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ON A PROBLEM OF P. ERDÖS

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P. Erdös asked the following problem: Does there exist an infinite sequence of integers $a_1 < \cdots$ satisfying for every $x \geq 1$

$$(1) \qquad A(x) = \sum_{a_i \leq x} 1 < \frac{c_1 x}{\log x}$$

so that every integer is of the form $2^k + a_i$ [1]. The analogous questions can easily be answered affirmatively if the powers of 2 are replaced by the rth power.

In this note we give a simple affirmative answer to the problem of Erdös. Let c_2 be a sufficiently small absolute constant. Our sequence A consists of all the integers of the form

$$(2) \qquad 5^u v \text{ and } 5^u v + 1, \text{ where } 5^u > c_2 \log v, \quad u = 1, 2, \ldots; \quad v = 1, \ldots$$

(2) clearly implies that (1) is satisfied for a sufficiently large c_1. To prove that every sufficiently large integer is of the form $2^k + a_i$ we only have to observe that for every r, 2 is a primitive root of 5^r, and choose $5^r \leq \log n < 5^{r+1}$, then we can find a $k < 5^r$ so that $n - 2^k$ or $n - 2^k - 1$ is of the form $5^r v$, or $n - 2^k$ is of the form (2). It is easy to see that for all n the number of solutions of

$$n = 2^k + a_i$$

a_i of the form (2) is less than an absolute constant c_3.

In this connection the following problem is of interest: Let $b_1 < \cdots$ be an infinite sequence of integers satisfying $B(x) > c_3 \log x$ for every x. Is there then a sequence A satisfying (1) so that every n can be written in the form $a_i + b_j$? I succeeded to prove, using a result in [1], that there exists a sequence $b_1 < \cdots$ satisfying $B(x) > c_3 \log x$ so that if every n can be written in the form $a_i + b_j$ then for infinitely many x

$$A(x) > c_4 \log \log x / \log x.$$

In view of a result of [2] this is best possible. This settles the problem in the negative. I will return to this subject at another occasion.

As to the constant c_1 in (1), we must clearly have $c_1 \geq \log 2$. Erdös conjectured that $c_1 > \log 2 + \epsilon$ for some fixed $\epsilon > 0$. The analogous conjecture for rth powers has been proven by Moser [3].

References

1. P. Erdös, *Some results on additive number theory*, Proc. Amer. Math. Soc. **5** (1954), 847–853 (see p. 853). See also Proc. of the Number Theory Conf. at Boulder, Colorado, 1963, Problem 33.

2. G. G. Lorentz, *On a problem of additive number theory*, Proc. Amer. Math. Soc. **5** (1954), 838–891.

3. L. Moser, *On the additive completion of sets of integers*, Proc. Symp. Pure Math., Amer. Math. Soc. **8** (1965), 175–180.

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