Proceedings of the Edinburgh Mathematical Society (1997) 40, 551–562 ©

TEST WORDS OF A FREE PRODUCT OF TWO FINITE CYCLIC GROUPS

by DANIEL A. VOCE

(Received 11th December 1995)

We characterize the test words of $\mathbb{Z}_m * \mathbb{Z}_n$. They are those elements not contained in a proper retract.

1991 *Mathematics subject classification:* 20E05, 20F55.

1. Introduction

An element w of a group Γ is a *test element* if every endomorphism of Γ fixing w is necessarily an automorphism. If Γ is a free group or a free product then the test elements are called *test words*. The element w is called a *test element for monomorphisms* if every monomorphism of Γ fixing w is necessarily an automorphism. Given a test element w, the endomorphism ϕ is an automorphism if and only if $\phi(w) = \alpha(w)$ for some automorphism α. Thus, the use of test elements provides a method for recognizing automorphisms of a particular group. In what follows we prove these results (terminology explained in Section 2).

Theorem 1. *If $G = \mathbb{Z}_m * \mathbb{Z}_n$ and ϕ is a monomorphism of G then the stable image of ϕ is a free factor of G.*

Corollary 2. *The test words for monomorphisms of G are those words of infinite order—i.e., those words not lying in a proper free factor.*

Theorem 2. *If $G = \mathbb{Z}_m * \mathbb{Z}_n$ and ϕ is an endomorphism of G then the stable image of ϕ is a retract of G.*

Corollary 3. *The test words of G are those words not lying in a proper retract.*

Specific examples of test words in a free product of two finite cyclic groups are given in Section 5. It should be noted that Turner [4] has proven the results listed above for the case when G is a finitely generated free group. I would like to acknowledge his contribution to this work as my dissertation advisor.

2. Preliminaries

Definition 1. [1] If $\phi : \Gamma \to \Gamma$ is an endomorphism of an arbitrary group Γ then the *stable image* of ϕ is

$$\phi^\infty(\Gamma) = \bigcap_{i=1}^{\infty} \phi^i(\Gamma), \quad \text{and} \quad \phi_\infty = \phi|\phi^\infty(\Gamma).$$

We shall see that the stable image plays an important part in our investigation of test elements. Suppose that w is a test element in a group Γ. Then w may not lie in a proper retract of Γ since otherwise, there would be a non-automorphism fixing w. Conversely, if w is not a test element then there exists an endomorphism $\phi : \Gamma \to \Gamma$ fixing w so that ϕ is not an automorphism. If Γ is Hopfian then ϕ cannot be a surjection and $\phi^\infty(\Gamma)$ is a proper subgroup containing w. We shall exhibit groups in which $\phi^\infty(\Gamma)$ is actually a proper retract containing w.

As motivation, we first examine the stable image of an endomorphism of a finite group T and provide a retract characterization for test words of T. Recall that a group satisfies the *ascending chain condition on subgroups* (ACC) if every ascending chain of subgroups eventually stabilizes. Clearly every finite group satisfies the ACC.

Lemma 1. *If $\phi : \Gamma \to \Gamma$ and Γ satisfies the ACC then ϕ_∞ is an automorphism.*

Proof. Consider the chain of maps

$$\Gamma \xrightarrow{\phi_1} \phi(\Gamma) \xrightarrow{\phi_2} \phi^2(\Gamma) \to \ldots \to \phi^{k-1}(\Gamma) \xrightarrow{\phi_k} \phi^k(\Gamma) \to \ldots$$

where ϕ_k is the restriction of ϕ to the subgroup $\phi^{k-1}(\Gamma)$. Let $\psi_k = \phi_k\phi_{k-1}\ldots\phi_1 : \Gamma \to \phi^k(\Gamma)$. We have an ascending chain of subgroups $\ker(\psi_1) < \ker(\psi_2) < \ldots < \Gamma$ so there exists an N such that $\ker(\psi_k) = \ker(\psi_N)$ for every $k \geq N$. This shows that the maps ϕ_k are eventually injective and hence ϕ_∞ is also injective. We now show that ϕ_∞ is surjective.

If $g \in \phi^\infty(\Gamma)$ then $g = \phi^n(g_n)$ for every n and for some $g_n \in \Gamma$. Choose N so that $\phi_N : \phi^{N-1}(\Gamma) \to \phi^N(\Gamma)$ is injective. For $n \geq N$ the elements $\phi^{n-1}(g_n)$ are in the subgroup $\phi^{N-1}(\Gamma)$. Furthermore, $\phi_N(\phi^{n-1}(g_n)) = g$ and by injectivity we get the equations

$$\phi^{N-1}(g_N) = \phi^N(g_{N+1}) = \phi^{N+1}(g_{N+2}) = \ldots$$

which means that $\phi^{N-1}(g_N) \in \phi^\infty(\Gamma)$ and that $\phi_\infty(\phi^{N-1}(g_N)) = \phi^N(g_N) = g$. □

Proposition 1. *If T is a finite group then $w \in T$ is a test element if and only if w does not lie in a proper retract of T.*

Proof. Suppose that w is not a test element and that ϕ is a non-automorphism fixing w. Since T is finite there exists an N such that $\phi^k(T) = \phi^N(T)$ for all $k \geq N$.

https://doi.org/10.1017/S0013091500024019 Published online by Cambridge University Press

Thus $\phi^\infty(T) = \phi^N(T)$ giving a retraction

$$T \xrightarrow{\phi^N} \phi^N(T) \xrightarrow{id} \phi^\infty(T) \xrightarrow{(\phi_\infty^{-1})^N} \phi^\infty(T).$$

Since ϕ is not surjective, $\phi^\infty(T)$ is a proper retract of T containing w. □

If Γ is any group containing a test element w then w cannot lie in a proper retract of Γ; in particular, the cyclic subgroup $\langle w\rangle$ cannot be a proper retract. The next proposition shows how to decide if a given element generates such a retract. We denote the exponent sum of an element w on a generator x_j by $|w|_{xj}$.

Proposition 2. *Suppose that Γ has the presentation*

$$\Gamma = \langle x_1, x_2, \ldots, x_s \mid r_1, r_2, \ldots, r_t\rangle$$

and $w \in \Gamma$. Let $|R|_X$ denote the $t \times s$ matrix whose ij^{th} entry is $|r_i|_{xj}$ and let $|w|_X$ denote the $1 \times s$ exponent sum vector of w on the generators x_j. Then w generates a retract of Γ if and only if there exists a solution to the equation:

$$\begin{pmatrix} |R|_X \\ -- \\ |w|_X \end{pmatrix} \begin{pmatrix} k_1 \\ \vdots \\ k_s \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

If w has finite order then this is an equation over $\mathbb{Z}_n$ where n is the order of w, otherwise it is over $\mathbb{Z}$ and $n = 0$.

Proof. Suppose that $\rho : \Gamma \to \langle w\rangle$ by $\rho(x_i) = w^{k_i}$. Then r_j is mapped to w^{l_j} where $l_j = k_1|r_j|_{x_1} + \ldots + k_s|r_j|_{x_s}$. Since ρ is a homomorphism, $l_j = 0 \pmod{n}$. The element w is mapped to w^l where $l = k_1|w|_{x_1} + \ldots + k_s|w|_{x_s}$. Since ρ is a retraction, $l = 1 \pmod{n}$. This argument reverses proving the converse. □

Corollary 1. *Suppose that G is a quotient of the free group of finite rank $F(X)$ admitting the presentation $G = \langle X \mid R\rangle$ where $R \subset [F, F]$. If $w \in G$ has infinite order then w generates a retract of G if and only if the entries of $|w|_X$ are relatively prime.*

Example 1. Suppose that $T = \langle x_1, x_2 \mid x_1^2, x_2^8, [x_1, x_2]\rangle$. Since T is abelian the retracts of T are precisely the direct factors of T. Any proper direct factor of T is cyclic (this is, in general, not true for any finite abelian group of rank 2). We first check which elements of T generate proper retracts. Choose $w \in T$ and suppose that $w = x_1^s x_2^t$ where $s \in \mathbb{Z}_2$ and $t \in \mathbb{Z}_8$. If $s = 0$ or $t = 0$ then w lies in a proper retract so we may ignore these cases. By Proposition 2, w generates a retract if and only if there exists a solution vector K over $\mathbb{Z}_{|w|}$ to the equation

$$\begin{pmatrix} 2 & 0 \\ 0 & 8 \\ 0 & 0 \\ s & t \end{pmatrix} \mathrm{K} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

This happens if and only if $w \neq x_1 x_2^2$ and $w \neq x_1 x_2^6$. These are the possible test elements of T. But since these elements are proper powers of only each other then neither lies in a proper retract. Thus, these elements are precisely the test elements of T.

3. Test words for monomorphisms of $\mathbb{Z}_m * \mathbb{Z}_n$

For the remainder of this note G will denote the group $\mathbb{Z}_m * \mathbb{Z}_n$ given the presentation

$$G = \langle x, y \mid x^m, y^n \rangle.$$

Any element w of G is defined by a unique *reduced* word

$$w = x^{a_1} y^{b_1} x^{a_2} y^{b_2} \dots x^{a_r} y^{b_r}$$

where the integers a_i, b_j are reduced modulo m and n respectively. All exponents are nonzero except possibly a_1 and b_r. The *length* of w, denoted $|w|$, is the number of nonzero powers of generators appearing in its reduced form. For example, in the group $\mathbb{Z}_4 * \mathbb{Z}_{13} = \langle x, y \mid x^4, y^{13} \rangle$ the length of $x^2 y^6$ is 2 and the length of $y^{12} x^3 y^{-1}$ is 3.

By the Normal Form Theorem for free products, the only elements of finite order in G are conjugates of powers of the generators x and y. Because of this, any endomorphism ϕ of G has one of the following four forms:

(1)	(2)	(3)	(4)
$x \mapsto gx^k g^{-1}$	$x \mapsto gx^k g^{-1}$	$x \mapsto gy^k g^{-1}$	$x \mapsto gy^k g^{-1}$
$y \mapsto hy^l h^{-1}$	$y \mapsto hx^l h^{-1}$	$y \mapsto hy^l h^{-1}$	$y \mapsto hx^l h^{-1}$

where g and h are arbitrary elements of G. We call the endomorphism in the i^{th} column a *type i* endomorphism, $1 \leq i \leq 4$.

Our main concern will be to prove that $\phi^\infty(G)$ is a retract of G. The previous paragraph suggests a proof of this result by using a case by case analysis on the conjugators g and h. This is in fact our approach. Arguments for type 3 maps are analogous to those for type 2 maps so we will omit mention of type 3 maps in our proofs. Notice that if ϕ is a type 4 map then ϕ^2 is type 1. From the definition of the stable image $\phi^\infty(G)$ it is clear that $\phi^\infty(G) = (\phi^2)^\infty(G)$. Thus to prove that the stable image is a retract we need only concern ourselves with type 1 and type 2 endomorphisms.

Theorem 1. *If $G = \langle x, y \mid x^m, y^n \rangle$ and ϕ is a monomorphism of G then the stable image of ϕ is a free factor of G.*

Proof. By previous comments we can assume that $\phi(x)$ is a conjugate of x^k and $\phi(y)$ is a conjugate of y^l or x^l for some k and l. Since ϕ is injective, k, is relatively prime to m (otherwise $\phi(x^r)=1$ where r is the order of x^k). But then x^k is an element of the multiplicative group of units of $\mathbb{Z}_m$ and $k^s=1 \pmod m$ for some s. Hence $\phi^s(x)$ is a conjugate of x. Again, since $\phi^\infty(G)=(\phi^s)^\infty(G)$ we may assume that ϕ has the form

$$\phi(x)=gxg^{-1}.$$

As for the image of y, if it is a conjugate of a power of y then we may assume as we did for x that

$$\phi(y)=hyh^{-1}.$$

To summarize thus far, we need only consider maps ϕ of one of the following types:

$$(1)\ \begin{array}{l} x\mapsto gxg^{-1}\\ y\mapsto hyh^{-1}\end{array} \qquad (2)\ \begin{array}{l} x\mapsto gxg^{-1}\\ y\mapsto hx^lh^{-1}\end{array}$$

In determining the structure of the stable image $\phi^\infty(G)$ we will first consider the map in the first column (a *type 1* monomorphism). For this, we will need to consider several different possibilities for g and h. In each case we will find that the stable image is either trivial, the entire group G, or a proper free factor. Type 2 monomorphisms will be considered after that.

Case 1: The map ϕ is a type 1 monomorphism, $g=1, h\neq 1$

$$x\mapsto x \qquad y\mapsto hyh^{-1}$$

Clearly we may assume that h does not end in a power of y. Also, if $h\in\langle x\rangle$ then ϕ is an inner automorphism and the stable image is G. Thus $h=\alpha yx^k$ or $h=\beta y^{-1}x^k$ where α and β are some elements of G not ending in y^{-1} or y respectively and $k\neq 0$. These two cases are similar so we shall deal only with the first case. We will not spell out such assumptions for the remainder of this proof. Thus ϕ is defined by

$$x\mapsto x \qquad y\mapsto(\alpha yx^k)y(x^{-k}y^{-1}\alpha^{-1}).$$

In this paragraph we will show that if $w\notin\langle x\rangle$ then $|\phi(w)|>|w|$. If $w=x^{a_1}y^{b_1}x^{a_2}y^{b_2}\dots x^{a_r}y^{b_r}$ then

$$\phi(w)=x^{a_1}\alpha yx^k(y^{b_1})x^{-k}y^{-1}\alpha^{-1}x^{a_2}\alpha yx^k(y^{b_2})x^{-k}y^{-1}\alpha^{-1}\dots x^{a_r}\alpha yx^k(y^{b_r})x^{-k}y^{-1}\alpha^{-1}.$$

When reducing this word the powers of y in parenthesis will never vanish. We may lose some of the original powers of x but only if α begins with a power of x. In this case, a power of x will be contributed by α^{-1}. This shows that $|\phi(w)| > |w|$.

It is clear that if $w \notin \langle x \rangle$ then $\phi(w) \notin \langle x \rangle$. But then we may apply the previous paragraph to such a w to obtain an increasing sequence of integers

$$|w| < |\phi(w)| < |\phi^2(w)| \ldots < |\phi^r(w)| < \ldots$$

Thus if $w \notin \langle x \rangle$ and $w \in \phi^n(G)$ then $|w| > n$ and so $w \notin \phi^\infty(G)$. We conclude that $\phi^\infty(G) = \langle x \rangle$ which is obviously a proper free factor of G.

Case 2: The map ϕ is a type 1 monomorphism, $g \neq 1, h \neq 1$

$$x \mapsto gxg^{-1} \qquad y \mapsto hyh^{-1}$$

Write $g = \alpha u$ and $h = \alpha v$ so that the product $u^{-1}v$ is reduced: α is the "common initial piece" of g and h. If both u and v are nontrivial then it is easy to see that the image of any word $w = x^{a_1}y^{b_1}x^{a_2}y^{b_2}\ldots x^{a_r}y^{b_r}$ grows in length and so $\phi^\infty(G)$ is trivial. The situation becomes more complicated if one of u or v is trivial (if both are trivial then ϕ is simply an inner automorphism). Without loss of generality we may assume that $u = 1$. In this case ϕ has the form

$$x \mapsto \alpha x \alpha^{-1} \qquad y \mapsto (\alpha v)y(v^{-1}\alpha^{-1}).$$

If w is written as above then

$$\phi(w) = \alpha x^{a_1} v y^{b_1}(v^{-1}x^{a_2}v)y^{b_2}v^{-1}\ldots(v^{-1}x^{a_r}v)y^{b_r}v^{-1}\alpha^{-1}.$$

Since the products $v^{-1}x^{a_i}v$ for $i = 2 \ldots r$ can never be powers of y the only possibility that $|\phi(w)| \leq |w|$ is that cancellation occurs at the beginning or end of this word. Whether such a reduction occurs or not depends on the word v. We will show that when $v = x^{-d}\alpha^{-1}x^d$ the stable image $\phi^\infty(G)$ is nontrivial but in all other cases the stable image is trivial.

Suppose that v is as above so that ϕ is defined by

$$x \mapsto \alpha x \alpha^{-1} \qquad y \mapsto (\alpha x^{-d}\alpha^{-1}x^d)y(x^{-d}\alpha x^d\alpha^{-1}).$$

Here, the element $x^d y x^{-d}$ is fixed so the stable image is clearly nontrivial. To determine its structure we simply make a change of basis for the group G. Specifically, let $\bar{x} = x^d y x^{-d}$ and $\bar{y} = x$. These elements are generators for the group and if we let $\bar{\alpha}$ denote the word α rewritten in terms of these new generators then ϕ is the map

$$\bar{x} \mapsto \bar{x} \qquad \bar{y} \mapsto \bar{\alpha}\bar{y}\bar{\alpha}^{-1}.$$

This map was studied in case 1 where we showed its stable image was a proper free factor of G.

In this final paragraph we assume $v \neq x^{-d}\alpha^{-1}x^d$ for any choice of d and show that $\phi^\infty(G)$ is trivial by a length argument. If $w = x^{a_1}y^{b_1}x^{a_2}y^{b_2}\dots x^{a_r}y^{b_r}$ then

$$\phi(w) = \alpha x^{a_1}vy^{b_1}(v^{-1}x^{a_2}v)y^{b_2}v^{-1}\dots(v^{-1}x^{a_r}v)y^{b_r}v^{-1}\alpha^{-1}.$$

It is possible that $|\phi(w)| \leq |w|$ only if $v = x^t\alpha^{-1}x^s$ for some t and s. When $t = -s \pmod m$ we have the situation in the previous paragraph. For all other choices of t and s it may be verified that $|\phi^2(w)| > |w|$. Thus for any element $w \in G$ we have an increasing sequence

$$|w| < |\phi^2(w)| < |\phi^4(w)| < \dots$$

This proves that the stable image of ϕ is trivial.

Case 3: The map ϕ is a type 2 monomorphism

$$x \mapsto gxg^{-1} \qquad y \mapsto hx^lh^{-1}$$

We will not give many details here as the approach is similar to the first two cases. It should be noted that not all type 2 maps are injective but this will not affect our arguments. First, if $g = h$ then $\phi^\infty(G) = \langle gxg^{-1}\rangle$ and there is nothing to show here. If $g = 1$ and $h \neq 1$ then another length argument will show that $w \notin \langle x\rangle$ implies that $|\phi^2(w)| > |w|$ so that $\phi^\infty(G) = \langle x\rangle$. If $g \neq 1$ and $h = 1$ then we may as well assume that g ends in a power of y. If $g = x^{-rl}y^r$ for some r then y^rxy^{-r} is fixed and another change of basis argument will revert this case back to the previous one. For all other choices of g the length of the image of a word is bigger than the length of the word and so $\phi^\infty(G)$ is trivial.

As before, the most complicated case is when both g and h are nontrivial. We write $g = \alpha u$ and $h = \alpha v$ so that $u^{-1}v$ is reduced. If both u and v are not the identity then $\phi^\infty(G)$ is trivial. If $u = 1$ and $v \neq 1$ then we may assume that v ends in a power of y. This is enough to show that the stable image is trivial by checking that $|\phi^2(w)| > |w|$ for every $w \in G$. Finally, assume $u \neq 1$ and $v = 1$. In this case,

$$x \mapsto \alpha uxu^{-1}\alpha^{-1} \qquad y \mapsto \alpha x^l\alpha^{-1}.$$

The situation here is similar to case 2. If $u = x^{-rl}\alpha^{-1}y^r$ for some r then y^rxy^{-r} is fixed and it may be shown by a change of basis that $\phi^\infty(G) = \langle y^rxy^{-r}\rangle$. In all other cases the lengths of words grow under the forward image of ϕ so that $\phi^\infty(G)$ is trivial. □

Corollary 2. *The test words for monomorphisms of G are those words of infinite order—i.e., those words not lying in a proper free factor.*

Proof. If w lies in a proper free factor then w is a power of a generator α since any proper free factor must be cyclic. Let β be a generator of G so that G is equal to the free product $\langle\alpha\rangle * \langle\beta\rangle$. The map

https://doi.org/10.1017/S0013091500024019 Published online by Cambridge University Press

$$\alpha \mapsto \alpha \qquad \beta \mapsto (\beta\alpha)\beta(\alpha^{-1}\beta^{-1})$$

is a monomorphism since $|\phi(g)| \geq |g|$ for every $g \in G$. Furthermore, ϕ is not surjective proving that w is not a test word for monomorphisms. □

4. Test words of $\mathbb{Z}_m * \mathbb{Z}_n$

Now that we have determined that the stable image of a monomorphism of $\mathbb{Z}_m * \mathbb{Z}_n$ is a free factor we may begin to examine the structure of the stable image when ϕ is an arbitrary endomorphism. We will not need to use a case by case analysis this time.

Lemma 2. *If $G = \langle x, y \mid x^m, y^n \rangle$ and ϕ is an endomorphism of G then the map $\phi_\infty : \phi^\infty(G) \to \phi^\infty(G)$ is an automorphism.*

Proof. To show that ϕ_∞ is an automorphism we need only prove that the maps $\phi_k : \phi^{k-1}(G) \to \phi^k(G)$ are eventually injective (see Lemma 1). We consider the chain

$$G \xrightarrow{\phi_1} \phi(G) \xrightarrow{\phi_2} \phi^2(G) \to \dots \to \phi^{k-1}(G) \xrightarrow{\phi_k} \phi^k(G) \to \dots$$

The subgroups $\phi^i(G)$ all have rank less than or equal to 2. By the Kurosh subgroup theorem [2], each subgroup in the chain must be a free product of at most two finite cyclic groups where the ranks on the free factors are divisors of m and n. There are only finitely many such groups up to isomorphism so we may choose $N > M$ with $\phi^N(G) \cong \phi^M(G)$. The composition

$$\phi^M(G) \xrightarrow{\phi_{M+1}} \phi^{M+1}(G) \to \dots \to \phi^{N-1}(G) \xrightarrow{\phi_N} \phi^N(G) \cong \phi^M(G)$$

is a surjective endomorphism of $\phi^M(G)$ and so it must be injective as well. This implies that the maps ϕ_k, $k > M$, are also injective. □

Theorem 2. *If $G = \mathbb{Z}_m * \mathbb{Z}_n$ and ϕ is an endomorphism of G then the stable image of ϕ is a retract of G.*

Proof. By the proof of Lemma 2 we may choose M so that the map $\phi_{M+1} : \phi^M(G) \to \phi^{M+1}(G)$ is a monomorphism. Regardless of the rank of $\phi^M(G)$, the subgroup $(\phi_{M+1})^\infty(\phi^M(G))$ is a retract of $\phi^M(G)$ (Theorem 1 or the proof of Proposition 1). However, it is clear that $\phi^\infty(G) = (\phi_{M+1})^\infty(\phi^M(G))$ so if ρ is the retraction mentioned above the composition

$$G \xrightarrow{\phi^M} \phi^M(G) \xrightarrow{\rho} \phi^\infty(G) \xrightarrow{(\phi_\infty^{-1})^M} \phi^\infty(G)$$

is a retraction of G onto the stable image of ϕ. □

Corollary 3. *The test words of G are those not lying in a proper retract.*

Example 2. Let $G = \langle x, y \mid x^6, y^{12} \rangle$ and suppose that ϕ is the map $x \mapsto x^3, y \mapsto y^4$. Then $\phi^\infty(G) = \phi(G) = \langle x^3, y^4 \rangle$. This gives an example of a proper retract of G which is not a proper free factor (since it has rank 2). In particular, the word x^3y^4 is a test word for monomorphisms but not a test word.

5. Retracts

In light of Corollary 3 it becomes interesting to determine the structure of the retracts of $\mathbb{Z}_m * \mathbb{Z}_n$. Example 2 shows that they are not just the free factors of this group.

Suppose that K is a retract of $G = \langle x, y \mid x^m, y^n \rangle$ and $\rho : G \to K$ is a retraction. Then K must have rank less than or equal to 2. Since we have already described cyclic retracts in general (Proposition 2), we will assume that K has rank 2 for the remainder of this section. Recall that ρ has one of the following forms:

(1) $x \mapsto gx^kg^{-1}$	(2) $x \mapsto gx^kg^{-1}$	(3) $x \mapsto gy^kg^{-1}$	(4) $x \mapsto gy^kg^{-1}$
$y \mapsto hy^lh^{-1}$	$y \mapsto hx^lh^{-1}$	$y \mapsto hy^lh^{-1}$	$y \mapsto hx^lh^{-1}$

Theorem 3. *Suppose that $G = \langle x, y \mid x^m, y^n \rangle$ and that K is a rank 2 retract of G with retraction ρ. Then ρ is type 1, 2, or 3.*

(a) *If ρ is a type 1 retraction then $K = \langle gx^kg^{-1}, hy^lh^{-1} \rangle$ such that*
 (1) $k^2 = k \pmod m$ *and* $l^2 = l \pmod n$, *and*
 (2) $g, h \in \langle\langle x^s, y^t \rangle\rangle$, *the normal subgroup generated by x^s and y^t, where s (resp. t) is the order of x^k (resp. y^l).*

(b) *If ρ is a type 2 retraction then $K = \langle gx^kg^{-1}, hx^lh^{-1} \rangle$ such that*
 (3) $k^2 = k \pmod m$ *and* $kl = l \pmod m$, *and*
 (4) $g \in \langle\langle x^s, y^t \rangle\rangle$ *where s (resp. t) is the order of x^k (resp. x^l).*

(c) *If ρ is a type 3 retraction then $K = \langle gy^kg^{-1}, hy^lh^{-1} \rangle$ such that*
 (5) $kl = k \pmod n$ *and* $l^2 = l \pmod n$, *and*
 (6) $h \in \langle\langle x^s, y^t \rangle\rangle$ *where s (resp. t) is the order of y^k (resp. y^l).*

Proof. Since $\rho^2 = \rho$, this map is clearly not type 4. Therefore, first suppose that ρ is a type 1 retraction given by

$$x \mapsto gx^kg^{-1} \qquad y \mapsto hy^lh^{-1}.$$

Again, since $\rho^2 = \rho$ we have the equation

$$\rho(g)gx^{k^2}g^{-1}\rho(g)^{-1} = gx^kg^{-1}$$

https://doi.org/10.1017/S0013091500024019 Published online by Cambridge University Press

and (1) clearly holds by abelianizing G. Replacing k^2 with k we see that

$$(g^{-1}\rho(g)g)x^k = x^k(g^{-1}\rho(g)g)$$

and a standard result of free products [3] states that the elements $g^{-1}\rho(g)g$ and x^k are in the same conjugate of a free factor of G or are both powers of the same element $w \in G$. In either case $g^{-1}\rho(g)g$ is some power of x. Our map ρ now looks like

$$x \mapsto (\rho(g)^{-1}gx^d)x^k(x^{-d}g^{-1}\rho(g)) = (\rho(g)^{-1}g)x^k(g^{-1}\rho(g))$$

and the conjugator may be assumed to be an element of the kernel. Hence ρ is a map defined by

$$x \mapsto g'x^k(g')^{-1} \qquad y \mapsto h'y^l(h')^{-1}$$

where $g', h' \in \ker(\rho)$.

We will now show that the kernel of ρ is normally generated by elements x^s and y^t as outlined in (2). If s and t are as stated then it is clear that $\langle\langle x^s, y^t\rangle\rangle$ is a normal subgroup of $\ker(\rho)$. We will prove that $\ker(\rho)/\langle\langle x^s, y^t\rangle\rangle$ is trivial. By Kurosh, $\rho(G) \cong \langle gx^kg^{-1}\rangle * \langle hy^lh^{-1}\rangle \cong \mathbb{Z}_s * \mathbb{Z}_t$ and by the Noether Isomorphism Theorems, $\rho(G) \cong G/\langle\langle x^s, y^t\rangle\rangle / \ker(\rho)/\langle\langle x^s, y^t\rangle\rangle$. Furthermore, $G/\langle\langle x^s, y^t\rangle\rangle = \langle x, y \mid x^m, y^n, x^s, y^t\rangle \cong \mathbb{Z}_s * \mathbb{Z}_t$ which finishes the proof in this case.

Now suppose that ρ is a type 2 retraction given by

$$x \mapsto gx^kg^{-1} \qquad y \mapsto hx^lh^{-1}.$$

The proof we give here will obviously work in the case that ρ is a type 3 retraction. Item (3) holds by previous arguments. The element g can be assumed to be an element of $\ker(\rho)$ as before. Finally, if $g \neq h$ then

$$K \cong \langle gx^kg^{-1}\rangle * \langle hx^lh^{-1}\rangle \cong \mathbb{Z}_s * \mathbb{Z}_t$$

and the proof of (2) also holds for (4). If $g = h$ then K is cyclically generated by the element gx^dg^{-1} where $d = gcd(k, l)$ which cannot happen since K has rank 2. □

Example 3. The element $w = xy$ is a test word of $\mathbb{Z}_m * \mathbb{Z}_n$.

It is evident that xy cannot lie in a cyclic retract or a rank 2 retract of type 2 or type 3. If xy is an element of a type 1 retract $K = \langle gx^kg^{-1}, hy^lh^{-1}\rangle$ where $k^2 = k \pmod m$ and $l^2 = l \pmod n$ then $k = l = 1$. This is easily seen by abelianizing G to get the equations $kd_1 = 1 \pmod m$ and $ld_2 = 1 \pmod n$ for some d_i. By Theorem 3, the conjugators $g, h \in \langle\langle x^m, y^n\rangle\rangle$ which implies that K is not proper.

Example 4. The commutator $[x, y]$ is a test word of $\mathbb{Z}_m * \mathbb{Z}_n$.

Every nontrivial element of a cyclic retract $\langle gx^k g^{-1}\rangle$ has nonzero exponent sum on x so it is impossible for such a retract to contain the commutator element.

Suppose

$$\rho : G \to K = \langle gx^k g^{-1}, hx^l h^{-1}\rangle$$

is a type 2 retraction and that $[x, y] \in K$. Then the image of $[x, y]$ is fixed under ρ so that

$$xyx^{-1}y^{-1} = gx^k(g^{-1}h)x^l(h^{-1}g)x^{-k}(g^{-1}h)x^{-l}h^{-1}.$$

If the right hand side of this equation has length 4 then $g^{-1}h$ must be a power of x. But then K would be cyclic and we have dealt with this case already. Clearly type 3 retractions may be disposed of in the same manner.

Finally, suppose

$$\rho : G \to K = \langle gx^k g^{-1}, hy^l h^{-1}\rangle$$

is a type 1 retraction. In case 2 of Theorem 1 we dealt with similar maps where $k = l = 1$. It was shown there that such maps had proper nontrivial stable images if and only if $h = [g, x^d]$, or symmetrically, $g = [h, y^d]$ for some d. The same arguments apply here for the map ρ. But any non-identity map of the form

$$x \mapsto gx^k g^{-1} \qquad y \mapsto [g, x^d]y^l[g, x^d]^{-1}$$

can never fix the commutator, proving that $[x, y]$ does not lie in K.

Example 5. The element $w = x^k y^k$ $(k > 1)$ is not a test word of $\mathbb{Z}_m * \mathbb{Z}_n$ for certain m and n.

The endomorphism of the group $\langle x \mid x^{10}\rangle$ which maps x to x^6 is a retract onto $\langle x^6\rangle$. Thus x^2y^2 lies in a proper retract of $\langle x, y \mid x^{10}, y^{10}\rangle$. If $k \geq 3$ then choose $m = k^2 - k$. In this case, the subgroup $\langle x^k, y^k\rangle$ is a proper retract of $\langle x, y \mid x^m, y^m\rangle$ containing $x^k y^k$.

Example 6. The test words of $\mathbb{Z}_{p_1^m} * \mathbb{Z}_{p_2^n}$ $(p_1 \neq p_2$ are primes) coincide with the test words for monomorphisms.

Suppose x and y are generators for $\mathbb{Z}_{p_1^m} * \mathbb{Z}_{p_2^n}$. We need to show that the retracts of this group are precisely the free factors. Any endomorphisms of this group must be a type 1 map so any retract is a type 1 retract. Let H be a nontrivial proper retract of $\mathbb{Z}_{p_1^m} * \mathbb{Z}_{p_2^n}$ generated by the elements $gx^k g^{-1}$ and $hy^l h^{-1}$ for some k and l. The equation $k^2 = k \pmod{p_1^m}$ implies that $k = 0, 1 \pmod{p_1^m}$. Similarly, $l = 0, 1 \pmod{p_2^n}$. Since H is a nontrivial subgroup, one of k or l is zero but not both. In particular, assuming $k = 0$, $H = \langle hyh^{-1}\rangle$ and is a proper free factor.

Example 7. $PSL(2,\mathbb{Z})$

The special linear group $SL(2,\mathbb{Z})$, of 2×2 integral matrices with determinant 1 is generated by the elements

$$x=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} \quad\text{and}\quad y=\begin{pmatrix} 0 & 1\\ -1 & 1\end{pmatrix}.$$

The modular group $PSL(2,\mathbb{Z})$ is the quotient of $SL(2,\mathbb{Z})$ by $\langle\langle x^2\rangle\rangle$ and has the presentation

$$PSL(2,\mathbb{Z})=\langle x,y \mid x^2,y^3\rangle.$$

It follows that this group contains the test words

$$xy=\begin{pmatrix} -1 & 1\\ 0 & -1\end{pmatrix} \quad\text{and}\quad [x,y]=\begin{pmatrix} 2 & -1\\ -1 & 1\end{pmatrix}.$$

REFERENCES

1. W. Imrich and E. C. Turner, Endomorphisms of free groups and their fixed points, *Math. Proc. Cambridge Philos. Soc* **105** (1989), 421–422.

2. R. Lyndon and P. Schupp, *Combinatorial Group Theory* (Springer-Verlag, 1970).

3. W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory* (Wiley, New York, 1966).

4. E. C. Turner, Test words for automorphisms of free groups, *Bull. London Math. Soc.*, to appear.

Department of Mathematics and Statistics
University at Albany
Albany, New York 12222
USA