SOME PROPERTIES OF BEATTY SEQUENCES II

Ian G. Connell

(received September 9, 1959)

1. Introduction. In a previous paper [1] we discussed a property of the complementary sequences

(1) $u_n = [n(1 + 1/\alpha)]$, $v_n = [n(1+\alpha)]$, $n = 1, 2, 3, \ldots$,

where square brackets denote the greatest integer function and α is any positive irrational. We called $\{u_n\}$ and $\{v_n\}$ Beatty sequences of argument α.

We now discuss some further properties which are connected with simple continued fractions. If $\alpha > 1$ has the continued fraction expansion

(2) $$\alpha = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots = [a_0, a_1, a_2, \ldots]$$

the n^{th} convergent to α is the ordinary fraction

(3) $$p_n/q_n = [a_0, a_1, \ldots, a_n], \qquad n = 0, 1, 2, \ldots .$$

The expansion for $1/\alpha$ is

(4) $$1/\alpha = [0, a_0, a_1, \ldots]$$

and if the convergents to $1/\alpha$ are p_n'/q_n' ,

(5) $$p_0' = 0,\ q_0' = 1,\ p_n' = q_{n-1},\ q_n' = p_{n-1} .$$

If $\alpha_n = [a_n, a_{n+1}, \ldots]$, then

Excerpt from Master of Science thesis, University of Manitoba, 1959.

Can. Math. Bull. vol. 3, no.1, Jan. 1960

https://doi.org/10.4153/CMB-1960-004-2 Published online by Cambridge University Press

(6) $$\alpha = [a_0, a_1, \ldots, a_{n-1}, \alpha_n] .$$

2. The subscript rules.

THEOREM 1. If $\{u_n\}$ and $\{v_n\}$ are the Beatty sequences of argument $\alpha > 1$ and if α has the convergents p_n/q_n, then

(7) $$u_{p_{2n}} = p_{2n} + q_{2n} - 1 ,$$

(8) $$u_{p_{2n+1}} = p_{2n+1} + q_{2n+1} ,$$

(9) $$v_{q_{2n}} = p_{2n} + q_{2n} ,$$

(10) $$v_{q_{2n+1}} = p_{2n+1} + q_{2n+1} - 1$$

for $n = 0, 1, 2, \ldots$.

Proof. By the well known formulas

$$\alpha = p_{2n}/q_{2n} + \gamma_1 , \quad 0 < \gamma_1 < 1/q_{2n}^2 ,$$

$$\alpha = p_{2n+1}/q_{2n+1} - \gamma_2 , \quad 0 < \gamma_2 < 1/q_{2n+1}^2 ,$$

$$1/\alpha = q_{2n}/p_{2n} - \gamma_3 , \quad 0 < \gamma_3 < 1/p_{2n}^2 ,$$

$$1/\alpha = q_{2n+1}/p_{2n+1} + \gamma_4 , \quad 0 < \gamma_4 < 1/p_{2n+1}^2 ,$$

we have

$$u_{p_{2n}} = [p_{2n}(1 + q_{2n}/p_{2n} - \gamma_3)]$$
$$= p_{2n} + q_{2n} - 1$$

since $-1 < -p_{2n}\gamma_3 < 0$. The rest follow just as easily.

NOTE. By equations (5) it is easily seen that the theorem is true for $0 < \alpha < 1$. For this essentially replaces α by $1/\alpha$ which interchanges $\{u_n\}$ and $\{v_n\}$ and (7)-(10) is replaced by an equivalent set.

For example if $\alpha = (1+\sqrt{5})/2$ the p_n and q_n are the Fibonacci numbers $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$ and inspecting Table I in [1] we see that

$$u_5 = 8 , \quad v_5 = 13 ,$$
$$u_8 = 12 , \quad v_8 = 20 , \text{ etc.}$$

The previous proof can be extended to prove the first part of

THEOREM 2. The sequences $\{u_n\}$ and $\{v_n\}$ each contain arithmetic progressions of arbitrary length, but neither contains one of infinite length.

Proof. As above,

$$u_{mp_{2n}} = [\, mp_{2n}(1 + q_{2n}/p_{2n} - \gamma_3\ )]$$
$$= m(p_{2n} + q_{2n}) - 1$$

for $1 \leq m \leq p_{2n}$. Thus $\{u_n\}$ contains an arithmetic progression of length p_{2n}, i.e., of arbitrary length. Similarly for $\{v_n\}$.

But suppose $\{u_n\}$ contained the infinite progression $am + b$, $m = 1, 2, 3, \ldots$. Since $u_n = [\, n(1 + 1/\alpha\ )]$,

$$u_n < n(1 + 1/\alpha\ ) < u_n + 1\ ,$$

or

$$n - \frac{\alpha}{\alpha + 1} < u_n \frac{\alpha}{\alpha + 1} < n\ .$$

That is

$$1 - \frac{\alpha}{\alpha + 1} < r\left\{u_n \frac{\alpha}{\alpha + 1}\right\} < 1,$$

where $r(x) = x - [x]$ is the fractional part of x. Hence

$$(11) \qquad 1 - \alpha/(\alpha + 1) < r\{(am + b)\alpha/(\alpha + 1)\} < 1$$

for all m. But the set of points $r\{ma\alpha\ /(\alpha + 1)\}$ is uniformly distributed in the unit interval since $a\alpha/(\alpha + 1)$ is irrational.

Thus the set

$$r\left\{m\frac{a\alpha}{\alpha + 1} + \frac{b\alpha}{\alpha + 1}\right\}$$

is uniformly distributed in the unit interval in contradiction to (11). The same proof goes through for $\{v_n\}$.

A proof of the statement about uniform distribution can be found in [2], chap. XXIII.

Note that Van der Waerden's theorem on arithmetic progressions (see [3], chap. 1) guarantees only that at least one

https://doi.org/10.4153/CMB-1960-004-2 Published online by Cambridge University Press

of the sequences contains an arithmetic progression of arbitrary length.

3. An algorithm. For an irrational number $\alpha > 1$ define the representation

(12) $$\psi(\alpha) = \{v_1, v_2, v_3, \ldots\}, \quad v_n = [n(1+\alpha)] .$$

It is clear that $\psi(\alpha)$ uniquely determines α (if it is a possible representation) and conversely.

Three obvious properties of the representation are

(13) $$[\alpha] = v_1 - 1 ,$$

(14) $$\psi(\alpha - k) = \{v_1 - k, v_2 - 2k, v_3 - 3k, \ldots\}$$

for any integer k, and

(15) $$\psi(1/\alpha) = \{u_1, u_2, u_3, \ldots\}, \quad u_n = [n(1 + 1/\alpha)] .$$

On the basis of these three properties we shall obtain

(a) an algorithm to determine the continued fraction expansion of α given $\psi(\alpha)$, and

(b) an algorithm for $\psi(\alpha)$ given the continued fraction expansion of α. This inverse algorithm is in effect an algorithm for the Beatty sequences of argument α.

(a) The usual continued fraction algorithm is an iterative procedure, each step consisting of taking an integral part and performing a division. To find the expansion $[a_0, a_1, a_2, \ldots]$ of α we calculate in succession

$$a_0 = [\alpha] , \quad \alpha_1 = 1/(\alpha - a_0)$$

$$a_1 = [\alpha_1] , \quad \alpha_2 = 1/(\alpha - a_1)$$

$$a_2 = [\alpha_2] , \text{ etc.}$$

In the present case we are given $\psi(\alpha) = \{v_1, v_2, \ldots\}$. Hence by (13) $a_0 = v_1 - 1$. Next we calculate $\psi(\alpha - a_0)$ by (14) and get $\psi(1/(\alpha - a_0)) = \{w_1, w_2, \ldots\}$ by the complementarity property (15). Using (13) again, $a_1 = w_1 - 1$. And so the process is iterated.

https://doi.org/10.4153/CMB-1960-004-2 Published online by Cambridge University Press

For example let $\alpha = e = 2.718$. Then $v_n = [3.718\, n]$ and

$$\psi(\alpha) = \{3, 7, 11, 14, 18, 22, 26, 29, 33, 37, 40, 44, 48, 52, 55, 59, \ldots\},$$

$$a_o = 2,$$

$$\psi(\alpha - a_o) = \{1, 3, 5, 6, 8, 10, 12, 13, 15, 17, 18, 20, 22, 24, 25, 27, \ldots\},$$

$$\psi(\alpha_1) = \{2, 4, 7, 9, 11, 14, 16, 19, 21, 23, 26, \ldots\},$$

$$a_1 = 1,$$

$$\psi(\alpha_1 - a_1) = \{1, 2, 4, 5, 6, 8, 9, 11, 12, 13, 15, \ldots\},$$

$$\psi(\alpha_2) = \{3, 7, 10, 14, \ldots\},$$

$$a_2 = 2,$$

$$\psi(\alpha_2 - a_2) = \{1, 3, 4, 6, \ldots\},$$

$$\psi(\alpha_3) = \{2, 5, \ldots\},$$

$$a_3 = 1,$$

$$\psi(\alpha_3 - a_3) = \{1, 3, \ldots\},$$

$$\psi(\alpha_4) = \{2, \ldots\},$$

$$a_4 = 1.$$

Therefore $e = [2, 1, 2, 1, 1, \ldots]$. In fact $e = [2, \overline{1, 2n, 1}]_{n=1}^{\infty}$ (c.f. [4], p. 134).

This algorithm avoids divisions with many significant figures. The disadvantage lies in having to calculate a large number of the v_n; but this becomes an advantage in the inverse algorithm.

(b) We are given $\alpha = [a_o, a_1, \ldots]$. Now $\alpha_n = [a_n, \ldots]$ and $\psi(\alpha_n) = [a_n + 1, \ldots] = \psi(1/(\alpha_{n-1} - a_{n-1}))$. Taking the complement we get $\psi(\alpha_{n-1} - a_{n-1}) = [1, 2, \ldots, a_n, \ldots]$. Hence $\psi(\alpha_{n-1}) = [1 + a_{n-1},\ 2 + 2a_{n-1}, \ldots]$. Iterating we finally arrive at $\psi(\alpha_o) = \psi(\alpha)$, the required result.

https://doi.org/10.4153/CMB-1960-004-2 Published online by Cambridge University Press

For example let $\alpha = \pi = [3,7,15,1,292,\ldots]$.

$$\psi(\alpha_2) = \{16,\ldots\}$$

$$\psi(\alpha_1 - 7) = \{1,2,3,\ldots,15,\ldots\}$$

$$\psi(\alpha_1) = \{8,16,24,\ldots,120,\ldots\}$$

$$\psi(\pi - 3) = \{1,2,3,4,5,6,7,9,10,11,\ldots\}$$

$$\psi(\pi) = \{4,8,12,16,20,24,28,33,37,41,\ldots\}$$

We have omitted most of the final answer in order to conserve space. The number of terms this method yields is often remarkable. Starting with $\alpha = \pi$, $\psi(\alpha_4) = \{293,\ldots\}$ one obtains $v_1 = 4,\ldots,v_{32988} = 136,622$, whence $u_1 = 1,\ldots,u_{103634} = 136,621$.

This algorithm fails for the one number $(1+\sqrt{5})/2$ and numbers equivalent ([2], chap. X) to it.

REFERENCES

1. I. Connell, Some properties of Beatty sequences I, Can. Math. Bull. 2(1959), 190-197.

2. G.H. Hardy and E.M. Wright, An Introduction to the Theory of Numbers, (Oxford, 1945).

3. A.Y. Khinchin, Three Pearls of Number Theory, (New York, 1952).

4. O. Perron, Kettenbrüche, (New York, 1950).

C.A.R.D.E., Quebec.