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ON A CLASS OF GENERALIZED SOLUBLE $\mathfrak{T}$-GROUPS

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Abstract. We study soluble groups in which every subgroup lying between a characteristic subgroup and its derived subgroup is normal.

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1. Introduction. Let $\mathfrak{T}_{\mathfrak{B}}$ denote the class of all groups G with the following property: *whenever C and D are characteristic subgroups of G such that $D \leq C$ and C/D is abelian, then each subgroup K of G such that $D \leq K \leq C$ is normal in G.* In other words G belongs to the class $\mathfrak{T}_{\mathfrak{B}}$ if and only if every subgroup lying between a characteristic subgroup and its derived subgroup is normal in G. The well known class of $\mathfrak{T}$-groups (i.e. groups in which each subnormal subgroup is normal) is contained in $\mathfrak{T}_{\mathfrak{B}}$.

The class $\mathfrak{T}_{\mathfrak{B}}$ was introduced by R. A. Bryce in [**1**] as a tool for the study of some generalized Wielandt subgroups. Recall that the Wielandt subgroup $w(G)$ of a group G is the normalizer of all the subnormal subgroups of G. It is easy to see that in any group G, $w(G)$ is a $\mathfrak{T}$-group. For each $m \geq 2$, Bryce defined the subgroup $u_m(G)$ as the normalizer of all the subnormal subgroups with defect at most m in G. He observed that $u_m(G)$ is still a $\mathfrak{T}$-group when $m \geq 3$, while it is a $\mathfrak{T}_{\mathfrak{B}}$-group if $m = 2$. Then he studied finite soluble $\mathfrak{T}_{\mathfrak{B}}$-groups, proving that their structure is quite restricted. From the definition it follows immediately that soluble $\mathfrak{T}_{\mathfrak{B}}$-groups are hypercyclic and hence locally supersoluble by a theorem of Baer (see [**16**]). Moreover by [**1**, Theorem 4.7] and [**7**, Lemma 4.1], if G is a soluble $\mathfrak{T}_{\mathfrak{B}}$-group, then G' is nilpotent of class at most 2. Thus in particular G has derived length at most 3. This bound is really achieved, as Bryce showed (see [**1**, p. 242–243]).

The aim of this paper is to continue the study of soluble $\mathfrak{T}_{\mathfrak{B}}$-groups. Note that for the study of infinite soluble $\mathfrak{T}_{\mathfrak{B}}$-groups we follow the pattern established by D. J. S. Robinson in [**13**] for the study of infinite soluble $\mathfrak{T}$-groups.

In Section 2 some elementary facts about soluble $\mathfrak{T}_{\mathfrak{B}}$-groups are proved. In Section 3 we consider periodic soluble $\mathfrak{T}_{\mathfrak{B}}$-groups. In Theorem 3.3 we prove that *soluble p-groups of the class $\mathfrak{T}_{\mathfrak{B}}$ are metabelian and they are even nilpotent of class at most* 2 *if $p > 2$.* On the other hand a soluble 2-group of the class $\mathfrak{T}_{\mathfrak{B}}$ need not be nilpotent and furthermore there is no bound for the nilpotency class of nilpotent 2-groups of the class $\mathfrak{T}_{\mathfrak{B}}$ (see Example 1 and Example 2). As an example of how to work with p-groups of the class $\mathfrak{T}_{\mathfrak{B}}$, in Proposition 3.7 we determine all 2-generator non-abelian finite p-groups of the class $\mathfrak{T}_{\mathfrak{B}}$.

The structure of generic periodic soluble $\mathfrak{T}_{\mathfrak{B}}$-groups is described by Theorem 3.8. In Theorem 3.9 finite soluble $\mathfrak{T}_{\mathfrak{B}}$-groups are characterized as particular semidirect products of finite nilpotent $\mathfrak{T}_{\mathfrak{B}}$-groups.

Non-periodic soluble $\mathfrak{T}_{\mathfrak{B}}$-groups are studied in Section 4. In Proposition 4.1 we show that *non-periodic nilpotent* $\mathfrak{T}_{\mathfrak{B}}$*-groups have class at most* 2. Non-nilpotent non-periodic soluble $\mathfrak{T}_{\mathfrak{B}}$-groups are described in Theorems 4.2 and 4.4. We have two different pictures according as the derived subgroup is periodic or not. In particular from Theorem 4.2 it follows that *a torsion-free soluble* $\mathfrak{T}_{\mathfrak{B}}$*-group is nilpotent of class at most* 2. From Theorem 4.4 we deduce that *a finitely generated soluble* $\mathfrak{T}_{\mathfrak{B}}$*-group with periodic derived subgroup is finite* (see Theorem 4.5).

A complete classification of soluble $\mathfrak{T}_{\mathfrak{B}}$-groups seems difficult to achieve, since characteristic subgroups, and hence automorphisms, are involved. We show with many examples that the results obtained actually give a realistic picture of the class of soluble $\mathfrak{T}_{\mathfrak{B}}$-groups.

For notation and terminology we refer mostly to [**14**, **15**]. By a p-divisible group, p a prime number, we mean a group in which every element has a pth root. A divisible group is a p-divisible group for every prime p.

2. Some elementary properties of $\mathfrak{T}_{\mathfrak{B}}$-groups. Recall that a power automorphism of a group is an automorphism which fixes every subgroup (see [**3**]). Therefore a group G belongs to $\mathfrak{T}_{\mathfrak{B}}$ if and only if it induces by conjugation power automorphisms on each of its characteristic abelian sections. If A is an abelian group and $P\mathrm{Aut}A$ denotes the group of power automorphisms of A, then $P\mathrm{Aut}A \leq Z(\mathrm{Aut}A)$ (see [**3**]). It follows that if G is a $\mathfrak{T}_{\mathfrak{B}}$-group, then G' centralizes every characteristic abelian section of G. Moreover note that the class $\mathfrak{T}_{\mathfrak{B}}$ is closed for characteristic subgroups and factor groups by characteristic subgroups.

We collect in the next lemma some known facts which will be repeatedly used throughout the paper.

LEMMA 2.1. *Let G be a group.*

(i) *If* $\alpha \in \mathrm{Aut}G$ *acts as a power automorphism on* G/G'*, then* α *acts as a power automorphism on each factor of the lower central series of G. In particular, if* α *has form* $x \mapsto x^n$ *on* G/G'*, then* α *has form* $x \mapsto x^{n^i}$ *on* $\gamma_i(G)/\gamma_{i+1}(G)$*, for each i.*

(ii) *If H is a normal nilpotent subgroup of G with class c and G/H is abelian, then* $C_G(H/H')$ *is nilpotent with class at most* $c+1$.

(iii) *If G is soluble and it centralizes its own derived series, then it is nilpotent with class at most* 2.

(iv) *Let A be an abelian group and let* α *be a power automorphism of A. If A is non-periodic, then* α *is either the identity or the inversion map. If A is a p-group of finite exponent, there is a positive integer l such that* $a^\alpha = a^l$ *for all a in A. If* α *is non-trivial and has order prime to p, then* α *is fixed-point-free.*

Proof. Point (i) is the content of Lemma 3 in [**5**]. To prove (ii) let H be a normal nilpotent subgroup of the group G, with class c and such that G/H is abelian. Then by a result of P. Hall [**9**] (or also by (i)) $C_G(H/H') \geq H \geq H' \geq \gamma_3(H) \geq \cdots \geq \gamma_c(H) \geq 1$ is a central series of $C_G(H/H')$ and (ii) follows. For (iii) note that if a soluble group G centralizes its own derived series, then it is nilpotent and $\gamma_3(G) = [G, G'] \leq G'' \leq \gamma_4(G) = 1$ (see [**7**, Lemma 4.1]). Point (iv) is the content of [**14**, 13.4.3]. □

The next lemma is the analogue of [**14**, Lemma 2.3.2].

LEMMA 2.2. *Let G be a* $\mathfrak{T}_{\mathfrak{B}}$*-group and let* $S = C_G(G^{(3)})$*. Then*

(i) *the derived series of G terminates with* $G^{(3)}$ *and* $G^{(3)} = \gamma_3(G')$*;*

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(ii) *S is the unique maximal characteristic soluble subgroup of G; S' is nilpotent with class at most* 2 *and* $[S, G'', G'] = 1$.

Proof. Part (i) follows at once from the fact that the derived subgroup of a soluble $\mathfrak{T}_{\mathfrak{B}}$-group is nilpotent with class at most 2 ([**1**, Theorem 4.7], [**7**, Lemma 4.1]). For the second part note that S' stabilizes the series $G \geq G' \geq G'' \geq G^{(3)} \geq 1$ and so it is nilpotent by a theorem of P. Hall [**9**]. Therefore S is a soluble $\mathfrak{T}_{\mathfrak{B}}$-group and S' has class at most 2. Let H be a characteristic soluble subgroup of G. Then $H \in \mathfrak{T}_{\mathfrak{B}}$, $H^{(3)} = 1$ and G' stabilizes the derived series of H. Hence $[H, G', G'] \leq H''$. Thus, by the Three Subgroups Lemma [**14**, 5.1.10], $[H, G''] \leq H''$, whence $[H, G'', G'] = 1$ and $[H, G^{(3)}] = 1$. It follows that S is the unique maximal characteristic soluble subgroup of G and $[S, G'', G'] = 1$. □

From now on we shall consider only soluble $\mathfrak{T}_{\mathfrak{B}}$-groups.

LEMMA 2.3. *Let G be a soluble $\mathfrak{T}_{\mathfrak{B}}$-group, and let L be the intersection of the terms of the lower central series of G. Then L and G'' are* 2*-divisible groups.*

Proof. Each characteristic abelian section of G with exponent 2 is a central section of G. Therefore $L^2 = L$. Moreover L is nilpotent, since $L \leq G'$, and thus it is 2-divisible by Corollary 1 to Theorem 9.23 in [**15**, Part 2]. Now let A be a characteristic subgroup of G. Then G acts on A/A' by means of power automorphisms and $A'/(A')^2$ is a central section of G. Consider $x \in A/(A')^2$, $g, h \in G$. Then there exist $\lambda, \mu \in \mathbb{Z}$ and $z_i \in A'/(A')^2$, $i = 1, 2$, such that $x^g = x^{\lambda} z_1$ and $x^h = x^{\mu} z_2$. Now if λ or μ is even, then $x \in A^2 A'/(A')^2$, a characteristic abelian section of G, and so $x^{[g,h]} = x$. Thus we may assume λ, μ odd, and we have:

$$x^{gh} = (x^{\lambda} z_1)^h = (x^{\mu} z_2)^{\lambda} z_1 = x^{\lambda\mu} z_2 z_1$$

and

$$x^{hg} = (x^{\mu} z_2)^g = (x^{\lambda} z_1)^{\mu} z_2 = x^{\lambda\mu} z_1 z_2 = x^{\lambda\mu} z_2 z_1$$

whence $x^{[g,h]} = x$. Hence $G' \leq C_G(A/(A')^2)$. Now if we choose $A = G'$ we get that $G' \leq C_G(G'/(G'')^2)$ and $G'' = [G', G'] \leq (G'')^2$. Since G'' is nilpotent, as above we conclude that G'' is 2-divisible. □

LEMMA 2.4. *Let G be a soluble $\mathfrak{T}_{\mathfrak{B}}$-group, and let K be a characteristic abelian subgroup of G with finite exponent. If G/K is a non-periodic abelian group, then $K \leq Z(G)$.*

Proof. Set $H = C_G(K)$. Then by [**13**, Section 4.1.3], H has finite index in G. Hence H is a non-periodic nilpotent group such that $H' \leq K \leq Z(H)$. Since K has finite exponent, it follows that $H/Z(H)$ has finite exponent and thus $Z(H)/K$ is a non-periodic central section of G. Then by Lemma 2.1 (iv), $Z(H) \leq Z(G)$ and so $K \leq Z(G)$ as claimed. □

3. Periodic soluble $\mathfrak{T}_{\mathfrak{B}}$-groups. In this section we describe the structure of periodic soluble $\mathfrak{T}_{\mathfrak{B}}$-groups. Extending a notation generally used for finite groups, if G is a p-group we denote by $\Omega_n(G)$, $n \geq 1$, the subgroup of G generated by the elements of order at most p^n.

LEMMA 3.1. *Let $G \in \mathfrak{T}_{\mathfrak{B}}$ be a nilpotent p-group, for some prime p, of class at most 2. Then G' is an elementary abelian p-group.*

Proof. Suppose that G' has finite exponent and in order to obtain a contradiction let us assume that G' has exponent greater than p. Taking eventually the quotient over $(G')^{p^2}$, we may assume directly that G' has exponent p^2. Then $[g^p, h^p] = [g, h]^{p^2} = 1$ for all $g, h \in G$ and so G^p is abelian. Set $H = G^pG'$ and $C = C_G(H)$. Clearly H is a characteristic abelian subgroup of G. If $C = G$, then $G^p \le Z(G)$ and hence G' has exponent at most p. So $C < G$. Then G acts non-trivially on H as a group of power automorphisms: in particular if $p = 2$, then each element of G acts on H as a power congruent to 1 modulo 4; thus [**13**, Section 4.1.3] implies that H has finite exponent and G/C is a cyclic group. Therefore $G = \langle x, C\rangle$, where C' has exponent at most p and $x^p \in H \le Z(C)$. Then for each $g, h \in G$ we can write $g = x^\alpha c_1$, $h = x^\beta c_2$, for some $c_1, c_2 \in C$, $1 \le \alpha, \beta \le p$ and thus $[g, h]^p = [x^\alpha c_1, x^\beta c_2]^p = [x^{p\alpha}, c_2][c_1, c_2,]^p[c_1, x^{p\beta}] = 1$. Hence G' has exponent at most p and we get the contradiction.

Now suppose that G' has infinite exponent. For each $n \ge 1$, set $H_n/G' = \Omega_n(G/G')$. Then H_n is a nilpotent p-group of the class $\mathfrak{T}_{\mathfrak{B}}$ with class at most 2 and $H_n^{p^n} \le Z(H_n)$. Hence H_n' has exponent at most p^n, and thus by the previous paragraph H_n' has exponent at most p. It follows that $G/\Omega_1(G')$ is abelian and we get a contradiction. □

We extend now Lemma 4.4 in [**1**] to soluble p-groups with the derived subgroup of finite exponent.

LEMMA 3.2. *Let $G \in \mathfrak{T}_{\mathfrak{B}}$ be a soluble p-group for some prime p and let G' have finite exponent. Then G is nilpotent. Furthermore either G has class at most 2 and G' has exponent at most p, or $p = 2$, $\gamma_2(G)/\gamma_3(G)^2$ has exponent 4 and there exists an element $x \in G$ which acts by conjugation on $\gamma_2(G)/\gamma_3(G)^2$ as the inversion map.*

Proof. First of all note that by [**13**, Section 4.1.3], each characteristic abelian section with exponent p is a central section of G. Therefore we can refine the derived series of G into a finite central series, and G is nilpotent. Set $\overline{G} := G/\gamma_3(G)^p\gamma_4(G)$. Then $\overline{G} \in \mathfrak{T}_{\mathfrak{B}}$, $\overline{G}' = \gamma_2(G)/\gamma_3(G)^p\gamma_4(G)$ is abelian and $\gamma_3(\overline{G})$ has exponent at most p. Moreover by Lemma 3.1, $\overline{G}'/\gamma_3(\overline{G})$ has exponent at most p. Thus $\overline{G}'$ has exponent at most p^2. Now if $p > 2$, for all $g, h \in \overline{G}$ we have

$$[g^p, h] = [g, h]^p[g, h, g]^{\binom{p}{2}} = [g, h]^p.$$

Then the first part of the proof of Lemma 3.1 still holds, proving that $\overline{G}'$ has exponent at most p. It follows that $\overline{G}' \le Z(\overline{G})$ and $\gamma_3(G) = \gamma_3(G)^p\gamma_4(G) = 1$.

Therefore if $\gamma_3(G) \ne 1$, then $p = 2$, $\overline{G}' = \gamma_2(G)/\gamma_3(G)^2$ and it is a non-central characteristic abelian subgroup of $\overline{G}$ with exponent 4. From the structure of the group of power automorphisms it follows that there exists an element $x \in G$ acting on $\overline{G}'$ as the inversion map. □

THEOREM 3.3. *Let $G \in \mathfrak{T}_{\mathfrak{B}}$ be a soluble p-group for some prime p. Then G is metabelian and either*

(i) *G is nilpotent of class at most 2 and G' has finite exponent at most p, or*

(ii) *$p = 2$ and $G = \langle x, C\rangle$ where $C = C_G(G')$, $x^2 \in Z(G)$ and x acts on G' as the inversion map. Moreover $C' \le Z(G)$ and $Z(G)$ has exponent at most 2.*

Proof. Let p be greater than 2. If G'/G'' has infinite exponent, then [**13**, Section 4.1.3] yields that G'/G'' is a central section of G. Hence by Lemma 2.1 (i), G centralizes the series $G' \geq G'' \geq 1$ and so it is nilpotent with class at most 2, by Lemma 2.1 (iii). But then Lemma 3.1 implies that G' has exponent at most p and we get a contradiction. Thus if $G \in \mathfrak{T}_{\mathfrak{B}}$ is a soluble p-group with p odd, then G' has finite exponent and (i) follows from Lemma 3.2.

Now let $p = 2$. Since G' is nilpotent of class at most 2, Lemma 3.1 implies that G'' has exponent at most 2. On the other hand G'' is a divisible group by Lemma 2.3. Hence $G'' = 1$ and G' is abelian. Moreover if G is nilpotent of class at most 2, then Lemma 3.1 implies (i) and we are done. So let us assume that G' is not contained in $Z(G)$ and let us prove that (ii) holds. Note that since C is nilpotent of class at most 2, C' has exponent at most 2 by Lemma 3.1 and thus $C' \leq Z(G)$. Furthermore $G'Z(G)$ is an abelian characteristic non-central subgroup of G and hence $Z(G)$ has exponent at most 2 provided there is an element of G acting on G' as the inversion map. Therefore we have only to show that $G = \langle x, C\rangle$, where $x^2 \in Z(G)$ and x acts on G' as the inversion map. If G' has infinite exponent, then by [**13**, Section 4.1.3], $|G : C| = 2$ and thus $G = \langle x, C\rangle$, where $x^2 \in C$ and x acts on G' as the inversion map. For each $c \in C$ we have $[c, x^2] = [c, x][c, x]^x = [c, x][c, x]^{-1} = 1$, whence $x^2 \in Z(G)$. If G' has finite exponent 2^n $(n \geq 2)$, then by Lemma 3.2 $G'/\gamma_3(G)^2$ has exponent 4 and there exists an element $x \in G$ such that x acts by conjugation on $G'/\gamma_3(G)^2$ as the inversion map. Set $D := C_G(G'/\gamma_3(G)^2)$. Then by [**13**, Section 4.1.3], $|G : D| = 2$. Each element of D acts on G', and so on D', as a power congruent to 1(mod 4); hence D has class at most 2 by Lemma 3.2. Then by Lemma 3.1 D' has exponent at most 2 and thus the proof of Lemma 2.3 (with D in the place of A) yields that $G' \leq C_G(D)$. Therefore, since $G' \not\leq Z(G)$, we have $D = C$. Clearly $x \notin C$, $G = \langle x, C\rangle$ and x acts on G' as a power congruent to -1 modulo 4. We claim that x acts on G' as the inversion map. This is trivial if $n = 2$, so assume $n > 2$. Since $x^2 \in C$, we have that x induces by conjugation on G' and on C/C' power automorphisms of order 2. Hence x acts on $G'/\Omega_1(G')$ as the inversion map and on $C/\Omega_1(G')$ (note that $C' \leq \Omega_1(G')$) as a power $-1 + \beta 2^{t-1}$, where $\beta \geq 0$ and 2^t is the exponent of $C/\Omega_1(G')$. Now $C^2 \leq G'$, since x acts on the central section C/G' as a power congruent to -1 modulo 4. Then $C/\Omega_2(G')$ has exponent at most 2^{t-1} and so it follows that for each $y \in C$, $(xy)^2 \in \langle x^2, \Omega_2(G')\rangle = T$. Therefore T is a characteristic subgroup of G, as it is generated by the characteristic subgroup $\Omega_2(G')$ and by the squares of the elements outside C, and it is also abelian. This yields that x^2 has order at most 2, as x acts on T as a power automorphism which is the inversion map on $\Omega_2(G')$. Thus since $\langle x^2\rangle$ is a normal subgroup of G, $x^2 \in Z(G)$. Then for each $c \in C$ we have $1 = [c, x^2] = [c, x][c, x]^x$, whence $[c, x]^x = [c, x]^{-1}$ and x acts as the inversion map on $[C, x]$. From $G' = [C, x]C'$ it follows that x acts as the inversion map on G', as wanted. □

We give now two examples which show that when $p = 2$ the result of Theorem 3.3 is best possible.

Example 1. Let $G = \langle x, A\rangle$, where A is abelian, $x^2 \in A$ and $a^x = a^{-1}$ for each $a \in A$. For each $b \in A$ there is an automorphism φ_b of G such that $x^{\varphi_b} = xb$ and $a^{\varphi_b} = a$, $a \in A$. It follows that every characteristic proper subgroup of G is contained in A and hence G acts on it as the inversion map. Thus G is a soluble $\mathfrak{T}_{\mathfrak{B}}$-group. In particular if A is a cyclic 2-group of order 2^n we get a finite nilpotent 2-group with class n. This shows that there is no bound for the nilpotency class of a finite 2-group of the class $\mathfrak{T}_{\mathfrak{B}}$. (Note that Example 5.5 in [**1**] shows that in general there is no bound for the nilpotency class of Sylow 2-subgroups of a finite soluble $\mathfrak{T}_{\mathfrak{B}}$-group of composite order.)

EXAMPLE 2. Let $Q = \langle a, b \mid a^4 = 1, b^2 = a^2, a^b = a^{-1}\rangle$ be the quaternion group of order 8 and let A be a quasicyclic 2-group. Let C be the central product of Q and A, where Q' and $\Omega_1(A)$ are amalgamated. Thus $C' = \langle a^2\rangle$. Set $G = \langle C, z\rangle$, where $z^2 = a^2$, z acts trivially on Q and as the inversion map on A. Clearly G is a non-nilpotent soluble 2-group with infinite exponent, $G' = A$, $Z(G) = C'$ and $C = C_G(G')$. We prove that G is a $\mathfrak{T}_{\mathfrak{B}}$-group. Let K be a characteristic subgroup of G. If K is not contained in C, then K contains an element acting on A as the inversion map. It follows that $G' = A = [A, K] \leq K$ and so $G' \leq [K, K] = K'$. Hence K/K' is a central section of G. So assume that $K \leq C$. If $K \leq Z(C) = AC'$, then K is abelian and G acts on it as the inversion map. Hence suppose that K is not contained in $Z(C)$. Then K contains an element $y = cx$, where $c \in Q$ is an element of order 4 and $x \in A$. Thus there exists an automorphism $\varphi \in \mathrm{Aut}Q$ such that $[c, c^\varphi] \neq 1$. Let us extend φ to an automorphism $\bar\varphi \in \mathrm{Aut}G$ by defining $q^{\bar\varphi} = q^\varphi$ for each $q \in Q$, $x^{\bar\varphi} = x$ for each $x \in A$ and $z^{\bar\varphi} = z$. Then $y^{\bar\varphi} = c^\varphi x \in K$ and thus $C' \leq K'$. Hence K/K' is a section of C/C' and thus G acts on K/K' as the inversion map.

Free groups of countable rank in the variety of nilpotent p-groups of class at most 2, exponent at most p^n, $n > 0$, and derived subgroup of exponent at most p, are soluble p-groups of the class $\mathfrak{T}_{\mathfrak{B}}$, as the following proposition shows.

PROPOSITION 3.4. *Let p be a prime and n a natural number. Every free group of countable rank in the variety of nilpotent p-groups of class at most 2, exponent at most p^n and derived subgroup of exponent at most p is a $\mathfrak{T}_{\mathfrak{B}}$-group.*

Proof. Let G be a free group of countable rank in the variety of nilpotent p-groups of class at most 2, exponent at most p^n and derived subgroup of exponent at most p. Then G/G' is the direct product of cyclic groups of order p^n and its characteristic subgroups are the subgroups $(G/G')^{p^k}$, $k = 0, \dots, n$. Since by [**12**, Theorem 4] each automorphism of G/G' is induced by an automorphism of G, it follows that if K is a characteristic subgroup of G not contained in G', then either $KG' = G$, and thus $K' = G'$, or $KG' \leq G^pG' \leq Z(G)$. Therefore in any case K/K' is a central section of G and G is a $\mathfrak{T}_{\mathfrak{B}}$-group. □

We want now to describe finite p-groups of exponent p in the class $\mathfrak{T}_{\mathfrak{B}}$, $p > 2$. Recall that a finite non-abelian p-group G is called a *special* p-group if $Z(G) = G' = \Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G, and G' is elementary abelian (see [**10**]). Moreover a p-group G is called *extra-special* if it is a special p-group and G' is cyclic.

LEMMA 3.5. *Let G be a finite group of exponent p in the class $\mathfrak{T}_{\mathfrak{B}}$, $p > 2$. Then G is the direct product of an elementary abelian p-group A and of a special p-group H belonging to $\mathfrak{T}_{\mathfrak{B}}$.*

Proof. Since $p > 2$, by Theorem 3.3 $G' \leq Z(G)$. Let A be a complement of G' in $Z(G)$ and let $H \geq G'$ be such that H/G' is a complement of $Z(G)/G'$ in G/G'. Then $G = HA$. Since $A \leq Z(G)$, $G' = H'$ and $H \cap A \leq G' \cap A = 1$. Thus G is the direct product of A and H; the equality $H'A = Z(G) = Z(H)A$ implies that $H' = Z(H)$.

Note that for each subgroup K of H, we have that $K/K' \leq KZ(H)/K' \leq KZ(G)/K' = KZ(G)/(KZ(G))'$. Therefore to prove that $H \in \mathfrak{T}_{\mathfrak{B}}$ it is enough to show that $KZ(G)$ is a characteristic subgroup of G whenever K is a characteristic subgroup

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of H containing $Z(H)$. So let $K \geq Z(H)$ be a characteristic subgroup of H. If σ is an automorphism of G, then A^σ is a complement of G' in $Z(G)$; thus G is the direct product of H and A^σ and there exists an automorphism $\tau \in \mathrm{Aut}G$ such that $h^\tau = h$ for each $h \in H$ and $a^\tau = a^{\sigma^{-1}}$ for each $a \in A$. It follows that $\sigma\tau$ leaves invariant every element of A and so it induces an automorphism on G/A, which acts on $G/Z(G)$ as σ does. Let σ_H be the automorphism of H induced by $\sigma\tau$ via the isomorphism $H \simeq G/A$. Then $(KZ(G))^\sigma = K^{\sigma_H}Z(G) = KZ(G)$, and thus $KZ(G)$ is a characteristic subgroup of G as wanted. □

Finite special p-groups of exponent p belonging to the class $\mathfrak{T}_{\mathfrak{B}}$ can be constructed in the following way. Let F be a finitely generated free group in the variety of nilpotent groups of class at most 2 and exponent at most p. Note that F is a special p-group and by Proposition 3.4 it is also a $\mathfrak{T}_{\mathfrak{B}}$-group. Let K be a subgroup of F' such that F/K is a special p-group (see [**4**] for a characterization of such subgroups K). By [**12**, Theorem 4] every automorphism of F/K is induced by an automorphism of F. Therefore $F/K \in \mathfrak{T}_{\mathfrak{B}}$ if and only if $C/C'K$ is a central section of F whenever C is a subgroup of F which contains K and is normal in $N_{\mathrm{Hol}F}(K)$. For example it is easy to verify that if K is the subgroup generated by a single commutator of any two free generators of F, then F/K is not a $\mathfrak{T}_{\mathfrak{B}}$-group, while if F is freely generated by the elements a, b, c, d and $K = \langle [a, b], [c, d]\rangle$, then $F/K \in \mathfrak{T}_{\mathfrak{B}}$.

The next proposition provides other examples of p-groups in the class $\mathfrak{T}_{\mathfrak{B}}$.

PROPOSITION 3.6. (i) *Direct products of finitely generated free groups in the variety of nilpotent groups of class at most* 2 *and exponent p and of abelian groups are* $\mathfrak{T}_{\mathfrak{B}}$*-groups.*
(ii) *Central products of extra-special p-groups of exponent p with cyclic centres are* $\mathfrak{T}_{\mathfrak{B}}$*-groups.*

Proof. To prove (i) clearly we may assume that $G = (\prod_{i\in I} G_i) \times A$ where G_i is a finitely generated free group in the variety of nilpotent groups of class at most 2 and exponent p for each $i \in I$ and A is abelian. For each $g \in G$ let g_i denote the ith component of g. We show that each characteristic subgroup $C \geq Z(G)$ of G is of the form $C = (\prod_{i\in J} G_i) \times (\prod_{i\in I\setminus J} G_i') \times A$ for some subset $J \subseteq I$. Since $Z(G) = G'A = (\prod_{i\in I} G_i') \times A$, we may assume that $C > Z(G)$. Then there exist an element $c \in C$ and an index $k \in I$ such that $c_k \notin G_k'$. Write $c = c_k x$ where $x \in (\prod_{i\neq k} G_i) \times A$. Note that each automorphism $\varphi \in \mathrm{Aut}(G_i)$, $i \in I$, extends to an automorphism $\overline{\varphi}$ of G such that $G_i^{\overline{\varphi}} = G_i$ and $\overline{\varphi}_{|G_i} = \varphi$, $((\prod_{i\neq j} G_i) \times A)^{\overline{\varphi}} = (\prod_{i\neq j} G_i) \times A$ and $\overline{\varphi}$ is the identity map on it. In particular it follows that ${c_k}^\varphi x \in C$ for each $\varphi \in \mathrm{Aut}(G_k)$ and thus ${c_k}^{-1}{c_k}^\varphi \in C$ for each $\varphi \in \mathrm{Aut}(G_k)$. Since $\mathrm{Aut}(G_k)$ acts on $G_k/Z(G_k)$ as the general linear group $GL(n_k, p)$, where $p^{n_k} = |G_k/Z(G_k)|$ (see [**6**, Section 20, Chapter A]), we have that $\mathrm{Aut}(G_k)$ acts transitively on the non-trivial elements of $G_k/Z(G_k)$ and hence we get that $G_k \leq C$. Now let J be the subset of the indices j of I such that there exists an element $g \in C$ with $g_j \notin G_j'$. Then as before we have that $G_j \leq C$ for each $j \in J$. Hence $C = (\prod_{i\in J} G_i) \times (\prod_{i\in I\setminus J} G_i') \times A$ as wanted. Then C/C' is a central section of G and so G is a $\mathfrak{T}_{\mathfrak{B}}$-group.

(ii) Assume now that G is a central product of the extra-special p-groups G_i of exponent p, $i \in I$. Note that each automorphism $\varphi \in \mathrm{Aut}(G_i)$, $i \in I$ which acts trivially on G_i', extends to an automorphism $\overline{\varphi}$ of G such that $G_j^{\overline{\varphi}} = G_j$ for each $j \in I$, $\overline{\varphi}_{|G_i} = \varphi$ and $\overline{\varphi}_{|G_j}$ is the identity map for $j \neq i$. Moreover the subgroup of the elements of $\mathrm{Aut}(G_i)$ which act trivially on G_i' acts on $G_i/Z(G_i)$ as the symplectic group $Sp(2n_i, p)$, where $p^{2n_i} = |G_i/Z(G_i)|$ (see [**6**, Theorem 20.8]), and so by [**10**, Satz 9.18] it acts transitively

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on the non-trivial elements of $G_i/Z(G_i)$. Therefore the argument used in the proof of (i) applies here to get the result claimed. □

Note that in particular by Proposition 3.6 the direct product of an extra-special p-group of exponent p and an abelian p-group of infinite exponent is a nilpotent p-group of the class $\mathfrak{T}_{\mathfrak{B}}$ with infinite exponent and nilpotency class 2 exactly.

As a further example of how one can work with finite p-groups of the class $\mathfrak{T}_{\mathfrak{B}}$, in the next proposition we determine all 2-generator non-abelian finite p-groups that are $\mathfrak{T}_{\mathfrak{B}}$-groups.

PROPOSITION 3.7. *A 2-generator non-abelian finite p-group is a $\mathfrak{T}_{\mathfrak{B}}$-group if and only if it is of one of the following types:*

(i) $\langle a, b \mid a^{p^n} = b^{p^n} = 1, [a, b]^p = 1, [a, b, a] = [a, b, b] = 1\rangle$, *where $n \geq 1$ and $n \neq 1$ if $p = 2$;*

(ii) $\langle a, b \mid a^{p^n} = b^{p^n} = 1, [a, b] = a^{p^{n-1}}\rangle$, *where $n \geq 1$ and $n \neq 1$ if $p = 2$;*

(iii) *the quaternion group Q_8;*

(iv) *the dihedral group $\langle a, b \mid a^2 = b^{2^n} = 1, b^a = b^{-1}\rangle$, where $n \geq 2$;*

(v) *the generalized quaternion group $\langle a, b \mid b^{2^n} = 1, a^2 = b^{2^{n-1}}, b^a = b^{-1}\rangle$, where $n \geq 3$;*

(vi) $\langle a, b \mid b^{2^n} = a^4 = 1, b^a = b^{-1}\rangle$, *where $n \geq 3$;*

(vii) $\langle a, b \mid b^{2^n} = a^4 = 1, b^a = b^{-1+2^{n-1}}\rangle$, *where $n \geq 3$.*

Proof. Let G be one of the groups in (i)–(vii). Then G is a 2-generator non-abelian finite p-group. If G is as in (i) or (ii) one can see that G does not contain characteristic proper subgroups properly containing $Z(G)$. It follows that if A is a characteristic proper subgroup of G not contained in $Z(G)$, then $AZ(G) = G$, so that $A' = G'$ and $A/A' \leq G/G'$. Hence G is a $\mathfrak{T}_{\mathfrak{B}}$-group. If $G = Q_8$, then G is a $\mathfrak{T}$-group. If G is one of the groups in (iv)–(vii), then $G' = \langle b^2\rangle$ and $C = C_G(G') = \langle a^2, b\rangle$. One can see that each characteristic proper subgroup of G is contained in C. Since G induces power automorphisms on C, G is a $\mathfrak{T}_{\mathfrak{B}}$-group.

Now let us assume that G is a 2-generator non-abelian finite p-group belonging to $\mathfrak{T}_{\mathfrak{B}}$. Suppose first that G has nilpotency class 2. Then G' has exponent p by Lemma 3.1 and so $G^p \leq Z(G)$. Hence the Frattini subgroup $\Phi(G) = G'G^p$ is contained in $Z(G)$. Since G is 2-generated and non-abelian we have that $\Phi(G) = Z(G)$ and $|G : \Phi(G)| = p^2$. We claim that if G is generated by two elements of different orders, then G is isomorphic to the dihedral group of order 8. Assume that $G = \langle a, b\rangle$, where $a, b \in G$, $|a| = p^n$, $|b| = p^m$ and $n > m$. Since G has class 2, for each $x, y \in G$ we have $(xy)^p = x^p y^p [x, y]^{\binom{p}{2}}$. Then $Z(G) = \Phi(G) = G'G^p = \langle a^p, b^p, G'\rangle$ and so $Z(G)$ has exponent p^{n-1} (note that $n > 1$). Thus $\Omega_{n-1}(G) \geq \langle b, Z(G)\rangle > Z(G)$. Now $\langle b, Z(G)\rangle$ is not a characteristic subgroup of G since it is a normal abelian subgroup on which a does not act by conjugation as a power automorphism. Therefore $\langle b, Z(G)\rangle < \Omega_{n-1}(G) = G$. Since for $p \neq 2$ G is a regular p-group and so $\Omega_{n-1}(G)$ has exponent p^{n-1}, we have $p = 2$. By the Burnside Basis Theorem ([**14**, 5.3.2]) there exist $a_1, b_1 \in G$ of order at most 2^{n-1} such that $G = \langle a_1, b_1\rangle$. Then $Z(G) = \langle {a_1}^2, {b_1}^2, G'\rangle$ and the exponent of $Z(G)$ is at most $\max\{2, 2^{n-2}\}$. Thus $2^{n-1} = \max\{2, 2^{n-2}\}$, whence $n = 2$. Hence G is isomorphic to the dihedral group of order 8.

Let us assume from now on that G is not isomorphic to the dihedral group of order 8. G/G' is a finite abelian group with 2 generators and so it is isomorphic to a direct product of two cyclic groups of order p^{n_1} and p^{n_2} respectively, with $n_2 \geq n_1 \geq 1$. Choose $a, b \in G$ such that $G/G' = \langle aG'\rangle \times \langle bG'\rangle$ and $|aG'| = p^{n_1}$, $|bG'| = p^{n_2}$. Then since $G' \leq \Phi(G)$, $G = \langle a, b\rangle$ and from what we have seen above $|a| = |b|$. If $\langle a\rangle \cap G' =$

$\langle b\rangle \cap G' = 1$, then $\langle a\rangle \cap \langle b\rangle = 1$ and $p^{n_1} = |aG'| = |a| = |b| = |bG'| = p^{n_2}$. Therefore G has the presentation (i). If $\langle a\rangle \cap G' \neq 1$ and $\langle a\rangle \cap \langle b\rangle = 1$, then we can choose a and b such that $[a, b] = a^{p^{n-1}}$ from which it follows that $n_1 = n - 1$ and $n_2 = n$. Thus G has the presentation (ii).

If $\langle a\rangle \cap G' \neq 1$ and $\langle a\rangle \cap \langle b\rangle \neq 1$, then $G' = \langle a\rangle \cap \langle b\rangle$ since G' has order p, and we can choose a and b such that $a^{p^{n-1}} = [a, b] = b^{p^{n-1}}$. Thus G has the presentation $G = \langle a, b \mid a^{p^n} = 1, a^{p^{n-1}} = b^{p^{n-1}} = [a, b]\rangle$. If $p > 2$ or $n > 2$, then every automorphism of G is of type

$$\sigma(\alpha, \beta, \gamma, \delta, z_1, z_2) : \begin{cases} a \mapsto a^{\alpha} b^{\beta} z_1 \\ b \mapsto a^{\gamma} b^{\delta} z_2 \end{cases}$$

where $z_1, z_2 \in Z(G)$, $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_p$, $\alpha\delta - \beta\gamma \neq 0$, $\alpha(\delta - 1) = \beta(\gamma + 1)$ and $\alpha + \beta = \gamma + \delta$. The subgroup $B = \langle ab^{-1}, Z(G)\rangle$ is an abelian characteristic subgroup of G and b does not induce by conjugation on B a power automorphism. Hence $p = 2$, $n = 2$ and G is isomorphic to Q_8.

Suppose now that G has nilpotency class greater than 2. Then by Theorem 3.3 G is a 2-group and $G = \langle x, C\rangle$ where $C = C_G(G')$, $x^2 \in Z(G)$ and x acts on G' as the inversion map. Since G is 2-generated, we can find two elements $a, b \in G$ such that $G = \langle a, b\rangle$, $a \notin C$ and $b \in C$. Then $G' = \langle [a, b]\rangle$. Furthermore $C = \langle a^2, b, [a, b]\rangle$ and so it is abelian. Thus a induces a power automorphism on C and so $b^a = b^{-1+2^t}$ where $|b| = 2^n$ and $t \geq n - 1$. Hence $C = \langle a^2, b\rangle$. It follows that if C is not cyclic, then G is one of the groups in (vi) and (vii). If C is cyclic, we can choose b such that $C = \langle b\rangle$. Then G is either as in (iv)–(v) or it is a quasi-dihedral group with the presentation $G = \langle a, b \mid a^2 = b^{2^n} = 1, b^a = b^{-1+2^{n-1}}\rangle$, $n \geq 3$. In the latter case every automorphism of G is of type

$$\sigma(\alpha, \beta) : \begin{cases} a \mapsto ab^{\alpha} \\ b \mapsto b^{\beta} \end{cases}$$

where $\alpha, \beta \in \mathbb{Z}$, α is even and β is odd. Therefore $H = \langle a, b^2\rangle$ is a characteristic subgroup of G, $H' = \langle b^4\rangle$ and H/H' is an elementary abelian non-central section of G. A contradiction since G is a $\mathfrak{T}_{\mathfrak{B}}$-group. □

For any group G, we denote by $\pi(G)$ the set of all primes p such that G has an element of order p. If G is nilpotent and $p \in \pi(G)$, G_p denotes the subgroup of the elements of order a power of p.

The following theorem extends to infinite periodic soluble $\mathfrak{T}_{\mathfrak{B}}$-groups Lemma 4.10 in [**1**] and may be compared to Theorem 4.2.2 in [**13**].

THEOREM 3.8. *Let G be a periodic soluble $\mathfrak{T}_{\mathfrak{B}}$-group, let L be the intersection of the terms of the lower central series of G and for $p \in \pi(L)$ set $C(p) = C_G(L_p/L_p')$.*

(i) *If p is an odd prime in the set $\pi(L)$, then $p \notin \pi(G/L)$, $G_p' = L_p$ and $G/C(p)$ is a non-trivial cyclic group whose order divides $p - 1$. Moreover if L_p is non-abelian, then it has exponent at most p^2 and $L_p{}' = Z(L_p) = \Phi(L_p)$.*

(ii) *If p is an odd prime in $\pi(G') \setminus \pi(L)$, then $G_p' \leq Z(G)$.*

(iii) *If $2 \in \pi(L)$, then 2 also belongs to $\pi(G/L)$, $G/C(2)$ has order 2 and G acts on L_2 as the inversion map.*

In particular $G'' = L'$ and it is a product of elementary abelian p-groups for $p \neq 2$.

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Proof. (i) Let p be an odd prime in $\pi(L)$. Let us show that if $O_{p'}(G) = 1$, then either G is a nilpotent p-group or $L = G' = O_p(G) = C(p)$ (for the definition of $O_p(G)$ and $O_{p'}(G)$ see [**11**]). Assume that G is not nilpotent. Then since G' and $C = C_G(G'/G'')$ are nilpotent, they are p-groups and by [**13**, Section 4.1.3], G/C is a cyclic group of order dividing $p^n(p-1)$ for a suitable $n \geq 0$. Thus $G/O_p(G)$ is a cyclic group of order dividing $p-1$. By Theorem 3.3, $O_p(G)$ is nilpotent of class at most 2. Hence by Lemma 2.1 (ii), $C_G(O_p(G)/O_p(G)')$ is a nilpotent p-group, and so it is equal to $O_p(G)$. Then each element of $G \setminus O_p(G)$ induces on $O_p(G)/O_p(G)'$ a power automorphism $a \mapsto a^w$ where w is a p-adic integer, $w \not\equiv 1 \bmod p$. It follows that if $O_p(G)/H$ is a central section of G, then $O_p(G) = H$. Therefore $L = O_p(G) = C(p) = G'$.

To treat now the general case we prove first that $LO_{p'}(G)/O_{p'}(G)$ is the least term of the lower central series of $G/O_{p'}(G)$. If $G/O_{p'}(G)$ is nilpotent, it is trivial. If $G/O_{p'}(G)$ is not nilpotent, then since $G/O_{p'}(G) \in \mathfrak{T}_{\mathfrak{B}}$ and $O_{p'}(G/O_{p'}(G)) = 1$, the previous discussion yields that $G'O_{p'}(G)/O_{p'}(G)$ is the least term of the lower central series of $G/O_{p'}(G)$. Moreover $G'O_{p'}(G)/LO_{p'}(G) \simeq_G G'/G' \cap LO_{p'}(G) \simeq_G G'_pL/L$, since G'/L is nilpotent and $(G' \cap LO_{p'}(G))/L$ is its p'-component. Thus $G'O_{p'}(G)/LO_{p'}(G)$ has a descending series which is central in G. Therefore $G'O_{p'}(G) = LO_{p'}(G)$ and $L_p = G'_p \simeq_G LO_{p'}(G)/O_{p'}(G)$. Now if $p \in \pi(L)$, then $G/O_{p'}(G)$ is not nilpotent and $p \in \pi(LO_{p'}(G)/O_{p'}(G))$. Thus by the previous paragraph $p \notin \pi(G/LO_{p'}(G))$ and hence $p \notin \pi(G/L)$. Note that $C(p) = LO_{p'}(G)$ to conclude that $G/C(p)$ is a cyclic group whose order divides $p-1$. Finally assume that L_p is non-abelian. Then by Lemma 3.2 $L_p{}'$ has exponent p and by what we have seen above there exists an element $g \in G$ acting on $L_p/L_p{}'$ fixed-point-freely. Thus if $n \geq 1$ is such that $\Omega_n(L_p)$ is non-abelian, we have that g acts on $\Omega_n(L_p)/\Omega_n(L_p)'$ as a power automorphism $a \mapsto a^m$, where m is an integer greater than 1 and relatively prime to p. Then by Lemma 2.1 (i), g acts on $\Omega_n(L_p)'$ as a power congruent to m^2 modulo p. Since m^2 is not congruent to m modulo p, it follows that $\Omega_n(L_p)' = Z(\Omega_n(L_p))$. This holds for each $n \geq 1$ such that $\Omega_n(L_p)$ is non-abelian and so it implies that $Z(L_p) = L_p{}'$. Hence $Z(L_p) = \Phi(L_p)$ and L_p has exponent at most p^2.

(ii) As a soluble periodic group, G/L is locally finite. Furthermore it is also hypocentral and thus locally nilpotent. Therefore it is the direct product of its p-components $(G/L)_p$. If p is an odd prime in $\pi(G') \setminus \pi(L)$, then by Theorem 3.3, $(G/L)_p$ is nilpotent of class at most 2, and so $[G'_p, G] \leq L \cap G'_p = 1$, that is $G'_p \leq Z(G)$.

(iii) By Lemma 2.3, L is a 2-divisible non-central nilpotent subgroup of G. If $2 \in \pi(L)$, then by [**15**, Theorem 9.23], L_2 is a non-trivial abelian divisible group and so it has infinite exponent. Thus by [**13**, Section 4.1.3], $|G : C(2)| = 2$ and each element of $G \setminus C(2)$ transforms every element of L_2 into its inverse. Since $L \leq C(2)$, $2 \in \pi(G/L)$.

To prove the final statement note that since G' is a periodic nilpotent group of class at most 2, G'' is the product of elementary abelian p-groups by Lemma 3.1. Then $(G'')_2 = 1$ by Lemma 2.3. Hence from (i) it follows that $G'' = L'$. □

At this point we can characterize finite soluble $\mathfrak{T}_{\mathfrak{B}}$-groups as particular semidirect products of finite nilpotent $\mathfrak{T}_{\mathfrak{B}}$-groups.

Let L be a finite nilpotent $\mathfrak{T}_{\mathfrak{B}}$-group of odd order whose non-abelian components are special p-groups, and let H be a finite nilpotent $\mathfrak{T}_{\mathfrak{B}}$-group whose order is relatively prime to that of L. As in [**5**] denote by $\mathrm{Aut}_\chi L$ the group of all automorphisms of L fixing all subgroups which lie between a characteristic subgroup of L and its derived subgroup. Let $\theta : H \to \mathrm{Aut}_\chi L$ be a group homomorphism and denote by K the kernel of θ and by Σ the subgroup of $\mathrm{Aut}H$ of all automorphisms σ of H such that $[H, \sigma] \leq K$.

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Finally assume that whenever C and D are Σ-invariant subgroups of H such that $D \trianglelefteq C$ and C/D is abelian, then each subgroup S of H such that $D \leq S \leq C$ is normal in H.

We form the semidirect product $G(L, H, \theta) = L \rtimes_\theta H$.

THEOREM 3.9. *The group $G(L, H, \theta)$ defined above is a finite soluble $\mathfrak{T}_{\mathfrak{V}}$-group. Every finite soluble $\mathfrak{T}_{\mathfrak{V}}$-group is isomorphic to some $G(L, H, \theta)$.*

Proof. Let $G = G(L, H, \theta)$, L, H, θ, K and Σ be as defined above. Clearly G is a finite soluble group. Let us show that G is a $\mathfrak{T}_{\mathfrak{V}}$-group. We need to show that for each characteristic subgroup C of G, G induces by conjugation on C/C' power automorphisms. To see this we may assume without loss of generality that C/C' is a p-group. If p does not divide $|L|$, then $C/C' \simeq_G CL/C'L$. Note that if φ is an automorphism of H such that $[H, \varphi] \leq K$, then φ extends to an automorphism $\overline{\varphi}$ of G defined by setting $(lh)^{\overline{\varphi}} = lh^{\varphi}$ for each $l \in L, h \in H$. Thus, since $CL/C'L$ is a characteristic section of G, it is H-isomorphic to an abelian Σ-invariant section of H. Hence H induces by conjugation power automorphisms on this section and so on C/C'. Assume now that p divides $|L|$. Then $C/C' \simeq_G C \cap L/C' \cap L$. By hypothesis G induces by conjugation power automorphisms on the abelian characteristic sections of L. Hence to conclude that G is a $\mathfrak{T}_{\mathfrak{V}}$-group it is enough to prove that $C \cap L/C' \cap L$ is G-isomorphic to a section of an abelian characteristic section of L. If $C \cap L$ is contained in the centre of the p-component L_p of L there is nothing to prove. If $[L_p, H] = 1$, then each automorphism α of L_p extends to an automorphism $\overline{\alpha}$ of G by setting $x^{\overline{\alpha}} = x^{\alpha}$ for each $x \in L_p$ and $y^{\overline{\alpha}} = y$ for each $y \in L_{p'} \cup H$. Hence $C \cap L$ and $C' \cap L$ are characteristic subgroups of L_p and we are done. So let us assume that $C \cap L$ is not contained in $Z(L_p)$ and that there is an element $h \in H$ acting non-trivially on L_p. Then L_p is non-abelian and by Theorem 3.8 (i) $Z(L_p) = L_p{}'$. By Lemma 2.1 (i) h acts as a power n on $C \cap L/(C \cap L_p{}')(C' \cap L)$ and as a power n^2 on $(C \cap L_p{}')(C' \cap L)/(C' \cap L)$, where $n \neq 1$ and $(n, p) = 1$. Hence $C' \cap L \geq C \cap L_p{}'$ and thus $C \cap L/C' \cap L$ is G-isomorphic to a section of $L_p/L_p{}'$. This proves that $G(L, H, \theta)$ is a $\mathfrak{T}_{\mathfrak{V}}$-group.

Conversely let G be a finite soluble $\mathfrak{T}_{\mathfrak{V}}$-group and let L be the nilpotent residual of G. Then by [**1**, Lemma 4.10] (or by Theorem 3.8) $(|L|, |G/L|) = 1$ and thus by a well known theorem associated with the names of Schur and Zassenhaus [**14**, 9.1.2], G is a semidirect product of L with a group $H \simeq G/L$. H and L are Hall subgroups of G and so by [**1**, Lemma 4.10] they are $\mathfrak{T}_{\mathfrak{V}}$-groups. Moreover by Theorem 3.8 non-abelian p-components of L are special p-groups. Each characteristic subgroup of L is characteristic in G and therefore every element of H acts by conjugation on L as an element of $\mathrm{Aut}_\chi L$. It remains to show that if Σ is the subgroup of $\mathrm{Aut}H$ of all automorphisms φ of H such that $[H, \varphi] \leq C_H(L)$, then H has the property that whenever A and B are Σ-invariant subgroups of H such that $B \trianglelefteq A$ and A/B is abelian, then each subgroup S of H such that $B \leq S \leq A$ is normal in H. First of all note that if we set $C = C_H(L/L')$, then $[G, \mathrm{Aut}G] \leq C$ by [**7**, Lemma 4.1] and its proof. Secondly $C_H(L/L') = C_H(L)$ by [**8**, Theorem 3.2] since $(|H|, |L|) = 1$. Thus it follows that $[H, \varphi] \leq C_H(L)$ for each automorphism φ induced on H by an automorphism of G via the isomorphism $H \simeq G/L$. Therefore for each Σ-invariant subgroup A of H, AL is characteristic in G. It follows that H induces power automorphisms on its Σ-invariant abelian sections and hence G is isomorphic to $G(L, H, \theta)$ for some θ. □

4. Non-periodic soluble $\mathfrak{T}_{\mathfrak{B}}$-groups. In this section we consider non-periodic soluble $\mathfrak{T}_{\mathfrak{B}}$-groups. An important role is played by the subgroup $C = C_G(G'/G'')$. Note that by Lemma 2.1 (i), C centralizes the derived series of G and hence it is nilpotent with class at most 3 and metabelian. Actually we shall see that C has class at most 2.

We say that a soluble $\mathfrak{T}_{\mathfrak{B}}$-group G is of *type* I if it is not nilpotent and $C = C_G(G'/G'')$ is non-periodic. We say that a soluble $\mathfrak{T}_{\mathfrak{B}}$-group G is *of type* II if it is non-nilpotent non-periodic, and $C = C_G(G'/G'')$ is periodic.

We begin with a result about non-periodic nilpotent $\mathfrak{T}_{\mathfrak{B}}$-groups.

PROPOSITION 4.1. *A non-periodic nilpotent $\mathfrak{T}_{\mathfrak{B}}$-group G has class at most 2. Moreover, for each prime p, $(G')^p$ is a p-divisible group.*

Proof. Let G be a non-periodic nilpotent $\mathfrak{T}_{\mathfrak{B}}$-group. To show that G has class at most 2 there is no loss of generality in assuming $\gamma_4(G) = 1$. Moreover, since nilpotent groups of class at most 2 form a variety, we may also assume $O_{p'}(G) = 1$ for a suitable prime p. Then $Z(G)$ is either non-periodic or a p-group with infinite exponent. Thus by Lemma 2.1 (iv), G acts by conjugation on $Z(G)G'$ as the identity, that is $G' \leq Z(G)$.

To prove the second part, let G be a non-periodic nilpotent $\mathfrak{T}_{\mathfrak{B}}$-group and let p be a prime. Since G has class at most 2, $(G^p)' \leq (G')^{p^2}$ and $G^pG'/(G^p)'$ is an abelian non-periodic characteristic section of G. Since G^pG'/G' is a central non-periodic section of G, from Lemma 2.1 (iv) it follows that $G^pG'/(G^p)'$ is a central section of G as well. Hence for each $x, y \in G$ we have $[x, y]^p = [x^p, y] \in (G^p)' \leq (G')^{p^2}$, and $(G')^p = (G')^{p^2}$ as claimed. □

The next result can be compared to Theorem 3.1.1 in [**13**].

THEOREM 4.2. *Let G be a soluble $\mathfrak{T}_{\mathfrak{B}}$-group of type* I *and let $C = C_G(G'/G'')$. Then $G = \langle x, C\rangle$, where $x^2 \in C$ and x acts on G'/G'' as the inversion map. Moreover $G' = C^2$, $G'' = (C')^2$, $C' \leq Z(G)$, and $Z(G)/G''$ has exponent at most* 2.

Proof. Let G be a soluble $\mathfrak{T}_{\mathfrak{B}}$-group of type I and set $D = C_G(C/C')$. Then by Lemma 2.1 (ii), D is nilpotent, and since C/C' is non-periodic, [**13**, Section 4.1.3] implies that $|G : D| = 2$. Since G is not nilpotent and D is not periodic, there exists an element of G acting on D/D' as the inversion map. Thus D/G' has exponent at most 2, G/G' has exponent at most 4 and G' is non-periodic. Then by [**13**, Section 4.1.3], $|G : C| = 2$ and $G = \langle x, C\rangle$, where $x^2 \in C$ and x acts by conjugation on G'/G'' as the inversion map. Furthermore x acts as the inversion map on C/C', since G is not nilpotent. By Proposition 4.1, $C' \leq Z(C)$ and then by Lemma 2.1 (i) x acts trivially on C'. Thus $C' \leq Z(G)$. The remaining facts follow easily. □

COROLLARY 4.3. *Let G be a torsion-free soluble $\mathfrak{T}_{\mathfrak{B}}$-group. Then G is nilpotent with class at most* 2, *G' is divisible and each characteristic abelian section of G is central in G.*

Proof. Assume first that G is nilpotent. Then by Proposition 4.1 G' is a central subgroup of G and $(G')^p$ is a p-divisible group for each prime p. Therefore, since G' is abelian torsion-free, $(G')^p = G'$ for each prime p and G' is divisible. Let H be a characteristic subgroup of G. Then $H/H \cap G'$ and $H \cap G'/H'$ are central sections of G, and at least one of them is non-periodic. Thus, by Lemma 2.1 (iv), H/H' is a central section of G.

It remains to prove that a torsion-free group cannot be a soluble $\mathfrak{T}_{\mathfrak{B}}$-group of type I. Suppose by contradiction that G is a torsion-free soluble $\mathfrak{T}_{\mathfrak{B}}$-group of type I.

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Then by Theorem 4.2, $G = \langle x, C\rangle$ where $C = C_G(G'/G'')$, $C' \leq Z(G)$, $x^2 \in C$ and x acts on C/C' as the inversion map. Since C is nilpotent, by the previous part of the proof, C' is divisible. Moreover $x^4 \in C'$. Thus the abelian group $\langle x, C'\rangle$ splits over C', and G contains a non-trivial element of order at most 4; a contradiction. □

Note that the group $U(3, \mathbb{Q})$ of all upper unitriangular matrices of degree 3 with coefficients in $\mathbb{Q}$ is a torsion-free soluble $\mathfrak{T}_{\mathfrak{B}}$-group. A more sophisticated soluble $\mathfrak{T}_{\mathfrak{B}}$-group of type I which shows that Theorem 4.2 cannot be improved is constructed in the next example.

Example 3. Let $V = \mathbb{Q}^2$ and let $f : V \times V \to \mathbb{Q}$ be the bilinear form defined by $f((x_1, x_2), (y_1, y_2)) = x_2y_1 - x_1y_2$. Then V, $\mathbb{Q}$ and f determine a nilpotent group N of class two that consists of the set $V \times \mathbb{Q}$ with the operation

$$(v, a)(w, b) = (v + w, a + b + f(v, w))$$

$v, w \in V$, $a, b \in \mathbb{Q}$ (see [**2**]). We have that $N' = \{(0, a) \mid a \in \mathbb{Q}\} \simeq \mathbb{Q}$ and $N/N' \simeq V$. In particular note that by [**15**, Theorem 9.23] N is a torsion-free divisible group. For each automorphism π of V, we define an automorphism $\overline{\pi}$ of N by setting $(v, a)^{\overline{\pi}} = (v^{\pi}, a \cdot \det\pi)$, $v \in V$, $a \in \mathbb{Q}$, where $\det\pi$ is the determinant of π. Then $\overline{\pi}$ acts as π on $N/N' \simeq V$ and as multiplication by $\det\pi$ on N'.

Let $\sigma \in \mathrm{Aut}V$ be multiplication by -1 in V; then $\overline{\sigma}$ is an automorphism of N of order two which acts trivially on N'. Set $H = N \rtimes \langle\overline{\sigma}\rangle \leq \mathrm{Hol}N$. Let $G = QH$ be the direct product of the quaternion group Q of order 8 and H. Then G is a soluble group with derived length 3, $G' = Q'N$ is non-periodic, $G'' = N'$, $C := C_G(G'/G'') = QN$ and $C' = Q'N'$. Moreover $G' = C^2$ and $G'' = (C')^2$. We claim that G is a $\mathfrak{T}_{\mathfrak{B}}$-group. Let K be a characteristic subgroup of G. If $K \not\leq QN$, then K contains an element which acts by conjugation on N as $\overline{\sigma}$ does. Hence $K \geq [K, N] \geq N$ and $K' \geq [K, N] \geq N$. Since G/N is a Dedekind group each subgroup lying between K and K' is normal in G and we are done. On the other hand, if $K \leq Q'N' \leq Z(G)$ we are done as well. So let $K \leq QN$ but $K \not\leq Q'N'$. Let qn, $q \in Q$, $n \in N$ be an element of K. If $q \notin Q'$, then there exists an automorphism $\varphi \in \mathrm{Aut}Q$ such that $[q, q^{\varphi}] \neq 1$. Hence, if Φ denotes the automorphism of G which acts on Q as φ and on H as the identity map, we have $1 \neq [q, q^{\varphi}] = [qn, q^{\varphi}n] = [qn, (qn)^{\Phi}] \in K'$, that is $Q' \leq K'$. Similarly suppose that $n \notin N'$. Note that since $\overline{\sigma}$ commutes with $\overline{\pi}$ for each $\pi \in \mathrm{Aut}V$, we can define an automorphism $\tilde{\pi}$ of H which extends $\overline{\pi}$ by setting

$$\begin{cases} x^{\tilde{\pi}} = x^{\overline{\pi}} & \text{for each } x \in N \\ \overline{\sigma}^{\tilde{\pi}} = \overline{\sigma} \end{cases}$$

Using this fact, it can be proved that for each element $m \in N'$ there exists an automorphism ψ of H depending on m, such that $[n, n^{\psi}] = m$. Thus, if Ψ denotes the automorphism of G acting on Q as the identity map and as ψ on H, we have $m = [n, n^{\psi}] = [qn, qn^{\psi}] = [qn, (qn)^{\Psi}] \in K'$, that is $N' \leq K'$. Therefore if $K \leq QN'$, then it contains an element of type qn where $q \notin Q'$ and so $Q' \leq K'$. Hence K/K' is a central section of G. If $K \leq Q'N$, then it contains an element of type qn where $n \notin N'$ and so $N' \leq K'$. Hence G acts on K/K' as the inversion map. If none of the previous occurs, then K contains an element of type qn where $q \notin Q'$ and $n \notin N'$ and thus $Q'N' \leq K'$ and G induces on K/K' the inversion map.

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Let us consider now soluble $\mathfrak{T}_{\mathfrak{B}}$-groups of type II. A first observation is that in a soluble $\mathfrak{T}_{\mathfrak{B}}$-group G of type II the set of all elements of finite order is a subgroup. In fact let P be the subgroup generated by the elements of finite order of G. Then $C \leq P$ and P/C is periodic since G/C is abelian. Therefore P is periodic and it coincides with the set of all elements of finite order.

We state now our main result about soluble $\mathfrak{T}_{\mathfrak{B}}$-groups of type II, which can be compared to [**13**, Theorem 4.3.1].

THEOREM 4.4. *Let G be a soluble $\mathfrak{T}_{\mathfrak{B}}$-group of type* II. *Then G' is abelian. Moreover if $C = C_G(G')$ and P is the subgroup of all elements of finite order, then*

(i) *$\gamma_3(G)$ is the least term of the lower central series of G and it is an abelian divisible group;*

(ii) *$\gamma_3(P)$ is the least term of the lower central series of P and it is an abelian divisible group;*

(iii) *each characteristic nilpotent subgroup $H \geq \gamma_3(G)$ has class at most 2;*

(iv) *$G' = \gamma_3(G) \times A$, where $A \leq Z(G)$, $\gamma_3(G) = \gamma_3(P) \times D$, where $D \leq Z(P)$ and $Z(C) = \gamma_3(G) \times B$, where $B \leq Z_2(G)$.*

Proof. (i) Note that $G/\gamma_4(G)$ is a non-periodic nilpotent $\mathfrak{T}_{\mathfrak{B}}$-group. Thus by Proposition 4.1, $\gamma_3(G) = \gamma_4(G)$ is the last term of the lower central series of G. In order to prove that G' is abelian and $\gamma_3(G)$ is an abelian divisible group, note that since G' is a periodic nilpotent group, it is enough to prove the result for the p-components of G' and of $\gamma_3(G)$. For $p = 2$ this follows from Lemma 2.3 and Lemma 3.1. Let $p \neq 2$: if $p \notin \pi(\gamma_3(G))$, then trivially G'_p is abelian. If $p \in \pi(\gamma_3(G))$, then $1 \neq \gamma_3(G)_p \simeq \gamma_3(G/O_{p'}(G))$ and so by Proposition 4.1, $G/O_{p'}(G)$ is a non-nilpotent non-periodic soluble $\mathfrak{T}_{\mathfrak{B}}$-group with periodic derived subgroup. Thus Theorem 4.2 implies that $G/O_{p'}(G)$ is of type II. Therefore we may directly assume that $O_{p'}(G) = 1$. Then G' is a nilpotent p-group of the class $\mathfrak{T}_{\mathfrak{B}}$ and thus by Theorem 3.3, G'' has exponent at most p and $(G')^p \leq Z(G')$. Hence applying Lemma 2.4 to the group $G/(G')^p$ we get that $\gamma_3(G) \leq (G')^p$. Thus $\gamma_3(G)$ is abelian. Then if $\gamma_3(G) = G'$ we have that G' is abelian and $G' = (G')^p$. Thus G' is divisible and we are done. So let us assume that $\gamma_3(G) < G'$. Clearly P is a characteristic subgroup of G. Hence P is a soluble periodic $\mathfrak{T}_{\mathfrak{B}}$-group and $O_{p'}(P) = 1$. Then C is a p-group and by what we have seen in the proof of Theorem 3.8, either P is a p-group or $P' = O_p(P) = C_P(P'/P'')$ is the least term of the lower central series of P. Thus if P is not a p-group, then $\gamma_3(P) \leq \gamma_3(G) \leq G' \leq C \leq O_p(G) = P' = \gamma_3(P)$ and $G' = \gamma_3(G)$. Therefore only the case when P is a p-group remains. Since G/C is non-periodic, from [**13**, Section 4.1.3] it follows that G'/G'' has infinite exponent and P/C is a cyclic group with order dividing $p - 1$. Hence $P = C$. Moreover since G'/G'' is not a central section of G, then by Lemma 2.1 (iii), $G'/\gamma_3(G)$, as a central section of G, has finite exponent. Thus $\gamma_3(G)$ has infinite exponent and from [**13**, Section 4.1.3] it follows that each element of $G \setminus C$ acts by conjugation on G'/G'' and on $\gamma_3(G)$ as λth powering, where λ is a p-adic integer, $\lambda \equiv 1 \bmod p$. Therefore $\gamma_3(G)/(\gamma_3(G))^p$ is a central section of G. Hence $\gamma_3(G) = (\gamma_3(G))^p$ and $\gamma_3(G)$ is divisible. Finally to prove that G' is abelian, note that since P is a p-group, Theorem 3.3 implies that $P/Z(P)$ has exponent at most p. Thus if $S/Z(P) = Z(G/Z(P))$, Lemma 2.4 implies that $P \leq S$. Since $G' \leq P$, $G/Z(P)$ is a nilpotent group of class at most 2 and derived subgroup of exponent at most p. Thus it follows that G/S has exponent at most p. Now since each element of $G \setminus C$ acts by conjugation on G'/G'' as a power $\lambda \equiv 1 \bmod p$, by [**13**, Section 4.1.3] we have that G/C

is isomorphic to a subgroup of the additive group of p-adic integers. Then G/S is cyclic with order at most p, and thus $G/Z(P)$ is abelian. Hence $G' \leq Z(P)$ is abelian.

(iii) Let H be a characteristic nilpotent subgroup of G such that $H \geq \gamma_3(G)$. If H is non-periodic the result follows from Proposition 4.1. So let H be periodic and note that in this case $G/\gamma_4(H)$ is non-periodic. If $G/\gamma_4(H)$ is nilpotent, then by Proposition 4.1, we get $\gamma_3(H) \leq \gamma_3(G) \leq \gamma_4(H)$ and we are done. If $G/\gamma_4(H)$ is not nilpotent, then Theorem 4.2 implies that it is a soluble $\mathfrak{T}_{\mathfrak{B}}$-group of type II. In this case we can therefore assume that $\gamma_4(H) = 1$. Let $p \in \pi(H)$. If $p \neq 2$, by Theorem 3.3, H_p has class at most 2. If $p = 2$ we distinguish two cases: if $2 \in \pi(\gamma_3(G))$, then by (i) H_2 has infinite exponent and by Theorem 3.3 H_2 has class at most 2. If $2 \notin \pi(\gamma_3(G))$, then $H_2 \cap \gamma_3(G) = 1$ and so $H_2 \leq Z_2(G)$. Thus in any case H_p has class at most 2 and we are done.

(ii) In order to prove that $\gamma_3(P)$ is the least term of the lower central series of P consider $G/\gamma_4(P)$ and use Proposition 4.1 if it is nilpotent and (iii) otherwise. To show that $\gamma_3(P)$ is an abelian divisible group, note that trivially $\gamma_3(P) \leq \gamma_3(G)$ and so it is abelian by (i). Moreover P is a soluble periodic $\mathfrak{T}_{\mathfrak{B}}$-group. Hence by Lemma 2.3, $(\gamma_3(P))_2$ is divisible. On the other hand, by Theorem 3.8, for each odd prime $p \in \pi(\gamma_3(P))$ we have $p \notin \pi(P/\gamma_3(P))$. Since $\gamma_3(G) \leq P$, this yields that $(\gamma_3(P))_p = (\gamma_3(G))_p$ and so by (i), each p-component of $\gamma_3(P)$ is divisible. Hence $\gamma_3(P)$ is divisible.

(iv) Since G' is abelian and $\gamma_3(G)$ is divisible we have that $G' = \gamma_3(G) \times A$, where $[A, G] \leq \gamma_3(G) \cap A = 1$. Moreover $G' \leq Z(C)$ and thus $Z(C) = \gamma_3(G) \times B$, where trivially $B \leq Z_2(G)$. Finally since $\gamma_3(P)$ is divisible, $\gamma_3(G) = \gamma_3(P) \times D$, where $D \leq Z_2(P)$. Then each p-component D_p of D is divisible. It follows that $D_p/(D_p \cap Z(P))$ is divisible. Then since $D_p/(D_p \cap Z(P))$ is a central section of P, Lemma 2.1(iv) yields that $D_p \leq Z(P)$. □

The next example exhibits a soluble $\mathfrak{T}_{\mathfrak{B}}$-group G of type II in which $\gamma_3(G) < G'$.

EXAMPLE 4. Let $H = A \rtimes \langle x \rangle$ where A is a quasicyclic p-group, p an odd prime, and x is an element of infinite order such that $a^x = a^{1+p}$ for each $a \in A$. Let E be an extra-special p-group of order p^3 and exponent p and let $G = EH$ be the direct product of E and H. Then $G' = E'A$, $C_G(G') = EA$ and $\gamma_3(G) = A$ is the least term of the lower central series of G. As in the previous examples it is not hard to see that G is a $\mathfrak{T}_{\mathfrak{B}}$-group.

We conclude with a result about finitely generated soluble $\mathfrak{T}_{\mathfrak{B}}$-groups. Recall that a finitely generated soluble $\mathfrak{T}$-group is finite or abelian ([**13**, Theorem 3.3.1]). By Corollary 4.3 a finitely generated soluble torsion-free group is a $\mathfrak{T}_{\mathfrak{B}}$-group if and only if it is abelian. In the general case we have that a finitely generated soluble $\mathfrak{T}_{\mathfrak{B}}$-group is abelian-by-finite.

THEOREM 4.5. *Let G be a finitely generated soluble $\mathfrak{T}_{\mathfrak{B}}$-group. Then G is supersoluble and one of the following conditions holds.*

(i) *G is finite;*

(ii) *G is a non-periodic nilpotent group with class at most 2, G' is finite and is the product of elementary abelian p-groups;*

(iii) *G is a $\mathfrak{T}_{\mathfrak{B}}$-group of type* I, *$G'$ is abelian and G/G' is a finite group with exponent at most* 4.

Proof. Trivially, as a finitely generated locally supersoluble group G is supersoluble. Therefore if $C = C_G(G'/G'')$ is periodic, then it is finite and G/C is finite as well.

Hence G is finite and (i) holds. So let C be non-periodic. If G is nilpotent, then by Proposition 4.1 it has class at most 2 and since G is supersoluble G' is finite and its p-components are elementary abelian p-groups. Hence in this case, (ii) holds. If G is not nilpotent, then it is of type I. Then by Theorem 4.2 G/G' has exponent at most 4 and so it is finite. If T is the torsion subgroup of G' then G'/T is a finitely generated nilpotent torsion-free group and so Corollary 4.3 yields that it is abelian. Thus G'' is finite. Since $C = C_G(G'/G'')$ is a nilpotent infinite group, $Z(C)$ is infinite and G acts on $Z(C)$ as the inversion map. From $G'' \leq Z(G)$ it follows that G'' has exponent at most 2 and hence Lemma 2.3 implies $G'' = 1$. □

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