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ON RADIAL VARIATION OF HOLOMORPHIC FUNCTIONS WITH l^p TAYLOR COEFFICIENTS

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Suppose $f(z)=\sum_{n=0}^{\infty}a_nz^n$ is holomorphic in $\Delta=\{z:|z|<1\}$ and $(a_n)\in l^p$ where $1\leqq p\leqq 2$. We prove that $\int_0^r|f^{(k)}(te^{i\theta})|^{1/k}\,dt=o(\log 1/(1-r))^{1-1/pk}$ for $k=1,2,\ldots$, and almost every θ. This result is sharp in the following sense: Let $p\in[1,2]$ and $\varepsilon(r)$ be a positive function defined on $[0,1)$ such that $\lim_{r\to 1^-}\varepsilon(r)=0$. Then there exists a function $f(z)=\sum_{n=0}^{\infty}a_nz^n$ holomorphic in Δ with $(a_n)\in l^p$ such that

$$\overline{\lim_{r\to 1^-}}\frac{\int_0^r \min_{|z|=t}|f^{(k)}(z)|^{1/k}\,dt}{\varepsilon(r)\left(\log\dfrac{1}{1-r}\right)^{1-1/pk}}=+\infty$$

for each $k>1/p$.

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Introduction

In this paper we determine the precise almost everywhere radial variation of all derivatives of the class of functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$ holomorphic in $\Delta=\{z:|z|<1\}$ and satisfying $(a_n)\in l^p$ where $1\leqq p\leqq 2$.

Radial variation

We first prove the following technical lemma.

Lemma 1. *For each $p\in[1,2]$ and $k=1,2,\ldots$ there is a constant $A=A_{p,k}$ depending only on p and k such that for each $(a_n)\in l^p$ we have*

$$\int_0^{2\pi}\int_0^1(1-t)^{pk-1}|f^{(k)}(te^{i\theta})|^p\,dt\,d\theta\leqq A\sum_{n=k}^{\infty}|a_n|^p \tag{1}$$

where $f(z)=\sum_{n=0}^{\infty}a_nz^n$, $z\in\Delta$.

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Proof. Let T be an operator defined by

$$T((a_n))=g \tag{2}$$

where g is a function on Δ defined by

$$g(z)=(1-|z|)^k\frac{d^{(k)}}{dz^{(k)}}\left(\sum_{n=0}^{\infty}a_nz^n\right). \tag{3}$$

Using the facts that $|f^{(k)}(te^{i\theta})|\leqq\sum_{n=k}^{\infty}n^k|a_n|t^{n-k}$ $(k=1,2,\ldots)$ and $\int_0^1(1-t)^{pk-1}t^{p(n-k)}\,dt=0(1/n^{pk})$ when $p=1$ or $p=2$ it is easy to prove that

$$\int_0^{2\pi}\int_0^r(1-t)^{pk-1}|f^{(k)}(te^{i\theta})|^p\,dt\,d\theta=0\left(\sum_{n=k}^{\infty}|a_n|^p\right) \tag{4}$$

when $p=1$ or $p=2$. It follows from (4) that T is a bounded linear operator from l^p to $L^p(\Delta,\mu)$ for $p=1$ or $p=2$ when $d\mu=1/(1-r)\,dr\,d\theta$. The Riesz–Thorin interpolation theorem [3] implies that T is a bounded linear operator from l^p to $L^p(\Delta,\mu)$ for all $p\in[1,2]$. Hence (1) holds and the proof is complete.

Corollary 2. *If* $(a_n)\in l^p$, $p\in[1,2]$ *and* $f(z)=\sum_{n=0}^{\infty}a_nz^n$, $z\in\Delta$ *then*

$$\int_0^1(1-t)^{pk-1}|f^{(k)}(te^{i\theta})|^p\,dt<+\infty \tag{5}$$

for $k=1,2,\ldots$ *and almost every* θ.

Proof. This follows directly from (1) by using Tonelli's theorem.

Theorem 3. *If* $p\in[1,2]$, $(a_n)\in l^p$, $k=1,2,\ldots$, $kp>1$ *and* $f(z)=\sum_{n=0}^{\infty}a_nz^n$, $z\in\Delta$, *then*

$$\int_0^r|f^{(k)}(te^{i\theta})|^{1/k}\,dt=o\left(\log\frac{1}{1-r}\right)^{1-1/pk} \tag{6}$$

for almost every θ.

Proof. Choose $\theta\in[0,2\pi]$ so that (5) holds. Given $\varepsilon>0$ for this θ there exists $r_0\in(0,1)$ so that

$$\int_{r_0}^r(1-t)^{pk-1}|f^{(k)}(te^{i\theta})|^p\,dt<\varepsilon \tag{7}$$

for all $r>r_0$. It follows easily from (7) and Hölder's inequality that

$$\frac{1}{\left(\log\frac{1}{1-r}\right)^{1-1/pk}}\int_0^r |f^{(k)}(te^{i\theta})|^{1/k}\,dt \leqq \frac{1}{\left(\log\frac{1}{1-r}\right)^{1-1/pk}}\left(\int_0^{r_0}|f^{(k)}(te^{i\theta})|^{1/k}\,dt\right)+\varepsilon \tag{8}$$

for all $r>r_0$. It is clear that (8) implies (6) for this θ and, since (5) holds for almost every θ, this completes the proof.

Remarks. When $p=1$ we have $\int_0^r|f^{(1)}(te^{i\theta})|\,dt=O(1)$ and $\int_0^r|f^{(k)}(te^{i\theta})|^{1/k}dt=o(\log 1/(1-r))^{1-1/k}$ for all $k\geqq 2$ and almost every θ. For $p=2$ we have $\int_0^r|f^{(k)}(te^{i\theta})^{1/k}\,dt=o(\log 1/(1-r))^{1-1/2k}$ for $k=1,2,\ldots$ and almost every θ. When $k=1$, this last result ($p=2$) was obtained by A. Zygmund in [2, p. 196].

We note that when $p=1$ both (1) and (5) and hence (6) can be sharpened by replacing $|f^{(k)}(te^{i\theta})|$ by $\max_{|z|=t}|f^{(k)}(z)|$.

When $p\in[1,2]$, $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $(a_n)\in l^p$ then it follows essentially from the Hausdorff–Young theorem that $f\in H^q$ when $1/p+1/q=1$ [1, Theorem 6.1]. Hence f has nontangential limits at $e^{i\theta}$ for almost every θ. It follows [2, p. 181–182] that $(1-r)^kf^{(k)}(z)\to 0$ as $z=re^{i\theta}$ tends nontangentially to $e^{i\theta_o}$ for $k=1,2,\ldots$ and almost every θ_o. For such an f it is easy to prove that $\int_0^r|f^{(k)}(te^{i\theta})|^{\lambda}\,dt=o(1/(1-r)^{\lambda k-1})$ for $k=1,2,\ldots,\lambda>1/k$ and almost every θ. It can be proved that given $p\in[1,2]$, $\varepsilon(r)$ a positive function defined on $[0,1)$ and satisfying $\lim_{r\to 1^-}\varepsilon(r)=0$ then there exists $f(z)=\sum_{n=0}^{\infty}a_nz^n$ holomorphic in Δ such that $(a_n)\in l^p$ and

$$\overline{\lim_{r\to 1^-}}\frac{(1-r)^{\lambda k-1}}{\varepsilon(r)}\int_0^r\min_{|z|=t}|f^{(k)}(z)|^{\lambda}\,dt=+\infty \text{ for } k=1,2,\ldots \text{ and each } \theta.$$

We now finish by proving that (6) is sharp in a strong sense.

Theorem 4. *Let $p\in[1,2]$ and $\varepsilon(r)$, $0\leqq r<1$ be a positive function satisfying $\lim_{r\to 1^-}\varepsilon(r)=0$. Then there exists a holomorphic function $f(z)=\sum_{n=0}^{\infty}a_nz^n$ in Δ with $(a_n)\in l^p$ such that*

$$\overline{\lim_{r\to 1^-}}\frac{\int_0^r\min_{|z|=t}|f^{(k)}(z)|^{1/k}\,dt}{\varepsilon(r)\left(\log\frac{1}{1-r}\right)^{1-1/pk}}=+\infty \tag{9}$$

for each $k>1/p$.

Proof. The function f will be constructed in the form

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$$f(z)=\sum_{l=1}^{\infty}(n_l2^l)^{-1/p}\sum_{n=n_l+1}^{2n_l}z^{2^{2^ln}}(z\in\Delta) \tag{10}$$

with a suitably chosen increasing sequence (n_l) of positive integers. Let $n_1=2$ and if $n_1,n_2,\dots,n_{l-1}$ are already selected then let n_l be such that

$$\varepsilon(1-2^{-2^{l+1}n_l})\leqq\frac{1}{l2^l} \tag{11}$$

and

$$\sum_{s=1}^{l-1}n_s2^{2^{s+1}n_s}\leqq\frac{1}{l}(n_l2^l)^{-1}2^{2^ln_l}. \tag{12}$$

Clearly such a choice is possible. It is obvious that the sequence of Taylor coefficients of f belongs to l^p.

Let

$$A_m=\left\{z\in\mathbb{C}:\frac{1}{m}\leqq1-|z|\leqq\frac{2}{m}\right\}\quad\text{for } m=2,3,\dots.$$

Let us fix a positive integer k such that $k>1/p$. First we prove that if l is sufficiently large then

$$|(n_l2^l)^{-1/p}(z^{2^{2^ln}})^{(k)}|^{1/k}\leqq2|f^{(k)}(z)|^{1/k} \tag{13}$$

for $n_l+1\leqq n\leqq2n_l$ and $z\in A_{2^{2^ln}}$.

To this end it is enough to prove that

$$|(n_l2^l)^{-1/p}(z^{2^{2^ln}})^{(k)}|^{1/k}\geqq2|f^{(k)}(z)-(n_l2^l)^{-1/p}(z^{2^{2^ln}})^{(k)}|^{1/k} \tag{14}$$

with n and z as in (13).

The left hand side of (14) can be estimated from below on $A_{2^{2^ln}}$ as follows (we assume here that $2^{2^l}>k$):

$$|(n_l2^l)^{-1/p}(z^{2^{2^ln}})^{(k)}|^{1/k}\geqq(n_l2^l)^{-1/pk}(2^{2^ln}-k)\left(1-\frac{2}{2^{2^ln}}\right)^{(2^{2^ln}-k)1/k}$$

$$\geqq2^{2^ln}(n_l2^l)^{-1/pk}(e^{-2/k}+\delta_l) \tag{15}$$

where $\delta_l\to0$ as $l\to+\infty$.

To estimate the right hand side of (14) from above on $A_{2^{2^ln}}$ we note that

$$|f^{(k)}(z)-(n_l2^l)^{-1/p}(z^{2^{2^l n}})^{(k)}|^{1/k}\leqq A^{1/k}+B^{1/k}+C^{1/k} \tag{16}$$

where

$$A=\left|\left(\sum_{s=1}^{l-1}(n_s2^s)^{-1/p}\sum_{m=n_s+1}^{2n_s}z^{2^{2^s m}}\right)^{(k)}\right|,$$

$$B=\left|\left((n_l2^l)^{-1/p}\sum_{n_l+1\leqq m>n}z^{2^{2^l m}}\right)^{(k)}\right| \quad \text{and}$$

$$C=\left|\left((n_l2^l)^{-1/p}\sum_{n>m\leqq 2n_l}z^{2^{2^l m}}+\sum_{s=l+1}^{\infty}(n_s2^s)^{-1/p}\sum_{m=n_s+1}^{2n_s}z^{2^{2^s m}}\right)^{(k)}\right|.$$

It follows that

$$A^{1/k}\leqq\left(\sum_{s=1}^{l-1}(n_s2^s)^{-1/p}n_s(2^{2^s\cdot 2n_s})^k\right)^{1/k}$$

$$\leqq\sum_{s=1}^{l-1}n_s2^{2^{s+1}n_s}\leqq\frac{1}{l}(n_l2^l)^{-1/pk}2^{2^l n} \tag{17}$$

where the last inequality follows from (12). Also we have

$$B^{1/k}\leqq(n_l2^l)^{-1/pk}\sum_{m=0}^{n-1}(2^{2^l})^m$$

$$=(n_l2^l)^{-1/pk}\frac{2^{2^l n}-1}{2^{2^l}-1}<\frac{1}{2^{2^l}-1}(n_l2^l)^{-1/pk}2^{2^l n}. \tag{18}$$

To estimate $C^{1/k}$ note that the exponents corresponding to $s>l$ are all different and all of the form $2^{2^l m}$ with some $m>2n_l$. Therefore

$$C^{1/k}\leqq(n_l2^l)^{-1/pk}\sum_{m=n+1}^{\infty}2^{2^l m}\left(1-\frac{1}{2^{2^l n}}\right)^{2^{2^l m}/k-1}$$

$$=(n_l2^l)^{-1/pk}2^{2^l n}\left(1-\frac{1}{2^{2^l n}}\right)^{-1}\sum_{m=1}^{\infty}2^{2^l m}\left(\left[\left(1-\frac{1}{2^{2^l n}}\right)^{2^{2^l n}}\right]^{1/k}\right)^{2^{2^l m}}$$

$$=(n_l 2^l)^{-1/pk} 2^{2^l n}\left(1-\frac{1}{2^{2^l n}}\right)^{-1} \sum_{m=1}^{\infty} 2^{2^l m}(e^{-1/k}+\gamma_l)^{2^{2^l m}}$$

$$\leqq (n_l 2^l)^{-1/pk} 2^{2^l n}\left(1-\frac{1}{2^{2^l n}}\right)^{-1} \sum_{j=2^{2^l}}^{\infty} j(e^{-1/k}+\gamma_l)^j$$

$$=(n_l 2^l)^{-1/pk} 2^{2^l n}\beta_l \tag{19}$$

where $\gamma_l \to 0$ and $\beta_l \to 0$ as $l \to +\infty$. It follows from (15), (16), (17), (18) and (19) that (14) holds. Hence (13) also holds for all sufficiently large values of l. Now fix such an l and set $r=1-2^{-2^{l+1}n_l}$. Then we have from (13) that

$$\int_0^r \min_{|z|=t} |f^{(k)}(z)|^{1/k}\, dt \geqq \sum_{n=n_l+1}^{2n_l} \int_{1-2^{-2^l n+1}}^{1-2^{-2^l n}} \min_{|z|=t} |f^{(k)}(z)|^{1/k}\, dt$$

$$\geqq \frac{1}{2} \sum_{n=n_l+1}^{2n_l} 2^{-2^l n} \min_{z \in A 2^{2^l n}} \left|(n_l 2^l)^{-1/p}(z^{2^{2^l n}})^{(k)}\right|^{1/k}$$

$$\geqq \frac{1}{2} \sum_{n=n_l+1}^{2n_l} 2^{-2^l n} 2^{2^l n}(n_l 2^l)^{-1/pk}(e^{-2/k}+\delta_l)$$

$$=n_l^{1-1/pk} 2^{-1-l/pk}(e^{-2/k}+\delta_l). \tag{20}$$

where $\delta_l \to 0$ as $l \to +\infty$.

It follows from (11) and (20) that

$$\frac{\int_0^r \min_{|z|=t} |f^{(k)}(z)|^{1/k}\, dt}{\varepsilon(r)\left(\log \frac{1}{1-r}\right)^{1-1/pk}} \geqq \frac{n_l^{1-1/pk} 2^{-1-l/pk}(e^{-2/k}+\delta_l)}{\varepsilon(1-2^{2^{l+1}n_l})(2^{l+1}n_l)^{1-1/pk}}$$

$$=\frac{(e^{-2/k}+\delta_l)\cdot l}{\varepsilon(1-2^{2^{l+1}n_l})\, l 2^l 2^{2-1/pk}} \geqq l \cdot 2^{1/pk-2}(e^{-2/k}+\delta_l). \tag{21}$$

since $l 2^{1/pk-2}(e^{-2/k}+\delta_l) \to +\infty$ as $l \to +\infty$ we see that (21) implies (9) and this completes the proof.

Remark. We note that for $p=2$, (9) is a sharpening of Theorem 3 in [**2**, p. 196].

REFERENCES

1. P. L. DUREN, *Theory of H^p Spaces* (Academic Press, 1970).

2. A. ZYGMUND, On certain integrals, *Trans. Amer. Math. Soc.* **55** (1944), 170–204.

3. A. ZYGMUND, *Trigonometric Series*, Vol 2, 2nd Ed. (Cambridge University Press, London and New York, 1968).

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