_1(p)$ and a Frobenius extension.

PROOF: That non-p-groups in $\mathcal{H}_1(p)$ have this structure is a consequence of Theorem 1. For, let P be a Sylow p-group of a group G in $\mathcal{H}_1(p)$, and let $N = O_{p'}(G)$. Then $G = PN$. Now let $y \in P \setminus H_p(G)$. Firstly, all such y have order p and hence $H_p(P) \subseteq P \cap H_p(G) \neq P$ so that $P \in \mathcal{H}_1(p)$. Secondly, y has no non-trivial fixed points in N by Lemma 2, so $C_N(P) = 1$. Also, using Theorem 1 and Lemma 2, $P \cap H_p(G) \subseteq C_P(N) \subseteq P \cap H_p(G)$ so $C_P(N) = P \cap H_p(G)$, and hence $|P : C_P(N)| = p$. We see also that G is a subdirect product of P, a p-group in $\mathcal{H}_1(p)$, and $G/P \cap H_p(G)$, a Frobenius extension.

Conversely, if G has the structure described, let $x \in G \setminus C_P(N)N$. Then $x = yn$ where $y \in P \setminus C_P(N)$ and $n \in N$. By hypothesis $y^p = 1$ and y has no fixed points in N. Hence $(yn)^p = 1$ by Lemma 3. Therefore $G \in \mathcal{H}_1(p)$.

This completes the proof of Theorem 3. ☐

We use Theorem 2 in a similar way to describe the groups in $\mathcal{H}_2(p)$.

THEOREM 4. *A finite non-p-group G is in $\mathcal{H}_2(p)$ if and only if it has a p-subgroup P and a nilpotent normal p'-subgroup N, with $G = PN$ satisfying the following conditions:*

(4.4) $C_N(P) = 1$;

(4.5)   *P has proper subgroups P_1, P_2 the complement of whose union*
            consists of elements of order p;

and either

(4.6) $C_P(N) = P_i$ and $|P : P_i| = p$, where i is one of 1 or 2;

or

(4.7) $N = N_1 \times N_2$, $P_i = C_P(N_i)$ and $|P : P_i| = p (i = 1, 2)$.

PROOF: Note first of all that a group in $\mathcal{H}_1(p)$ comes under the alternative (4.6). This follows from Theorem 3, with the choice of P_2, say, as 1.

Hence suppose that G is in $\mathcal{H}_2(p)$ but not in $\mathcal{H}_1(p)$. There are in G subgroups A_1, A_2 whose union is not G, and the complement of whose union consists of elements of order p. If A, B are chosen minimally, as in the proof of Theorem 2, then $AB = G$ (or else G would be in $\mathcal{H}_1(p)$), A, B are normal in G, and they are incomparable. Write $D = A \cap B$.

Write $N = O_{p'}(G)$. Let P be a Sylow p-subgroup of G. Then, by Theorem 2, $G = PN$. Since P is not contained in $A \cup B$ it follows from Lemma 2 that $C_N(P) = 1$

confirming (4.4). Write $P_1 = P \cap A_1$ and $P_2 = P \cap A_2$; both are proper and the complement in P of their union consists of elements of order p. This confirms (4.5).

Let H/K be a chief factor of G contained in N, and write $C = C_G(H/K)$. It must be that $D \subseteq C$ by Lemma 1, since $D \trianglelefteq G$. Moreover $C \subseteq A \cup B$ by Lemma 2. Since C is not a union of two proper subgroups we have either $C \subseteq A$ or $C \subseteq B$. It cannot be that C is contained in D, as otherwise $C = D$, and G/C is a faithfully represented p-group with non-cyclic centre. So suppose that $D \subset C \subseteq B$. Now G acts on H/K with kernel C, that is as $A/D \times B/C$. However a faithfully and irreducibly represented nilpotent group has cyclic centre, so it follows that B/C is trivial, since A/D is not. That is, $C = B$.

We have proved that the centraliser of a chief factor of G contained in N is either A or B. If $C_0 = C_P(N)$ then NC_0 centralisers every chief factor of G in N. If NC_0 is not contained in D therefore, NC_0 is in A or in B, say in B. However if C is the centraliser of some chief factor of G in N, then we have $D \subset NC_0 \subseteq C \subseteq B$. It follows that $C = B$, and this for every p'-chief factor centraliser. Now

$$\begin{aligned}
B &= \cap(C_G(H/K) : H/K \text{ a } p'\text{-chief factor}) \\
&= \cap(NC_P(H/K) : H/K \text{ a } p'\text{-chief factor}) \\
&= N[\cap(C_P(H/K) : H/K \text{ a } p'\text{-chief factor})] \\
&= NC_0.
\end{aligned}$$

We deduce also that A/D acts fixed-point-freely on each chief factor in N since, for every $a \in A \setminus D$, and for $b \in B \setminus D$, $ab \notin A \cup B$ so it must act fixed-point-freely on N, and therefore fixed-point-freely on every chief factor in N. Therefore, since A/D has exponent p by (3.4), A/D is of order p. One consequence of this is that B has index p in G, and hence $B = A_2$. Also $NC_0 = A_2$. Hence $P_2 = A_2 \cap P = NC_0 \cap P = C_P(N)$. This takes care of (4.6).

There remains the possibility that $C_0 \subseteq D$. Then $NC_0 = D$. Moreover the kernel of the action of G on a chief factor H/K in N is either A or B, and A/D or B/D acts faithfully and fixed-point-freely on H/K, as the case may be. Furthermore there is at least one such chief factor not centralised by A, and at least one not centralised by B. Therefore A/D, B/D are both of order p. Hence both A, B are of index p in G, in fact $A = A_1$, $B = A_2$; so both P_1, P_2 are normal and have index p in P. Note that $C_P(N) = P_1 \cap P_2$, and $P = P_1 P_2$.

Let $N_1 = C_N(P_1)$, $N_2 = C_N(P_2)$. Then $N = N_1 \times N_2$. This is because, firstly, $N = N_1 \times [N, P_1]$, since P_1 acts on N as a group of co-prime order; secondly $[N, P_1, P_2] = [N, P_2, P_1]$, since P_1, P_2 commute in their action on N; thirdly $[N, P_1, P_2, \ldots, P_1, P_2] = 1$ for sufficiently many entries P_1, P_2 since one

of P_1, P_2 centralises each chief factor of G in N; and fourthly, therefore, $1 = [N, P_1, \ldots, P_1, P_2, \ldots, P_2] = [N, P_1, P_2]$. Hence $[N, P_1] \subseteq C_N(P_2) = N_2$, so $N = N_1 N_2 = N_1 \times N_2$ since $N_1 \cap N_2 \subseteq C_N(P) = 1$. Finally note that $C_P(N_i) = P_i$ ($i = 1, 2$) since $P_i \subseteq C_P(N_i) \neq P$.

Conversely suppose that a group G satisfies the conditions of Theorem 4. Let $A_i = P_i N$ ($i = 1, 2$). Both are proper subgroups of G, so their union is not G. Consider first the case (4.6). Let $x \in P \setminus (P_1 \cup P_2)$. Then x has no non-trival fixed points in N, because of (4.4). Hence, by Lemma 3, every element of the form xn ($x \in P \setminus (P_1 \cup P_2)$, $n \in N$) has order p. However all elements of G not in $A_1 \cup A_2$ are of this form. Therefore $G \in \mathcal{H}_2(p)$.

Finally suppose that G satisfies (4.7). If $P_1 = P_2$ then the conditions of Theorem 3 are satisfied with $P_1 N = H_p(G)$, and so $G \in \mathcal{H}_1(p)$. So suppose that $P_1 \neq P_2$. Every element of G not in $A_1 \cup A_2$ has the form $y = x_1 x_2 n$ where x_1, x_2 are in $P \setminus P_1$ and $P \setminus P_2$ respectively, and n is in N. However $x_1 x_2 \notin P_1 \cup P_2$ and so has no fixed points in N by (4.4). By Lemma 3 therefore, y has order p. This means that G is in $\mathcal{H}_2(p)$.

The proof of Theorem 4 is complete.   ▯

COROLLARY. *A group in $\mathcal{H}_2(p)$ is a sub-direct product of a p-group in $\mathcal{H}_2(p)$ and either one or two Frobenius extensions.*

PROOF: This follows at once from Theorem 4. The case (4.6) involves one Frobenius extension, the case (4.7), two.

It will be observed that the examples discussed in section 2 are, in the case $n = 2$, special cases of the structure described by Theorem 4.   ▯

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