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RESEARCH ARTICLE

Likelihood ratio comparisons and logconvexity properties of p-spacings from generalized order statistics

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Abstract
Due to the importance of generalized order statistics (GOS) in many branches of Statistics, a wide interest has been shown in investigating stochastic comparisons of GOS. In this article, we study the likelihood ratio ordering of p-spacings of GOS, establishing some flexible and applicable results. We also settle certain unresolved related problems by providing some useful lemmas. Since we do not impose restrictions on the model parameters (as previous studies did), our findings yield new results for comparison of various useful models of ordered random variables including order statistics, sequential order statistics, k-record values, Pfeifer's record values, and progressive Type-II censored order statistics with arbitrary censoring plans. Some results on preservation of logconvexity properties among spacings are provided as well.

1. Introduction

Stochastic comparisons of order statistics (OS), record values, and their spacings have been studied extensively by many authors during the last 20 years. However, some of them remained incomplete. In this article, we focus on one of the most important tasks the likelihood ratio ordering of spacings. It is known that the spacings of ordered random variables appear in many branches of statistical theory with applications to many fields such as reliability or life testing. As a general framework for models of ordered random variables, Kamps [23,24] introduced the concept of generalized order statistics (GOS). So, it is natural and interesting to study comparisons of ordered random variables and their spacings into the model of GOS with flexible choices for its parameters.

Let X be a nonnegative random variable with cumulative distribution function (cdf) $F(x)$, survival (or reliability) function $\bar{F}(x) = 1-F(x)$, and probability density function (pdf) $f(x)$. Let $h(x) = f(x)/\bar{F}(x)$ and $\kappa(x) = f(x)/F(x)$ be the hazard rate and reversed hazard rate functions of X, respectively. The random variables $X_{(r,n,\tilde{m}_n,k)}$, $r = 1,\dots,n$, are called GOS of X if their joint density function is given by

$$\mathbf{f}(x_1,\dots,x_n) = k\left(\prod_{j=1}^{n-1}\gamma_{(j,n,\tilde{m}_n,k)}\right)\left(\prod_{i=1}^{n-1}[\bar{F}(x_i)]^{m_i}f(x_i)\right)[\bar{F}(x_n)]^{k-1}f(x_n),$$

for all $F^{-1}(0) < x_1 \le x_2 \le \cdots \le x_n < F^{-1}(1-)$, where $n \in \mathbb{N}$, $k > 0$, and $m_1,\dots,m_{n-1} \in \mathbb{R}$ are such that $\gamma_{(r,n,\tilde{m}_n,k)} = k+n-r+\sum_{j=r}^{n-1} m_j > 0$ for all $r \in \{1,\dots,n-1\}$, and $\tilde{m}_n = (m_1,\dots,m_{n-1})$, if $n \ge 2$ ($\tilde{m}_n \in \mathbb{R}$ is arbitrary, if $n = 1$).

This general model contains several useful models. For example, if $m_1 = \cdots = m_{n-1} = 0$ and $k = 1$, or $m_1 = \cdots = m_{n-1} = -1$ and $k \in \mathbb{N}$, then the GOS would convert into OS and k-record values, respectively. Sequential order statistics (which describes the lifetime of a sequential $(n-r+1)$-out-of-n systems) under a proportional hazard rate model are also included in GOS. Indeed, the specific choice of distribution functions

$$F_i(x) = 1 - (1 - F(x))^{\alpha_i}, \quad i = 1, \ldots, n, \tag{1}$$

with a cdf F and positive real numbers $\alpha_1, \ldots, \alpha_n$ lead to the model of GOS with parameters $k = \alpha_n$ and $m_i = (n-i+1)\alpha_i - (n-i)\alpha_{i+1} - 1$ (and hence $\gamma_i = (n-i+1)\alpha_i$). In the literature, (1) is usually referred to the proportional hazard rate assumption (see [17,31] for new extensions of the proportional hazard rate model). We refer the reader to Table 1 of Kamps [23] for complete information on various submodels.

We denote the generalized spacings of GOS by $D_{(r,s,n,\tilde{m}_n,k)} = X_{(s,n,\tilde{m}_n,k)} - X_{(r-1,n,\tilde{m}_n,k)}$ for $1 \leq r \leq s \leq n$, with $X_{(0,n,\tilde{m}_n,k)} \equiv 0$. For $s = r$, they are simple spacings and for $s = r + p - 1$, p-spacings (denoted by $D^{(p)}_{(r,n,\tilde{m}_n,k)}$).

We say that X (with pdf f) is smaller than Y (with pdf g) in likelihood ratio order (denoted by $X \leq_{\mathrm{lr}} Y$) if $g(x)/f(x)$ is increasing in x in the union of their supports (cf. Shaked and Shanthikumar [33]). Throughout the paper, the word increasing (decreasing) is used for nondecreasing (nonincreasing) and all expectations are implicitly assumed to exist whenever they are written. Also, X (or F) is said to be increasing likelihood ratio (ILR) if its pdf exists and is logconcave. If it is logconvex, then it is called decreasing likelihood ratio (DLR).

Now, consider the following problems:

(P1) $X \in \mathrm{DLR} \Rightarrow D^{(p)}_{(r,n,\tilde{m}_n,k)} \leq_{\mathrm{lr}} D^{(p)}_{(r+1,n,\tilde{m}_n,k)}$;
(P2) $X \in \mathrm{DLR} \Rightarrow D^{(p)}_{(r,n+1,\tilde{m}_n,k)} \leq_{\mathrm{lr}} D^{(p)}_{(r,n,\tilde{m}_n,k)}$;
(P3) $X \in \mathrm{DLR} \Rightarrow D^{(p)}_{(r,n,\tilde{m}_n,k)} \leq_{\mathrm{lr}} D^{(p)}_{(r+1,n+1,\tilde{m}_n,k)}$;
(P4) $X \in \mathrm{DLR} \Rightarrow D^{(p)}_{(r,n,\tilde{m}_n,k)} \leq_{\mathrm{lr}} D^{(p)}_{(r',n',\tilde{m}_n,k)}$, $r \leq r'$, $n' - r' \leq n - r$;
(P5) $X \in \mathrm{ILR} \Rightarrow D^{(p)}_{(r,n,\tilde{m}_n,k)} \geq_{\mathrm{lr}} D^{(p)}_{(r+1,n+1,\tilde{m}_n,k)}$;
(P6) $X \in \mathrm{ILR} \Rightarrow D^{(p)}_{(r,n,\tilde{m}_n,k)} \leq_{\mathrm{lr}} D^{(p+1)}_{(r-1,n,\tilde{m}_n,k)}$;
(P7) $X \in \mathrm{ILR} \Rightarrow D^{(p)}_{(r,n,\tilde{m}_n,k)} \leq_{\mathrm{lr}} D^{(p')}_{(r',n,\tilde{m}_n,k)}$, $p + 1 \leq p'$, $r' \leq r - 1$, $p + r = p' + r'$;
(P8) $X \in \mathrm{DLR} \Rightarrow D^{(p)}_{(r,n,\tilde{m}_n,k)} \geq_{\mathrm{lr}} D^{(p+1)}_{(r,n+1,\tilde{m}_n,k)}$.

For OS, Misra and van der Meulen [28] obtained (P_1) and (P_2) and Hu and Zhuang [22] proved (P_3)–(P_6). For GOS, Hu and Zhuang [21] obtained (P_1)–(P_6) under the condition $m_1 = m_2 = \cdots = m_{n-1}$ in which the marginal and joint pdf of GOS have a closed form representation. Finally, in an interesting article, Xie and Hu [37] proved (P_1)–(P_4) without the condition $m_1 = m_2 = \cdots = m_{n-1}$ using some conditionally results about GOS. We also note that the likelihood ratio ordering of spacings of GOS in the conditional case was studied in Xie and Hu [36].

In this article, we obtain new finding for these problems. It is planned as follows. In Section 2, we give some preliminary results and useful lemmas that can be also on interest in the study of other topics. In Section 3, we obtain our main results for very flexible cases of GOS with different parameters $\tilde{m}_n$ and $\tilde{m}'_{n'}$. This enables us to compare the p-spacings of submodels of GOS among themselves. More generally, we can compare the p-spacings obtained from different submodels (we refer the reader to Franco *et al.* [20], Belzunce *et al.* [9], Esna-Ashari *et al.* [18] and Alimohammadi *et al.* [4], for some stochastic orderings of GOS with different parameters $\tilde{m}_n$ and $\tilde{m}'_{n'}$). Specifically, we extend (P_1)–(P_4) in the unifying Theorem 3.1 for different parameters m_i and m'_i. However, we note that (P_5) and (P_6) remained as open problems for unequal m_i and m'_i. We also extend (P_5) in Theorem 3.3 for different m_i and m'_i but just for simple spacings, that is, for $p = 1$. Also, we extend it in Theorem 3.4 for arbitrary p-spacings and for $m'_i = m_i$, but unequal m_i. Property (P_7) (which is more general than (P_6)) is extended as well for different parameters m_i and m'_i in Theorem 3.5. At the end of this section, we discuss the new problem (P_8) in Theorem 3.7. In Section 4, we first derive some preliminary results about the relationships among the logconvexity properties of $f(x)$, $h(x)$, and $\kappa(x)$. These findings may be of independent interest. Then, the preservation properties of the logconvexity among spacings are discussed.

It is known that the multivariate likelihood ratio order is preserved under marginalization (cf. [33]). But, our main results can not be deduced from the existing multivariate results. For example, Fang *et al.* [19] gave the results for simple spacings while we obtain the results for general spacings and also for very flexible case via different parameters m_i and m_i'. Sharafi *et al.* [34] considered the two-sample problem with some restrictions on m_i while we considered the one-sample problem.

2. Preliminary results and useful lemmas

There exist several representations for the marginal density functions of GOS (see, e.g., [15,23]). Cramer *et al.* [16] obtained the expression

$$f_{X_{(r,n,\tilde{m}_n,k)}}(x) = c_{r-1}[\bar{F}(x)]^{\gamma_{(r,n,\tilde{m}_n,k)}-1} g_r(F(x)) f(x), \tag{2}$$

where $c_{r-1} = \prod_{i=1}^{r} \gamma_{(i,n,\tilde{m}_n,k)}$, $r = 1, \ldots, n$, $\gamma_{(n,n,\tilde{m}_n,k)} = k$, and g_r is a particular Meijer's G-function. For the joint pdf of $X_{(r,n,\tilde{m}_n,k)}$ and $X_{(s,n,\tilde{m}_n,k)}$, $1 \le r < s \le n$, Tavangar and Asadi [35] established the expression

$$\begin{aligned} f_{X_{(r,n,\tilde{m}_n,k)},X_{(s,n,\tilde{m}_n,k)}}(x,y) &= c_{s-1}[\bar{F}(x)]^{\gamma_{(r,n,\tilde{m}_n,k)}-\gamma_{(s,n,\tilde{m}_n,k)}-1} g_r(F(x)) \\ &\quad \times [\bar{F}(y)]^{\gamma_{(s,n,\tilde{m}_n,k)}-1} \psi_{s-r-1}\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right) f(x) f(y), \end{aligned} \tag{3}$$

for $x < y$ (zero elsewhere), where $\psi_0(t) = 1$, $\psi_1(t) = \delta_{m_{r+1}}(1-t)$,

$$\psi_l(t) = \int_t^1 \int_{u_{l-1}}^1 \cdots \int_{u_2}^1 \delta_{m_{r+1}}(1-u_1) \prod_{i=1}^{l-1} u_i^{m_{r+i+1}} du_1 \ldots du_{l-2}\, du_{l-1}, \quad 0 \le t \le 1,\ l = 2, 3, \ldots$$

and

$$\delta_m(t) = \begin{cases} \dfrac{1}{m+1}(1-(1-t)^{m+1}), & m \neq -1, \\ -\ln(1-t), & m = -1, \end{cases}$$

for $t \in (0, 1)$.

According to Lemmas 2.1 and 3.1 in Alimohammadi and Alamatsaz [1], we have the following recursive formulas:

$$g_1(t) = 1, \quad g_r(t) = \int_0^t g_{r-1}(u)[1-u]^{m_{r-1}}\, du, \quad 0 \le t \le 1,\ r = 2, \ldots, n, \tag{4}$$

and

$$\psi_0(t) = 1, \quad \psi_l(t) = \int_t^1 \psi_{l-1}(u) u^{m_{r+l}}\, du, \quad 0 \le t \le 1,\ l = 1, 2, \ldots \tag{5}$$

For each $y \in \mathbb{R}_+$, denote $\bar{F}_y(x) = \bar{F}(x+y)/\bar{F}(y)$, $x \in \mathbb{R}_+$. Now, substituting r with $r-1$ in (3) and after some calculations, we obtain

$$\begin{aligned} f_{D_{(r,s,n,\tilde{m}_n,k)}}(x) &= c_{s-1} \int_0^{+\infty} [\bar{F}(x+y)]^{\gamma_{(s,n,\tilde{m}_n,k)}-1} \psi_{s-r}(\bar{F}_y(x)) f(x+y) \\ &\quad \times [\bar{F}(y)]^{\gamma_{(r-1,n,\tilde{m}_n,k)}-\gamma_{(s,n,\tilde{m}_n,k)}-1} g_{r-1}(F(y)) f(y)\, dy, \quad x \ge 0 \end{aligned} \tag{6}$$

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for $2 \le r \le s \le n$, where, according to (5) for $r-1$,

$$\psi_{s-r}(\bar{F}_y(x)) = \int_{\bar{F}_y(x)}^{1} \psi_{s-r-1}(u) u^{m_{s-1}}\, du, \quad r+1 \le s \le n, \tag{7}$$

with $\psi_0(t) = 1$ and, for $r=1$, we have $f_{D_{(1,s,n,\tilde{m}_n,k)}}(x) = f_{X_{(s,n,\tilde{m}_n,k)}}(x)$.

Many researchers have paid attention to various aspects of GOS. The majority of such results have been obtained under restrictions on the parameters of the model of GOS as the condition $m_1 = \cdots = m_{n-1}$. Notice that the pdf of GOS has a closed form representation in this case. We will try to avoid this assumption.

Let us review now the definition of logconvexity/logconcavity and a useful result about inheritance of them from a function to its right and left side integrals.

Definition 2.1 (Barlow and Proschan [7]). *A function $\lambda : \mathbb{R} \longmapsto \mathbb{R}_+$ is said to be logconvex (logconcave) if $\lambda(\alpha x + (1-\alpha)y) \le (\ge)[\lambda(x)]^{\alpha}[\lambda(y)]^{1-\alpha}$, for all $x, y \in \mathbb{R}$ and $\alpha \in (0,1)$.*

Theorem 2.2 (Alimohammadi *et al.* [3]). *Let λ be an integrable function, ω be a differentiable increasing function, and ω' be logconvex on (a,b). If $\lambda \circ \omega$ is logconvex on (a,b), then $\int_{\omega(x)}^{b} \lambda(u)\, du$ is logconvex and $\int_{a}^{\omega(x)} \lambda(u)\, du$ is logconcave provided that $-\infty < a$, $b = \infty$, and $\omega(\infty) = \infty$.*

An important special case of this theorem is that, if a pdf $f(x)$ with support (a,∞) is logconvex, then $\bar{F}(x)$ is logconvex and $F(x)$ is logconcave. In particular, this property holds for lifetime random variables with support $(0,\infty)$. Another important function satisfying the conditions of Theorem 2.2 is $\omega(x) = e^x$. Alimohammadi *et al.* [3] also proved that if logconvex is replaced by logconcave in Theorem 2.2, then $\int_{a}^{\omega(x)} \lambda(u)\, du$ and $\int_{\omega(x)}^{b} \lambda(u)\, du$ are logconcave provided that $\omega^{-1}(a)$ and $\omega^{-1}(b)$ are defined, respectively.

We also recall the following definition about the very useful concept of total positivity.

Definition 2.3 (Karlin [26]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of the real line $\mathbb{R}$. A function $\lambda : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is said to be totally positive of order 2 (TP_2) (resp. reverse regular of order 2 (RR_2)) if*

$$\lambda(x_1, y_1)\lambda(x_2, y_2) - \lambda(x_1, y_2)\lambda(x_2, y_1) \ge (\le) 0,$$

for all $x_1 \le x_2$ in $\mathcal{X}$ and all $y_1 \le y_2$ in $\mathcal{Y}$.

Note that the TP_2 (RR_2) property is equivalent to $\lambda(x_2, y)/\lambda(x_1, y)$ is increasing (decreasing) in y when $x_1 \le x_2$, whenever this ratio exists. Also note that the product of two TP_2 (RR_2) functions is TP_2 (RR_2). Moreover, if $\lambda(x,y)$ is TP_2 (RR_2) in (x,y), then $\lambda_1(x)\lambda(x,y)\lambda_2(y)$ is TP_2 (RR_2) in (x,y) when λ_1 and λ_2 are two nonnegative functions (cf. Karlin [26]).

The part (i.a) of the following theorem was established by Karlin [26] and the others by Esna-Ashari *et al.* [18]. It was called the *extended basic composition theorem.*

Theorem 2.4 (Extended basic composition theorem). *Let $\lambda_1 : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}_+$, $\lambda_2 : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}_+$, and $\lambda : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}_+$ be Borel-measurable functions satisfying*

$$\lambda(x,y) = \int_{\mathcal{Z}} \lambda_1(x,y,z)\lambda_2(x,y,z)\, d\mu(z),$$

where μ denotes a sigma-finite measure defined on $\mathcal{Z}$.

(i.a) If λ_1 and λ_2 are TP_2 in each pairs of variables, then λ is TP_2 in (x,y);
(i.b) If λ_1 and λ_2 are RR_2 in (y,z) and (x,z), and λ_1 and λ_2 are TP_2 in (x,y), then λ is TP_2 in (x,y);

(ii.a) *If* λ_1 *and* λ_2 *are* RR_2 *in* (y,z) *and* (x,y), *and* λ_1 *and* λ_2 *are* TP_2 *in* (x,z), *then* λ *is* RR_2 *in* (x,y);
(ii.b) *If* λ_1 *and* λ_2 *are* RR_2 *in* (x,y) *and* (x,z), *and* λ_1 *and* λ_2 *are* TP_2 *in* (y,z), *then* λ *is* RR_2 *in* (x,y).

The lemma below, due to Misra and van der Meulen [28], is often used in establishing the monotonicity of a fraction in which the numerator and denominator are integrals or summations.

Lemma 2.5 (Misra and van der Meulen [28]). *Let* Θ *be a subset of the real line* $\mathbb{R}$ *and let* U *be a nonnegative random variable having a cdf belonging to the family* $\mathcal{P} = \{G(\cdot\,|\,\theta), \theta \in \Theta\}$ *which satisfies that, for* $\theta_1, \theta_2 \in \Theta$,

$$G(\cdot\,|\,\theta_1) \le_{st} (\ge_{st}) G(\cdot\,|\,\theta_2), \quad \text{whenever } \theta_1 \le \theta_2.$$

Let $\phi(u,\theta)$ *be a real-valued function defined on* $\mathbb{R} \times \Theta$, *which is measurable in* u *for each* θ *such that* $E[\phi(U,\theta)]$ *exists. Then,*

(i) $E[\phi(U,\theta)]$ *is increasing in* θ, *if* $\phi(u,\theta)$ *is increasing in* θ *and increasing (decreasing) in* u;
(ii) $E[\phi(U,\theta)]$ *is decreasing in* θ, *if* $\phi(u,\theta)$ *is decreasing in* θ *and decreasing (increasing) in* u.

The following lemmas play a crucial role for obtaining our main results. They are also useful on their own. The proofs of these lemmas are given in the Appendix.

In whole of the paper, we consider the following two assumptions:

Assumption A (A′). $m_i \ge 0$ *for all* i, *and* f *is logconvex (logconcave).*

Assumption B (B′). $-1 \le m_i < 0$ *for all* i, f, *and* h *are logconvex (logconcave).*

Lemma 2.6. *Let* $s \ge r+1$.

(i) *The function* $\psi_{s-r}(\bar{F}_y(x))$ *is* TP_2 (RR_2) *in* (x,y) *provided that at least one of the two assumptions A or B (A′ or B′) is satisfied;*
(ii) *The function* $\psi_{s-r}(\bar{F}_y(x)) \cdot [\bar{F}(y)]^{m_{s-1}}$ *is* TP_2 (RR_2) *in* (y,s) *provided that* m_i *is decreasing (increasing) in* i *and that at least one of the two assumptions A or B is satisfied;*
(iii) *The function* $\psi_{s-r}(\bar{F}_y(x))$ *is* TP_2 (RR_2) *in* (x,s) *provided that* m_i *is decreasing (increasing) in* i.

Remark 2.7. *Cramer [13] proved that* $(X_{(1,n,\tilde{m}_n,k)}, \ldots, X_{(n,n,\tilde{m}_n,k)})$ *are multidimensional* TP_2 *without any condition (for definition and properties of multidimensional* TP_2, *we refer the reader to Karlin and Rinott [27]). Using the preservation of this property under marginalization, we have the* TP_2 *property of* $f_{X_{(r,n,\tilde{m}_n,k)},X_{(s,n,\tilde{m}_n,k)}}(x,y)$ *in* (x,y). *Thus, according to (3), the function* $\psi_{s-r-1}(\bar{F}(y)/\bar{F}(x))$ *is* TP_2 *in* (x,y) *without any condition. Also, Burkschat [10] obtained a result about the multidimensional* TP_2 *property of* $(D_{(1,1,n,\tilde{m}_n,k)}, \ldots, D_{(n,n,n,\tilde{m}_n,k)})$. *So,* $f_{D_{(r,r,n,\tilde{m}_n,k)},D_{(s,s,n,\tilde{m}_n,k)}}(x,y)$ *is* TP_2 *in* (x,y). *We note that these properties do not imply part (i) of Lemma 2.6.*

Lemma 2.8. *Let* Y *be a nonnegative random variable having a cdf belonging to the family* $\mathcal{P} = \{G(\cdot\,|\,x), x \in \mathbb{R}_+\}$ *with corresponding pdf*

$$\begin{aligned} g(y\,|\,x) = {} & c(x)[\bar{F}(x+y)]^{\gamma(s,n,\tilde{m}_n,k)-1}\psi_{s-r}(\bar{F}_y(x))f(x+y) \\ & \times [\bar{F}(y)]^{\gamma(r-1,n,\tilde{m}_n,k)-\gamma(s,n,\tilde{m}_n,k)-1} g_{r-1}(F(y))f(y), \end{aligned} \tag{8}$$

for $y \ge x$, *where*

$$\begin{aligned} c(x) = \Bigg[\int_0^{+\infty} & [\bar{F}(x+z)]^{\gamma(s,n,\tilde{m}_n,k)-1}\psi_{s-r}(\bar{F}_z(x))f(x+z) \\ & \times [\bar{F}(z)]^{\gamma(r-1,n,\tilde{m}_n,k)-\gamma(s,n,\tilde{m}_n,k)-1} g_{r-1}(F(z))f(z)\,dz \Bigg]^{-1}, \end{aligned}$$

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is the normalizing constant. Then, for $x_1, x_2 \in \mathbb{R}_+$,

$$G(\cdot \mid x_1) \leq_{lr} (\geq_{lr}) G(\cdot \mid x_2), \quad \text{whenever } x_1 \leq x_2,$$

provided that at least one of the two assumptions A (A′) with $\gamma_{(s,n,\tilde{m}_n,k)} \geq 1$ *or B (B′) is satisfied.*

Lemma 2.9. *For* $s \geq 2$*, the function* $\psi_{s-1}(\bar{F}_y(x))/g_s(F(x))$ *is increasing (decreasing) in x provided that at least one of the two assumptions A or B (A′ or B′) is satisfied.*

We also need some results for different parameters m_i and m'_i. From now on, we consider $\tilde{m}'_{n'} = (m'_1, \ldots, m'_{n'-1})$ with $\gamma_{(r',n',\tilde{m}'_{n'},k')} = k' + n' - r' + \sum_{j=r'}^{n'-1} m'_j > 0$. First, we recall the following lemma about the function g_r.

Lemma 2.10 (Esna-Ashari *et al.* [18])**.** *If* $r \leq r'$ *and* $m'_{r'-i} \leq m_{r-i}$ *for* $1 \leq i \leq r-1$*, then* $\breve{g}_{r'}(t)/g_r(t)$ *is increasing in t, where* g_r *and* $\breve{g}_{r'}$ *are defined as in (4) with parameters* m_i *and* m'_i*, respectively.*

Remark 2.11. *According to the method of proof used in Lemma 3.2 of [18], we get that*

(i) if $m'_{r-i} \geq m_{r-i}$ *for* $1 \leq i \leq r-1$*, then* $\breve{g}_r(t)/g_r(t)$ *is decreasing in t;*
(ii) if $r' \leq r$ *and* $m_{r-i} \geq m_{r'-i}$ *for* $1 \leq i \leq r'-1$*, then* $g_{r'}(t)/g_r(t)$ *is decreasing in t.*

Now, we obtain the following result about the function ψ.

Lemma 2.12. *Suppose that* ψ *and* $\breve{\psi}$ *are defined as in (7) with parameters* m_i *and* m'_i*, respectively. Let us assume* $s' - r' = s - r \geq 1$ *and* $s \leq s'$*. Then, the function*

$$\Delta_{(r,r',s,s')}(x,y) = \frac{\breve{\psi}_{s'-r'}(\bar{F}_y(x))}{\psi_{s-r}(\bar{F}_y(x))} \cdot [\bar{F}(y)]^{m'_{s'-1} - m_{s-1}} \tag{9}$$

(i) is increasing (decreasing) in y provided that $m'_j \leq m_i$ $(m'_j \geq m_i)$ *for all* $i \leq j$*, and that at least one of the two assumptions A or B is satisfied;*
(ii) is increasing (decreasing) in x provided that $m'_j \leq m_i$ $(m'_j \geq m_i)$ *for all* $i \leq j$*.*

3. Likelihood ratio comparisons

In this section, we study the preservation of the likelihood ratio order among spacings of GOS. It is worth mentioning that the direct studying of likelihood ratio ordering of spacings of GOS by means of its marginal pdf is rather complicated (since the pdf has not a closed form). Thus, some authors imposed the restriction $m_1 = \cdots = m_{n-1}$ on the model in which the marginal and joint pdf of GOS have the closed form or study conditionally results about GOS. However, we obtain our main results directly. This enable us to have a more flexible choice of parameters to compare them.

Theorem 3.1. *Let* $X_{(r,n,\tilde{m}_n,k)}$*,* $r = 1, \ldots, n$ *and* $X_{(r',n',\tilde{m}'_{n'},k')}$*,* $r' = 1, \ldots, n'$*, be the GOS based on a common absolutely continuous cdf F. If* $r \leq r'$*,* $s \leq s'$ *and* $s' - r' = s - r$*, then*

$$D_{(r,s,n,\tilde{m}_n,k)} \leq_{lr} D_{(r',s',n',\tilde{m}'_{n'},k')}$$

provided that $m'_j \leq m_i$ *for all* $i \leq j$*,* $\gamma_{(s',n',\tilde{m}'_{n'},k')} \leq \gamma_{(s,n,\tilde{m}_n,k)}$ *and at least one of the following three conditions is satisfied: assumption A with* $\gamma_{(s,n,\tilde{m}_n,k)} \geq 1$ *for* $r \geq 2$*, assumption A with* $\gamma_{(s',n',\tilde{m}'_{n'},k')} \geq 1$ *for* $r = 1$*, or assumption B.*

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Proof. We give the proof in two cases.

Case 1: $r \geq 2$. From (6), we have

$$\frac{f_{D_{(r',s',n',\tilde{m}'_{n'},k')}}(x)}{f_{D_{(r,s,n,\tilde{m}_n,k)}}(x)} = E[\phi(Y,x)],$$

where

$$\begin{aligned}
\phi(y,x) &\propto [\bar{F}(x+y)]^{\gamma_{(s',n',\tilde{m}'_{n'},k')}-\gamma_{(s,n,\tilde{m}_n,k)}} \frac{\breve{\psi}_{s'-r'}(\bar{F}_y(x))}{\psi_{s-r}(\bar{F}_y(x))} \frac{\breve{g}_{r'-1}(F(y))}{g_{r-1}(F(y))} \\
&\quad \times [\bar{F}(y)]^{(\gamma_{(r'-1,n',\tilde{m}'_{n'},k')}-\gamma_{(s',n',\tilde{m}'_{n'},k')})-(\gamma_{(r-1,n,\tilde{m}_n,k)}-\gamma_{(s,n,\tilde{m}_n,k)})}, \\
&= [\bar{F}(x+y)]^{\gamma_{(s',n',\tilde{m}'_{n'},k')}-\gamma_{(s,n,\tilde{m}_n,k)}} \cdot \Delta_{(r,r',s,s')}(x,y) \cdot \frac{\breve{g}_{r'-1}(F(y))}{g_{r-1}(F(y))} \\
&\quad \times [\bar{F}(y)]^{(\gamma_{(r'-1,n',\tilde{m}'_{n'},k')}-\gamma_{(s',n',\tilde{m}'_{n'},k')})-(\gamma_{(r-1,n,\tilde{m}_n,k)}-\gamma_{(s,n,\tilde{m}_n,k)})-(m'_{s'-1}-m_{s-1})} \\
&= [\bar{F}(x+y)]^{\gamma_{(s',n',\tilde{m}'_{n'},k')}-\gamma_{(s,n,\tilde{m}_n,k)}} \cdot \Delta_{(r,r',s,s')}(x,y) \cdot \frac{\breve{g}_{r'-1}(F(y))}{g_{r-1}(F(y))} \\
&\quad \times [\bar{F}(y)]^{(\sum_{j=r'-1}^{s'-2} m'_j)-(\sum_{j=r-1}^{s-2} m_j)},
\end{aligned} \tag{10}$$

$\Delta_{(r,r',s,s')}(x,y)$ is defined as in (9), and Y is a nonnegative random variable having a cdf belonging to the family $\mathcal{P} = \{G(\cdot\,|\,x), x \in \mathbb{R}_+\}$ with the pdf defined as in (8). It is seen that the following properties hold in (10):

- The first term is increasing in x and y since $\gamma_{(s',n',\tilde{m}'_{n'},k')} \leq \gamma_{(s,n,\tilde{m}_n,k)}$;
- The second term is increasing in x and y due to the assumptions of the theorem and Lemma 2.12;
- The third term is increasing in y due to Lemma 2.10 since $r \leq r'$ and $m'_j \leq m_i$ for all $i \leq j$;
- The fourth term is increasing in y since $m'_j \leq m_i$ for all $i \leq j$.

Furthermore, according to the assumptions of the theorem and Lemma 2.8, we have $G(\cdot\,|\,x_1) \leq_{st} G(\cdot\,|\,x_2)$ for $x_1 \leq x_2$. Now, part (i) of Lemma 2.5 implies that $E[\phi(Y,x)]$ is increasing in x.

Case 2: $r = 1$. From (2) and (6), we have

$$\begin{aligned}
\frac{f_{D_{(r',s',n',\tilde{m}'_{n'},k')}}(x)}{f_{D_{(1,s,n,\tilde{m}_n,k)}}(x)} &= \int_0^{+\infty} \frac{[\bar{F}(x+y)]^{\gamma_{(s',n',\tilde{m}'_{n'},k')}-1}}{[\bar{F}(x)]^{\gamma_{(s,n,\tilde{m}_n,k)}-1}} \frac{\breve{\psi}_{s'-r'}(\bar{F}_y(x))}{g_s(F(x))} \frac{f(x+y)}{f(x)} \cdot \nu(y)\,dy \\
&= \int_0^{+\infty} \left[\frac{\bar{F}(x+y)}{\bar{F}(x)}\right]^{\gamma_{(s',n',\tilde{m}'_{n'},k')}-1} [\bar{F}(x)]^{\gamma_{(s',n',\tilde{m}'_{n'},k')}-\gamma_{(s,n,\tilde{m}_n,k)}} \\
&\quad \times \frac{\breve{\psi}_{s'-r'}(\bar{F}_y(x))}{g_s(F(x))} \frac{f(x+y)}{f(x)} \cdot \nu(y)\,dy
\end{aligned} \tag{11}$$

$$\begin{aligned}
&= \int_0^{+\infty} \left[\frac{\bar{F}(x+y)}{\bar{F}(x)}\right]^{\gamma_{(s',n',\tilde{m}'_{n'},k')}} [\bar{F}(x)]^{\gamma_{(s',n',\tilde{m}'_{n'},k')}-\gamma_{(s,n,\tilde{m}_n,k)}} \\
&\quad \times \frac{\breve{\psi}_{s'-r'}(\bar{F}_y(x))}{g_s(F(x))} \frac{h(x+y)}{h(x)} \cdot \nu(y)\,dy,
\end{aligned} \tag{12}$$

where $\nu(y)$ does not depend on x. Now, according to the assumptions of the theorem, to prove that (11) and (12) are increasing in x, it is sufficient to show that $\breve{\psi}_{s'-r'}(\bar{F}_y(x))/g_s(F(x))$ is increasing in x. To do this, we write it as

$$\frac{\breve{\psi}_{s'-r'}(\bar{F}_y(x))}{g_s(F(x))} = \frac{\breve{\psi}_{s'-r'}(\bar{F}_y(x))}{\psi_{s-r}(\bar{F}_y(x))} \frac{\psi_{s-r}(\bar{F}_y(x))}{g_s(F(x))}.$$

Here, the first and second terms are increasing in x by part (ii) of Lemmas 2.12 and 2.9 with $r = 1$, respectively. Therefore, the proof is completed. □

Remark 3.2. *Xie and Hu [37] proved the statement of Theorem 3.1 (which also contains the previous findings of Hu and Zhuang [21]) in their separate Theorems 3.1, 3.2, and 20 and Corollary 3.4 under the following conditions for the parameters:*

$$k = k', \; m_i = m_i', \; m_i \text{ is decreasing in } i, \text{ and } r' - r \geq n' - n. \tag{13}$$

By choosing $s = r + p - 1$ and $s' = r' + p - 1$, one can see that (13) implies the conditions in Theorem 3.1.

Theorem 3.3. *Let $X_{(r,n,\tilde{m}_n,k)}$, $r = 1, \ldots, n$, and $X_{(r',n',\tilde{m}'_{n'},k')}$, $r' = 1, \ldots, n'$, be the GOS based on a common absolutely continuous cdf F. If $r \leq r'$, then*

$$D_{(r,r,n,\tilde{m}_n,k)} \geq_{lr} D_{(r',r',n',\tilde{m}'_{n'},k')}$$

provided that:

Case 1: For $r \geq 2$, $m'_j \leq m_i$ for all $i \leq j$ and $\gamma_{(r',n',\tilde{m}'_{n'},k')} = \gamma_{(r,n,\tilde{m}_n,k)}$ hold and at least one of the following conditions hold: assumption A' with $\gamma_{(r,n,\tilde{m}_n,k)} \geq 1$ or assumption B'.
Case 2: For $r = 1$, $\gamma_{(r',n',\tilde{m}'_{n'},k')} \geq \gamma_{(1,n,\tilde{m}_n,k)}$ holds and at least one of the following conditions hold: assumption A' with $\gamma_{(r',n',\tilde{m}'_{n'},k')} \geq 1$ or assumption B'.

Proof. Case 1: $r \geq 2$. From (6), we have

$$\frac{f_{D_{(r',r',n',\tilde{m}'_{n'},k')}}(x)}{f_{D_{(r,r,n,\tilde{m}_n,k)}}(x)} = E[\phi(Y,x)],$$

where

$$\phi(y,x) \propto [\bar{F}(x+y)]^{\gamma_{(r',n',\tilde{m}'_{n'},k')} - \gamma_{(r,n,\tilde{m}_n,k)}} \frac{\breve{g}_{r'-1}(F(y))}{g_{r-1}(F(y))} [\bar{F}(y)]^{m'_{r'-1} - m_{r-1}}, \tag{14}$$

and Y is a nonnegative random variable having a cdf belonging to the family $\mathcal{P} = \{G(\cdot\,|\,x), x \in \mathbb{R}_+\}$ with the pdf defined as in (8). It is seen that the following properties hold in (14):

- The first term is constant with respect to x and y since $\gamma_{(r'+p-1,n',\tilde{m}'_{n'},k')} = \gamma_{(r+p-1,n,\tilde{m}_n,k)}$;
- The second term is increasing in y due to Lemma 2.10 since $r \leq r'$, $m'_j \leq m_i$ for all $i \leq j$;
- The third term is increasing in y since $m'_j \leq m_i$ for all $i \leq j$.

Furthermore, according to Lemma 2.8, we have $G(\cdot\,|\,x_1) \geq_{st} G(\cdot\,|\,x_2)$ for $x_1 \leq x_2$. Now, part (ii) of Lemma 2.5 implies that $E[\phi(Y,x)]$ is decreasing in x.

Case 2: $r = 1$. From (2) and (6), we have

$$\begin{aligned}
&\frac{f_{D_{(r',r',n',\tilde{m}'_{n'},k')}}(x)}{f_{D_{(1,r,n,\tilde{m}_n,k)}}(x)} \\
&= \int_0^{+\infty} \left[\frac{\bar{F}(x+y)}{\bar{F}(x)}\right]^{\gamma_{(r',n',\tilde{m}'_{n'},k')}-1} [\bar{F}(x)]^{\gamma_{(r',n',\tilde{m}'_{n'},k')} - \gamma_{(1,n,\tilde{m}_n,k)}} \frac{f(x+y)}{f(x)} \cdot \nu(y)\, dy \\
&= \int_0^{+\infty} \left[\frac{\bar{F}(x+y)}{\bar{F}(x)}\right]^{\gamma_{(r',n',\tilde{m}'_{n'},k')}} [\bar{F}(x)]^{\gamma_{(r',n',\tilde{m}'_{n'},k')} - \gamma_{(1,n,\tilde{m}_n,k)}} \frac{h(x+y)}{h(x)} \cdot \nu(y)\, dy,
\end{aligned}$$

where $\nu(y)$ does not depend on x. So, the result follows from the assumptions of the theorem. □

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Theorem 3.4. *Let $X_{(r,n,\tilde{m}_n,k)}$, $r = 1, \ldots, n$, and $X_{(r',n',\tilde{m}_{n'},k')}$, $r' = 1, \ldots, n'$, be the GOS based on a common absolutely continuous cdf F. If $r + 1 \leq r'$ and $s' - r' = s - r$, then*

$$D_{(r,s,n,\tilde{m}_n,k)} \geq_{lr} D_{(r',s',n',\tilde{m}_{n'},k')}$$

provided that:

Case 1: For $r \geq 2$, $m_j \leq m_i$ for all $i \leq j$ and $\gamma_{(s',n',\tilde{m}_{n'},k')} = \gamma_{(s,n,\tilde{m}_n,k)}$ hold and at least one of the following conditions hold: assumption A' with $\gamma_{(s,n,\tilde{m}_n,k)} \geq 1$ or assumption B'.
Case 2: For $r = 1$, $\gamma_{(s',n',\tilde{m}_{n'},k')} \geq \gamma_{(s,n,\tilde{m}_n,k)}$ holds at least one of the following conditions hold: assumption A' with $\gamma_{(s',n',\tilde{m}_{n'},k')} \geq 1$ or assumption B'.

Proof. *Case 1*: $r \geq 2$. From (6), we have

$$\frac{f_{D_{(r',s',n',\tilde{m}_{n'},k')}}(x)}{f_{D_{(r,s,n,\tilde{m}_n,k)}}(x)} = E[\phi(Y,x)],$$

where

$$\begin{aligned}\phi(y,x) &\propto [\bar{F}(x+y)]^{\gamma_{(s',n',\tilde{m}_{n'},k')}-\gamma_{(s,n,\tilde{m}_n,k)}} \frac{g_{r'-1}(F(y))}{g_{r-1}(F(y))} \\ &\quad \times [\bar{F}(y)]^{(\gamma_{(r'-1,n',\tilde{m}_{n'},k')}-\gamma_{(s',n',\tilde{m}_{n'},k')})-(\gamma_{(r-1,n,\tilde{m}_n,k)}-\gamma_{(s,n,\tilde{m}_n,k)})}, \\ &= [\bar{F}(x+y)]^{\gamma_{(s',n',\tilde{m}_{n'},k')}-\gamma_{(s,n,\tilde{m}_n,k)}} \frac{g_{r'-1}(F(y))}{g_{r-1}(F(y))} [\bar{F}(y)]^{(\sum_{j=s}^{s'-1} m_j)-(\sum_{j=r-1}^{r'-2} m_j)},\end{aligned} \tag{15}$$

and Y is a nonnegative random variable having a cdf belonging to the family $\mathcal{P} = \{G(\cdot\,|\,x), x \in \mathbb{R}_+\}$ with the pdf defined as in (8). It is seen that the following hold in (15):

- The first term is constant with respect to x and y because of $\gamma_{(s',n',\tilde{m}_{n'},k')} = \gamma_{(s,n,\tilde{m}_n,k)}$;
- The second term is increasing in y due to Lemma 2.10 because of $r + 1 \leq r'$, $m_j \leq m_i$ for all $i \leq j$;
- The third term is increasing in y because of $m_j \leq m_i$ for all $i \leq j$.

Furthermore, according to the assumptions of the theorem and Lemma 2.8, we have $G(\cdot\,|\,x_1) \geq_{st} G(\cdot\,|\,x_2)$ for $x_1 \leq x_2$. Now, part (ii) of Lemma 2.5 implies that $E[\phi(Y,x)]$ is decreasing in x.
Case 2: $r = 1$. From (2) and (6), we have

$$\begin{aligned}\frac{f_{D_{(r',s',n',\tilde{m}_{n'},k')}}(x)}{f_{D_{(1,s,n,\tilde{m}_n,k)}}(x)} &= \int_0^{+\infty} \left[\frac{\bar{F}(x+y)}{\bar{F}(x)}\right]^{\gamma_{(s',n',\tilde{m}_{n'},k')}-1} [\bar{F}(x)]^{\gamma_{(s',n',\tilde{m}_{n'},k')}-\gamma_{(s,n,\tilde{m}_n,k)}} \\ &\quad \times \frac{\psi_{s'-r'}(\bar{F}_y(x))}{g_s(F(x))} \frac{f(x+y)}{f(x)} \cdot \nu(y)\,dy \\ &= \int_0^{+\infty} \left[\frac{\bar{F}(x+y)}{\bar{F}(x)}\right]^{\gamma_{(s',n',\tilde{m}_{n'},k')}} [\bar{F}(x)]^{\gamma_{(s',n',\tilde{m}_{n'},k')}-\gamma_{(s,n,\tilde{m}_n,k)}} \\ &\quad \times \frac{\psi_{s'-r'}(\bar{F}_y(x))}{g_s(F(x))} \frac{h(x+y)}{h(x)} \cdot \nu(y)\,dy,\end{aligned}$$

where $\nu(y)$ does not depend on x. First, note that

$$\frac{\psi_{s'-r'}(\bar{F}_y(x))}{g_s(F(x))} = \frac{\psi_{s-1}(\bar{F}_y(x))}{g_s(F(x))}$$

is decreasing in x by Lemma 2.9. Now, according to the assumptions of the theorem, $f_{D_{(r',s',n',\tilde{m}_{n'},k')}}(x)/f_{D_{(1,s,n,\tilde{m}_n,k)}}(x)$ is decreasing in x. Thus, the proof is completed. □

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Now, we study the preservation of likelihood ratio ordering among p-spacings for different values of p in the next two theorems.

Theorem 3.5. *Let $X_{(r,n,\tilde{m}_n,k)}$, $r = 1,\ldots,n$, and $X_{(r',n',\tilde{m}_{n'},k')}$, $r' = 1,\ldots,n'$, be the GOS based on a common absolutely continuous cdf F. If $r' \le r-1$ and $s'-r' = s-r$, then*

$$D_{(r,s,n,\tilde{m}_n,k)} \le_{lr} D_{(r',s',n',\tilde{m}_{n'},k')}$$

provided that at least one of the following conditions hold: assumption A' with $\gamma_{(s,n,\tilde{m}_n,k)} \ge 1$ or assumption B' and in the following cases:

Case 1: For $r \ge 3$, $m_j \le m_i$ for all $i \le j$ and $\gamma_{(s',n',\tilde{m}_{n'},k')} = \gamma_{(s,n,\tilde{m}_n,k)}$ hold.
Case 2: For $r = 2$, $\gamma_{(s',n',\tilde{m}_{n'},k')} \le \gamma_{(s,n,\tilde{m}_n,k)}$ holds.

Proof. *Case 1*: $r \ge 3$. From (6), we have

$$\frac{f_{D_{(r',s',n',\tilde{m}_{n'},k')}}(x)}{f_{D_{(r,s,n,\tilde{m}_n,k)}}(x)} = E[\phi(Y,x)],$$

where $\phi(y,x)$ is the same as (15), and Y is a nonnegative random variable having a cdf belonging to the family $\mathcal{P} = \{G(\cdot\,|\,x), x \in \mathbb{R}_+\}$ with the pdf defined as in (8). Here, it is seen that the following hold in (15):

- The first term is constant with respect to x and y since $\gamma_{(s',n',\tilde{m}_{n'},k')} = \gamma_{(s,n,\tilde{m}_n,k)}$;
- The second term is decreasing in y due to Remark 2.11, (ii), since $m_j \ge m_i$ for all $i \le j$;
- The third term is decreasing in y since $m_j \ge m_i$ for all $i \le j$.

Furthermore, according to the assumptions of the theorem and Lemma 2.8, we have $G(\cdot\,|\,x_1) \ge_{st} G(\cdot\,|\,x_2)$ for $x_1 \le x_2$. Now, part (i) of Lemma 2.5 implies that $E[\phi(Y,x)]$ is increasing in x.

Case 2: $r = 2$. From (2) and (6), we have

$$\begin{aligned}\left[\frac{f_{D_{(1,s',n',\tilde{m}_{n'},k')}}(x)}{f_{D_{(2,s,n,\tilde{m}_n,k)}}(x)}\right]^{-1} &= \int_0^{+\infty}\left[\frac{\bar{F}(x+y)}{\bar{F}(x)}\right]^{\gamma_{(s,n,\tilde{m}_n,k)}-1}[\bar{F}(x)]^{\gamma_{(s,n,\tilde{m}_n,k)}-\gamma_{(s',n',\tilde{m}_{n'},k')}}\\ &\quad\times\frac{\psi_{s-2}(\bar{F}_y(x))}{g_{s'}(F(x))}\,\frac{f(x+y)}{f(x)}\cdot\nu(y)\,dy\\ &= \int_0^{+\infty}\left[\frac{\bar{F}(x+y)}{\bar{F}(x)}\right]^{\gamma_{(s,n,\tilde{m}_n,k)}}[\bar{F}(x)]^{\gamma_{(s,n,\tilde{m}_n,k)}-\gamma_{(s',n',\tilde{m}_{n'},k')}}\\ &\quad\times\frac{\psi_{s-2}(\bar{F}_y(x))}{g_{s'}(F(x))}\,\frac{h(x+y)}{h(x)}\cdot\nu(y)\,dy,\end{aligned}$$

where $\nu(y)$ does not depend on x. First, note that

$$\frac{\psi_{s-2}(\bar{F}_y(x))}{g_{s'}(F(x))} = \frac{\psi_{s'-1}(\bar{F}_y(x))}{g_{s'}(F(x))}$$

is decreasing in x by Lemma 2.9. Now, according to the assumptions of the theorem, $f_{D_{(1,s',n',\tilde{m}_{n'},k')}}(x)/f_{D_{(2,s,n,\tilde{m}_n,k)}}(x)$ is increasing in x. Thus, the proof is completed. □

Remark 3.6. *With the restrictions $k = k'$ and $m_i = m_i'$, Hu and Zhuang [21] proved the statements of Theorems 3.3 and 3.4 under the additional condition $m_1 = \cdots = m_{n-1}$ in their Theorem 4.1(c). Also, by choosing $s = r+p-1$ and $s' = r'+(p+1)-1$ with $r' = r-1$ in our Theorem 3.5, one can see that Theorem 4.4 of Hu and Zhuang [21] is a special case of Theorem 3.5.*

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Theorem 3.7. *Let $X_{(r,n,\tilde{m}_n,k)}$ and $X_{(r,n',\tilde{m}'_{n'},k')}$, $r = 1,\dots,\max\{n,n'\}$, be the GOS based on a common absolutely continuous cdf F. If $r = r' \geq 2$ and $s+1 \leq s'$, then*

$$D_{(r,s,n,\tilde{m}_n,k)} \geq_{lr} D_{(r',s',n',\tilde{m}'_{n'},k')}$$

provided that $m'_j \geq m_i$ for all $i \leq j$, m'_i is increasing in i, $\gamma_{(s',n',\tilde{m}'_{n'},k')} = \gamma_{(s,n,\tilde{m}_n,k)}$ and

$$\sum_{j=r-1}^{s-2}(m'_j - m_j) + \sum_{j=s-1}^{s'-2} m'_j \geq 0$$

holds and that at least one of the following conditions hold: assumption A with $\gamma_{(s,n,\tilde{m}_n,k)} \geq 1$, or assumption B.

Proof. From (6), we have

$$\frac{f_{D_{(r,s',n',\tilde{m}'_{n'},k')}}(x)}{f_{D_{(r,s,n,\tilde{m}_n,k)}}(x)} = E[\phi(Y,x)],$$

where

$$\begin{aligned}
\phi(y,x) &\propto [\bar{F}(x+y)]^{\gamma_{(s',n',\tilde{m}'_{n'},k')}-\gamma_{(s,n,\tilde{m}_n,k)}}\frac{\check{\psi}_{s'-r}(\bar{F}_y(x))}{\psi_{s-r}(\bar{F}_y(x))}\frac{\check{g}_{r-1}(F(y))}{g_{r-1}(F(y))}\\
&\quad\times[\bar{F}(y)]^{(\gamma_{(r-1,n',\tilde{m}'_{n'},k')}-\gamma_{(s',n',\tilde{m}'_{n'},k')})-(\gamma_{(r-1,n,\tilde{m}_n,k)}-\gamma_{(s,n,\tilde{m}_n,k)})},\\
&= [\bar{F}(x+y)]^{\gamma_{(s',n',\tilde{m}'_{n'},k')}-\gamma_{(s,n,\tilde{m}_n,k)}}\cdot\Delta_{(r,r,s,s)}(x,y)\cdot\frac{\check{\psi}_{s'-r}(\bar{F}_y(x))[\bar{F}(y)]^{m'_{s'-1}}}{\check{\psi}_{s-r}(\bar{F}_y(x))[\bar{F}(y)]^{m'_{s-1}}}\frac{\check{g}_{r-1}(F(y))}{g_{r-1}(F(y))}\\
&\quad\times[\bar{F}(y)]^{(\gamma_{(r-1,n',\tilde{m}'_{n'},k')}-\gamma_{(s',n',\tilde{m}'_{n'},k')})-(\gamma_{(r-1,n,\tilde{m}_n,k)}-\gamma_{(s,n,\tilde{m}_n,k)})-(m'_{s-1}-m_{s-1})-(m'_{s'-1}-m'_{s-1})}\\
&= [F(x+y)]^{\gamma_{(s',n',\tilde{m}'_{n'},k')}-\gamma_{(s,n,\tilde{m}_n,k)}}\cdot\Delta_{(r,r,s,s)}(x,y)\cdot\frac{\check{\psi}_{s'-r}(\bar{F}_y(x))[\bar{F}(y)]^{m'_{s'-1}}}{\check{\psi}_{s-r}(\bar{F}_y(x))[\bar{F}(y)]^{m'_{s-1}}}\frac{\check{g}_{r-1}(F(y))}{g_{r-1}(F(y))}\\
&\quad\times[\bar{F}(y)]^{(\sum_{j=r-1}^{s-2} m'_j-m_j)+(\sum_{j=s-1}^{s'-2} m'_j)},
\end{aligned}\tag{16}$$

$\Delta_{(r,r,s,s)}(x,y)$ is defined as in (9), and Y is a nonnegative random variable having a cdf belonging to the family $\mathcal{P} = \{G(\cdot\,|\,x), x \in \mathbb{R}_+\}$ with the pdf defined as in (8). It is seen that the following hold in (16):

- The first term is constant with respect to x and y because of $\gamma_{(s',n',\tilde{m}'_{n'},k')} = \gamma_{(s,n,\tilde{m}_n,k)}$;
- The second term is decreasing in x and y due to Lemma 2.12 since $m'_j \geq m_i$ for all $i \leq j$;
- The third term is decreasing in x and y due to parts (ii) and (iii) of Lemma 2.6, since m'_i is increasing in i;
- The fourth term is decreasing in y due to part (i) of Remark 2.11, since $m'_j \geq m_i$ for all $i \leq j$;
- The fifth term is decreasing in y because of $\sum_{j=r-1}^{s-2}(m'_j - m_j) + \sum_{j=s-1}^{s'-2} m'_j \geq 0$.

Furthermore, according to the assumptions of the theorem and Lemma 2.8, we have $G(\cdot\,|\,x_1) \leq_{st} G(\cdot\,|\,x_2)$ for $x_1 \leq x_2$. Now, part (ii) of Lemma 2.5 implies that $E[\phi(Y,x)]$ is decreasing in x. □

Let $D^{(p)}_{(r,n)}$ denote the p-spacings of OS. As a corollary of the above theorem, if $X \in$ DLR, then $D^{(p)}_{(r,n)} \geq_{\text{lr}} D^{(p+1)}_{(r,n+1)}$, because of $m'_i = m_i = 0$ and

$$1+(n+1)-(r+(p+1)-1) = \gamma_{(s',n',\tilde{m}'_{n'},k')} = \gamma_{(s,n,\tilde{m}_n,k)} = 1+n-(r+p-1).$$

Remark 3.8. *By choosing the parameters of GOS appropriately, our general results can be used to compare the p-spacings from submodels of GOS. In this way, we can obtain results for p-spacings*

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from k-record values (cf. [23,24]), progressive Type-II right censored order statistics with arbitrary censoring schemes (cf. [6]), and order statistics under multivariate imperfect repair (cf. [9]). More generally, we can compare p-spacings from different submodels. For example, we can compare that obtained from OS and sequential order statistics (cf. Cramer and Kamps [14]), from Pfeifer's record values and the epoch times of a nonhomogeneous Poisson process (cf. [8]), and so forth.

4. Relations among logconvexity properties

The notion of logconcavity/logconvexity plays an important role not only in different areas of Statistics but also in Mathematics, Economics, etc. In Reliability Theory, the different aspects of this concept have been studied so far (see, e.g., [3,29,30], and references therein). First, we give some results relating to logconvexity. Pellerey *et al.* [32] proved that if the hazard rate function $h(x)$ is logconcave, then the pdf $f(x)$ is logconcave. For the reversed hazard rate function $\kappa(x)$, Alimohammadi *et al.* [2] proved that if $\kappa(x)$ is logconcave, then the pdf $f(x)$ is logconcave. For logconvexity, we have the following implications.

Proposition 4.1. *Let X be a random variable with an absolutely continuous distribution and support (a, ∞) for a real number a.*

(i) If h is logconvex and decreasing, then f is logconvex;
(ii) If f is logconvex, then κ is logconvex.

Proof. (i) As $h(x) = f(x)/\bar{F}(x)$, we get

$$\ln f(x) = \ln h(x) + \ln \bar{F}(x) = \ln h(x) - \int_a^x h(u)\,du. \tag{17}$$

Since h is decreasing, it follows that $\int_a^x h(u)\,du$ is concave which means that $-\int_a^x h(u)\,du$ is convex. So, (17) becomes the sum of two convex functions. Therefore, $\ln f$ is convex.

(ii) According to Theorem 2.2, if f is logconvex, then F is logconcave which in turn implies that $1/F$ is logconvex. Thus, $\kappa(x) = f(x) \cdot (F(x))^{-1}$ is logconvex. □

Moreover, it is well known that if f is logconvex with support (a, ∞), then κ is decreasing (see e.g., p. 315 in [30] and Figure 2 of [3]).

We complete the analysis of the implications in Proposition 4.1 with some examples. The following example satisfies the assumptions in that proposition.

Example 4.2. *Suppose that X has a Pareto distribution with pdf $f(x) = (1 + x)^{-2}$, $x \geq 0$. By some direct calculations, one can see that $h(x) = (1+x)^{-1}$ is logconvex and decreasing. Also, f is logconvex and $\kappa(x) = (x^2+x)^{-1}$ is logconvex and decreasing. Also, one can consider the exponential distribution as an obvious example.*

The following counterexamples show that the implications (ii) and (i) in Proposition 4.1 do not hold in the reverse direction, respectively.

Counterexample 4.3. *Suppose that X has the generalized exponential distribution with pdf $f(x) = \beta e^{-x}(1 - e^{-x})^{\beta-1}$, $x \geq 0$, $\beta > 0$. It is easy to see that $\kappa(x) = \beta(e^x - 1)^{-1}$ is logconvex and decreasing for all $\beta > 0$. However, for $\beta > 1$, f is strictly logconcave (see, e.g., Table 1 of [3]).*

Counterexample 4.4. *Suppose that X has a truncated Cauchy distribution with pdf $f(x) = (4/\pi)(1 + x^2)^{-1}$ for $x \geq 1$. It is logconvex over $x > 1$. The hazard rate function is $h(x) = (1+x^2)^{-1}/(\pi/2 - \arctan(x))$ for $x \geq 1$. However, Figure 1 shows that $(\ln h(x))''$ takes negative values and, therefore, h is not logconvex.*

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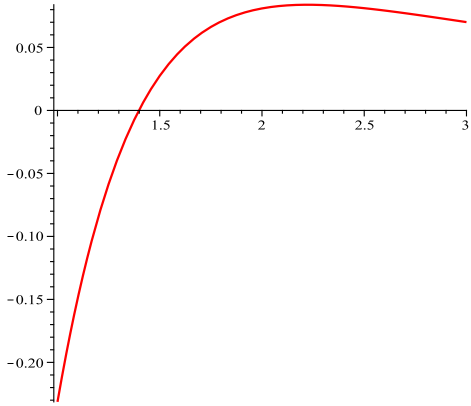

Figure 1. *Plot of* $(\ln h(x))''$ *in Counterexample 4.4.*

Remark 4.5. *We should note that according to the previous logconcavity findings and Proposition 4.1, there are two differences as follows:*

(i) If h is logconcave, then f *is logconcave, and, if h is logconvex (with the condition that it is decreasing), then* f *is logconvex. But, for inheritance of logconcavity/logconvexity among* f *and* κ*, we have*

$$\kappa \underset{\text{logconvexity}}{\overset{\text{logconcavity}}{\rightleftarrows}} f;$$

(ii) Pellerey et al. [32] proved that if h is logconcave, then it must be increasing. Using this result, they showed that logconcavity of h implies that of f*. However, logconvexity of h does not necessarily imply that it is decreasing. As a counterexample, suppose that* $f(x) = e^{x+1-e^x}$, $x \geq 0$*. The corresponding hazard rate function* $h(x) = e^x$ *is logconvex and increasing. Also, since* f *is logconcave (see, e.g., [32]), this counterexample shows that the condition h is decreasing can not be relaxed in Proposition 4.1.*

Now, we consider the preservation of logconvexity among spacings of GOS. For OS, Misra and van der Meulen [28] proved that if X is DLR, then the simple spacings are also DLR, and if X is ILR, then the p-spacings are also ILR. According to their Remark 3.1, the result for p-spacings ($p \geq 2$) is not valid for the DLR case. Then, Hu and Zhuang [21] extended these results for GOS under the condition $m_1 = \cdots = m_{n-1}$. Finally, Chen *et al.* [11] gave the result without condition $m_1 = \cdots = m_{n-1}$ just for ILR case. The following theorem states the result for the DLR case without condition $m_1 = \cdots = m_{n-1}$.

Theorem 4.6. *Let* $X_{(r,n,\tilde{m}_n,k)}$, $r = 1, \ldots, n$*, be the GOS based on an absolutely continuous cdf F and let us assume* $\gamma_{(r,n,\tilde{m}_n,k)} \geq 1$ *for* $r = 1, \ldots, n$*. If X is DLR, then the simple spacings* $D_{(r,r,n,\tilde{m}_n,k)}$ *are also DLR for* $r = 1, \ldots, n$*.*

Proof. We give the proof in two cases.

Case 1: $r \geq 2$. First note that a function f is logconvex if, and only if, $f(x+\varepsilon)/f(x)$ is increasing in x for all $\varepsilon > 0$. From (6), we have

$$\frac{f_{D_{(r,r,n,\tilde{m}_n,k)}}(x+\varepsilon)}{f_{D_{(r,r,n,\tilde{m}_n,k)}}(x)} = E[\phi(\tilde{Y}, x)],$$

where

$$\phi(y, x) = \left[\frac{\bar{F}(x+\varepsilon+y)}{\bar{F}(x+y)}\right]^{\gamma_{(r,n,\tilde{m}_n,k)}-1} \frac{f(x+\varepsilon+y)}{f(x+y)},$$

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and $\tilde{Y}$ is a nonnegative random variable having a cdf belonging to the family $\tilde{\mathcal{P}} = \{\tilde{G}(\cdot\,|\,x), x \in \mathbb{R}_+\}$ with corresponding pdf

$$\tilde{\zeta}(y\,|\,x) = c(x)[\bar{F}(x+y)]^{\gamma_{(r,n,\tilde{m}_n,k)}-1} f(x+y)[\bar{F}(y)]^{m_{r-1}} g_{r-1}(F(y)) f(y),$$

where $c(x)$ is the normalizing constant. According to Theorem 2.2, $\phi(y,x)$ is increasing in x and y. Also, it is easy to see that $\tilde{G}(\cdot\,|\,x_1) \leq_{\mathrm{lr}} \tilde{G}(\cdot\,|\,x_2)$ for $x_1 \leq x_2$. Thus, part (i) of Lemma 2.5 implies that $E[\phi(\tilde{Y},x)]$ is increasing in x.

Case 2: $r = 1$. According to (2) and Theorem 2.2, $f_{D_{(1,1,n,\tilde{m}_n,k)}}$ becomes the product of logconvex functions and, thus, it is logconvex. Therefore, the proof is completed. □

As seen in Appendix, the DLR property is not preserved by GOS for different values of parameter m_i including record values. However, the record values and, more generally, the k-record values from the exponential distribution (which is both logconvex and logconcave) are logconcave. The pdf of the rth k-record value X_r^* is given by

$$f_{X_r^*}(x) = k^r(r-1)![\bar{F}(x)]^{k-1}[-\ln \bar{F}(x)]^{r-1} f(x), \tag{18}$$

(see, e.g., [5]). For the exponential distribution with parameter $\beta > 0$, we have $-\ln \bar{F}(x) = \beta x$ which is logconcave. In another direction, if f is logconcave (logconvex), then $\bar{F}$ is logconcave (logconvex). So, (18) becomes the product of logconcave functions and, therefore, $f_{X_r^*}$ is logconcave as claimed. This is also deduced from Corollary 2.4 of Chen *et al.* [11] where it is stated that the logconcavity of f is not sufficient for the logconcavity of record values and that we need a stronger condition that h is logconcave. Since h is both logconcave and logconvex in the exponential distribution, this example also reveals that the record values could not be logconvex even under the logconvexity of hazard rate h.

5. Discussion

In this article, we have studied the likelihood ratio ordering of p-spacings of ordered random variables under the GOS model. We not only have strengthened and complemented some previous findings but also have obtained some new results. In Table 1, we summarize the previous findings and the new results obtained with respect of the problems stated in the introduction.

There are still two open problems:

(I) Is problem (P_5) valid for GOS with different m_i and m_i' and $p \geq 2$?
(II) Is problem (P_6) [or (P_7)] valid for GOS with different m_i and m_i'?

Table 1. *Problems on likelihood ratio ordering of p-spacings.*

	Order statistics	GOS: equal m_i	GOS: unequal m_i	GOS: different m_i and m_i'
(P_1)	Misra and van der Meulen [28]	Hu and Zhuang [21]	Xie and Hu [37]	Theorem 3.1
(P_2)	Misra and van der Meulen [28]	Hu and Zhuang [21]	Xie and Hu [37]	Theorem 3.1
(P_3)	Hu and Zhuang [22]	Hu and Zhuang [21]	Xie and Hu [37]	Theorem 3.1
(P_4)	Hu and Zhuang [22]	Hu and Zhuang [21]	Xie and Hu [37]	Theorem 3.1
(P_5)	Hu and Zhuang [22]	Hu and Zhuang [21]	Theorem 3.4	Theorem 3.3 for $p = 1$
(P_6)	Hu and Zhuang [22]	Hu and Zhuang [21]	Theorem 3.5	?
(P_7)	–	–	Theorem 3.5	?
(P_8)	–	–	–	Theorem 3.7

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However, according to our numerous computations (two of which are given in Example 5.3), we conjecture that both problems are valid as follows:

Conjecture 5.1. *Theorem 3.4 including different* $\tilde{m}_n$ *and* $\tilde{m}'_{n'}$ *is valid with the following additional condition for* $r \geq 2$: $m'_j \leq m_i$ *for all* $i \leq j$.

Conjecture 5.2. *Theorem 3.5 including different* $\tilde{m}_n$ *and* $\tilde{m}'_{n'}$ *is valid with the following additional condition for* $r \geq 3$: $m'_j \geq m_i$ *for all* $i \leq j$.

Example 5.3. *Let* X *be the uniform distribution with pdf* $f(x) = 1, 0 \leq x \leq 1$, *which is logconcave. First, for Conjecture 5.1 consider* $r = 2$, $n = 3$, $s = 3$ *(and hence* $p = 2$*),* $\tilde{m}_2 = \{2, 1\}$, $k = 1$ *and* $r' = r + 1 = 3$, $n' = n + 1 = 4$, $s' = 4$ *(and hence* $p = 2$*),* $\tilde{m}'_3 = \{1, 0, 0\}$, $k' = 1$. *One can see that the parameters satisfy the conditions of Conjecture 5.1. Figure 2 shows that* $\rho_1(x) = f_{D_{(3,4,4,\{1,0,0\},1)}}(x)/f_{D_{(2,3,3,\{2,1\},1)}}(x)$ *is decreasing in* $x \in (0, 1)$, *and, thus* $D_{(2,3,3,\{2,1\},1)} \geq_{lr} D_{(3,4,4,\{1,0,0\},1)}$ *holds.*

Now, for Conjecture 5.2, consider $r = 3$, $n = 3$, $s = 3$ *(and hence* $p = 1$*),* $\tilde{m}_2 = \{0, 1\}$, $k = 1$ *and* $r' = r - 1 = 2$, $n' = n = 3$, $s' = 3$ *(and hence* $p = 2$*),* $\tilde{m}'_2 = \{1, 2\}$, $k' = 1$. *Obviously, the parameters satisfy the conditions of Conjecture 5.2. Figure 3 shows that* $\rho_2(x) = f_{D_{(2,3,3,\{1,2\},1)}}(x)/f_{D_{(3,3,3,\{0,1\},1)}}(x)$ *is increasing in* $x \in (0, 1)$, *and thus* $D_{(3,3,3,\{0,1\},1)} \leq_{lr} D_{(2,3,3,\{1,2\},1)}$ *holds.*

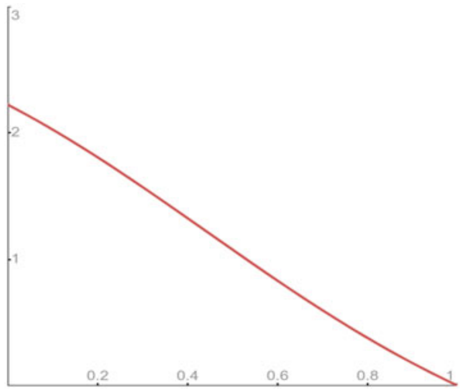

Figure 2. *Plot of* $\rho_1(x)$ *in Example 5.3.*

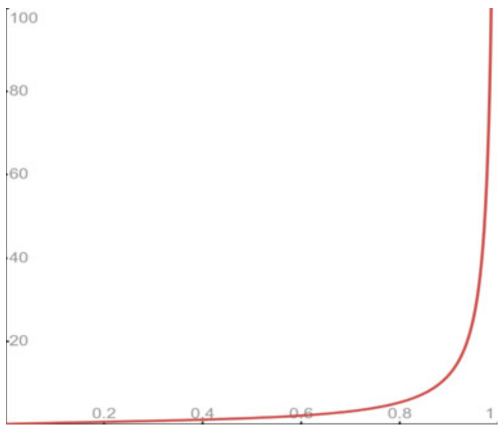

Figure 3. *Plot of* $\rho_2(x)$ *in Example 5.3.*

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Appendix

In this appendix, we first give the proofs of the lemmas included in Section 2. Then, we investigate the preservation of logconvexity among GOS.

Proof of Lemma 2.6. (i) From (7), for all $y_1 \le y_2$, we have

$$\begin{aligned}\frac{\psi_{s-r}(\bar{F}_{y_2}(x))}{\psi_{s-r}(\bar{F}_{y_1}(x))} &= \frac{\int_{\mathbb{R}} I_{\{0\le u\le x\}}\psi_{s-r-1}(\bar{F}_{y_2}(u))[\bar{F}_{y_2}(u)]^{m_{s-1}}\frac{f(u+y_2)}{\bar{F}(y_2)}\,du}{\int_{\mathbb{R}} I_{\{0\le u\le x\}}\psi_{s-r-1}(\bar{F}_{y_1}(u))[\bar{F}_{y_1}(u)]^{m_{s-1}}\frac{f(u+y_1)}{\bar{F}(y_1)}\,du}\\ &= E[\phi(U,x)],\end{aligned}$$

where I_A is the indicator function,

$$\phi(u,x) \propto \frac{\psi_{s-r-1}(\bar{F}_{y_2}(u))}{\psi_{s-r-1}(\bar{F}_{y_1}(u))}\left[\frac{\bar{F}(u+y_2)}{\bar{F}(u+y_1)}\right]^{m_{s-1}}\frac{f(u+y_2)}{f(u+y_1)} \tag{A.1}$$

$$= \frac{\psi_{s-r-1}(\bar{F}_{y_2}(u))}{\psi_{s-r-1}(\bar{F}_{y_1}(u))}\left[\frac{\bar{F}(u+y_2)}{\bar{F}(u+y_1)}\right]^{m_{s-1}+1}\frac{h(u+y_2)}{h(u+y_1)}, \tag{A.2}$$

and U is a nonnegative random variable having a cdf belonging to the family $\mathcal{P} = \{G(\cdot\,|\,x,y_1), x, y_1 \in \mathbb{R}_+\}$ with corresponding pdf

$$g(u\,|\,x,y_1) = c(x,y_1) I_{\{0\le u\le x\}}\psi_{s-r-1}(\bar{F}_{y_1}(u))[\bar{F}_{y_1}(u)]^{m_{s-1}}\frac{f(u+y_1)}{\bar{F}(y_1)}, \tag{A.3}$$

in which

$$c(x,y_1) = \left[\int_0^x \psi_{s-r-1}(\bar{F}_{y_1}(z))[\bar{F}_{y_1}(z)]^{m_{s-1}}\frac{f(z+y_1)}{\bar{F}(y_1)}\,dz\right]^{-1},$$

is the normalizing constant. First, note that, for $x_1 \le x_2$,

$$\frac{g(u\,|\,x_2,y_1)}{g(u\,|\,x_1,y_1)} \propto \frac{I_{\{0\le u\le x_2\}}}{I_{\{0\le u\le x_1\}}},$$

is increasing in u because $I_{\{0\le u\le x\}}$ is TP_2 in (x,u). Let $s-r=1$. According to (A.1), one can see that $\phi(u,x)$ is increasing (decreasing) in u when f is logconvex (logconcave) and $m_i \ge 0$. Also, according to (A.2), $\phi(u,x)$ is increasing (decreasing) in u when f and h are logconvex (logconcave) and $-1 \le m_i < 0$ (note that if h is logconcave, then f is so, see [32]). Furthermore, $\phi(u,x)$ is constant with respect to x. Thus, the desired result follows by induction and Lemma 2.5.

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(ii) From (7), we have

$$\psi_{s-r}(\bar{F}_y(x))[\bar{F}(y)]^{m_{s-1}} = \int_0^x \psi_{s-r-1}(\bar{F}_y(u))[\bar{F}(u+y)]^{m_{s-1}} \frac{f(u+y)}{\bar{F}(y)}\, du \tag{A.4}$$

$$= \int_0^x \psi_{s-r-1}(\bar{F}_y(u))[\bar{F}(u+y)]^{m_{s-1}+1} \frac{h(u+y)}{\bar{F}(y)}\, du. \tag{A.5}$$

We prove the stated result only under assumptions A or A′ by using (A.4). The proof under assumption B or B′ from (A.5) is similar and, thus, omitted. Consider $[\bar{F}(u+y)]^{m_{s-1}}$ as a function of three variables (u, y, s).

First, suppose that m_i is decreasing in i. Then, $[\bar{F}(u+y)]^{m_{s-1}}$ is TP_2 in (u, s) and (y, s). If f is logconvex and $m_i \geq 0$, then it is also TP_2 in (u, y). Also, when f is logconvex, then $f(u+y)$ is TP_2 in (u, y). By induction and part (i.a) of Theorem 2.4, we can conclude that the integral in (A.4) is TP_2 in (y, s).

Let us assume now that m_i is increasing in i. By means of a similar approach and using part (ii.b) of Theorem 2.4 this time, we can conclude that the integral in (A.4) is RR_2 in (y, s).

(iii) From (7), we have

$$\psi_{s-r}(\bar{F}_y(x)) = \int_{\mathbb{R}} I_{\{0 \leq u \leq x\}} \psi_{s-r-1}(\bar{F}_y(u)) \left[\bar{F}_y(u)\right]^{m_{s-1}} \frac{f(u+y)}{\bar{F}(y)}\, du.$$

If m_i is decreasing (increasing) in i, then $[\bar{F}(u+y)]^{m_{s-1}}$ is TP_2 (RR_2) in (u, s). Now, since $I_{\{0 \leq u \leq x\}}$ is TP_2 in (x, u), we have the desired result using induction and part (i.a) (part (ii.a)) of Theorem 2.4. □

Proof of Lemma 2.8. To prove $G(\cdot\,|\,x_1) \leq_{\mathrm{lr}} (\geq_{\mathrm{lr}}) G(\cdot\,|\,x_2)$ for $x_1 < x_2$, we consider the ratio

$$\frac{\zeta(y\,|\,x_2)}{\zeta(y\,|\,x_1)} \propto \left[\frac{\bar{F}(x_2+y)}{\bar{F}(x_1+y)}\right]^{\gamma(s,n,\tilde{m}_n,k)-1} \frac{\psi_{s-r}(\bar{F}_y(x_2))}{\psi_{s-r}(\bar{F}_y(x_1))} \frac{f(x_2+y)}{f(x_1+y)}$$

$$= \left[\frac{\bar{F}(x_2+y)}{\bar{F}(x_1+y)}\right]^{\gamma(s,n,\tilde{m}_n,k)} \frac{\psi_{s-r}(\bar{F}_y(x_2))}{\psi_{s-r}(\bar{F}_y(x_1))} \frac{h(x_2+y)}{h(x_1+y)}.$$

Now, the result follows according to the assumptions stated in the lemma and part (i) of Lemma 2.6. □

Proof of Lemma 2.9. From (4) and (7), we have

$$\frac{\psi_{s-1}(\bar{F}_y(x))}{g_s(F(x))} = E[\phi(U, x)],$$

where

$$\phi(u, x) \propto \frac{\psi_{s-2}(\bar{F}_y(u))}{g_{s-1}(F(u))} \frac{[\bar{F}(u+y)]^{m_{s-1}}}{[\bar{F}(u)]^{m_{s-1}}} \frac{f(u+y)}{f(u)}$$

$$= \frac{\psi_{s-2}(\bar{F}_y(u))}{g_{s-1}(F(u))} \left[\frac{\bar{F}(u+y)}{\bar{F}(u)}\right]^{m_{s-1}+1} \frac{h(u+y)}{h(u)},$$

and U is a nonnegative random variable having a cdf belonging to the family $\mathcal{P} = \{G(\cdot\,|\,x), x \in \mathbb{R}_+\}$ with corresponding pdf

$$g(u\,|\,x) = c(x) I_{\{0 \leq u \leq x\}} g_{s-1}(F(u)) [\bar{F}(u)]^{m_{s-1}} f(u),$$

in which

$$c(x) = \left[\int_0^x g_{s-1}(F(z)) [\bar{F}(z)]^{m_{s-1}} f(z)\, dz\right]^{-1},$$

is the normalizing constant. Obviously, $G(\cdot\,|\,x_1) \leq_{\text{lr}} G(\cdot\,|\,x_2)$ for $x_1 < x_2$, and, according to the conditions of lemma, $\phi(u, x)$ is increasing (decreasing) in u. Also, $\phi(u, x)$ is constant with respect to x. Thus, the result follows by induction and part (i) (part (ii)) of Lemma 2.5. □

Proof of Lemma 2.12. (i) From (7), we have

$$\Delta_{(r,r',s,s')}(x, y) = E[\phi(U, y)],$$

where

$$\begin{aligned}\phi(u, y) &\propto \frac{\check{\psi}_{s'-r'-1}(\bar{F}_y(u))}{\psi_{s-r-1}(\bar{F}_y(u))} \left[\frac{\bar{F}(u+y)}{\bar{F}(y)}\right]^{m'_{s'-1}-m_{s-1}} \cdot [\bar{F}(y)]^{m'_{s'-1}-m_{s-1}} \\ &= \frac{\check{\psi}_{s'-r'-1}(\bar{F}_y(u))}{\psi_{s-r-1}(\bar{F}_y(u))} [\bar{F}(u+y)]^{m'_{s'-1}-m_{s-1}},\end{aligned}$$

and U is a nonnegative random variable having a cdf belonging to the family $\mathcal{P} = \{G(\cdot\,|\,x, y), x, y \in \mathbb{R}_+\}$ with the pdf given in (A.3). By assumptions, we have $G(\cdot\,|\,x, y_1) \leq_{\text{lr}} G(\cdot\,|\,x, y_2)$ for $y_1 < y_2$. Let $s' - r' = s - r = 1$. Because of $m'_j \leq m_i$ ($m'_j \geq m_i$) for all $i \leq j$, $\phi(u, y)$ is increasing (decreasing) in both u and y. So, the desired result follows by induction and part (i) (part (ii)) of Lemma 2.5.

(ii) The method of proof is similar to that of part (i) and so it is omitted. □

Finally, we examine the preservation of logconvexity among GOS. Cramer [12] and Chen *et al.* [11] proved the closure of ILR property among GOS without the condition $m_1 = \cdots = m_{n-1}$. In the following counterexample, we show that the DLR property is not preserved among GOS. To do this, obviously we do not need that m_i's are not equal, and so we assume $m_i = m$ for all i. We consider four cases: $m = -1$ (record values), $-1 < m < 0$, $m = 0$ (OS), and $m > 0$. Before giving the counterexample, according to (2) and Theorem 2.2, it is worth noting that the DLR property is preserved by the smallest GOS if $\gamma_{(1,n,\tilde{m}_n,k)} \geq 1$. This contains the lifetime of series systems and, more generally, sequential series systems with parameter $\alpha_1 \geq 1/n$ (cf. model in (1)).

Counterexample A.1. *Consider the pdf* $f(x) = (e^{-x} + 2e^{-2x})/2$ *for* $x \geq 0$. *Since the mixture of logconvex densities is logconvex (cf. [7] p. 103), it follows that* f*, being a mixture of exponential densities, is logconvex. Consider* $k = 1$, $r = 2$ *and* $n = 3$. *As shown in Figures A.1–A.4,* $(\ln f(x))''$ *is not nonnegative for different values of* m.

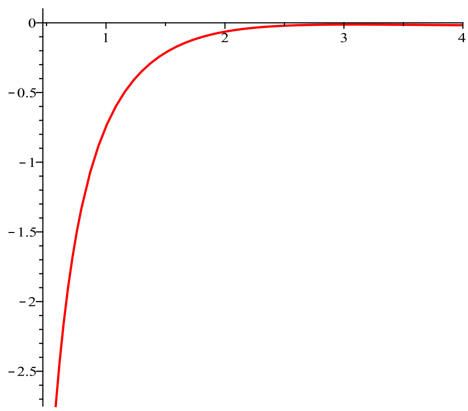

Figure A.1. *Plot of* $(\ln f(x))''$ *for* $m = -1$.

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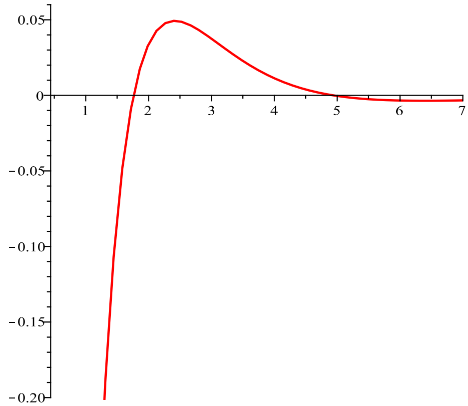

Figure A.2. *Plot of* $(\ln f(x))''$ *for* $m = -0.5$.

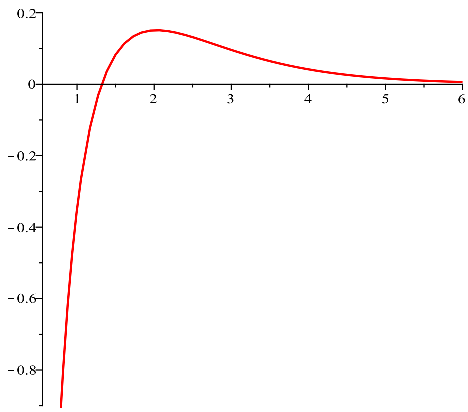

Figure A.3. *Plot of* $(\ln f(x))''$ *for* $m = 0$.

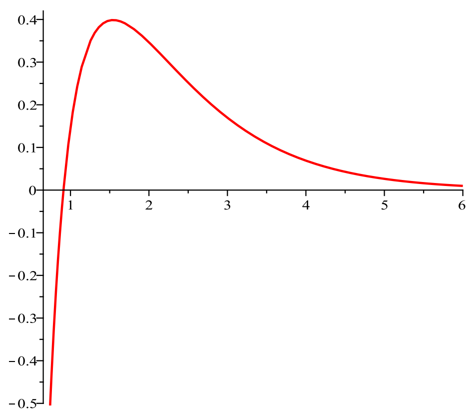

Figure A.4. *Plot of* $(\ln f(x))''$ *for* $m = 1$.

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