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A PROPERTY OF COMPLETELY MONOTONIC FUNCTIONS

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Abstract

A non-negative function $f(t)$, $t > 0$, is said to be *completely monotonic* if its derivatives satisfy $(-1)^n f^{(n)}(t) \geqslant 0$ for all t and $n = 1, 2, \ldots$. For such a function, either $f(t+\delta)/f(t)$ is strictly increasing in t for each $\delta > 0$, or $f(t) = ce^{-dt}$ for some constants c and d, and for all t. An application of this result is given.

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The motivation for this note is the following result which arose in connection with work on stability of numerical methods for singular integral equations; see [1].

PROPOSITION A. *For* $t > 0$, $\delta > 0$, $h > 0$, $n = 0, 1, \ldots$, $p < n$ *and* $p \neq 0, 1, \ldots, n-1$, *the function*

$$G(t) = \frac{\Delta_h^n (t+\delta)^p}{\Delta_h^n t^p}$$

is strictly increasing in t.

Here, $\Delta_h^0 f(t) = f(t)$ and $\Delta_h^{n+1} f(t) = \Delta_h^n f(t+h) - \Delta_h^n f(t)$ for $n = 1, 2, \ldots$. To simplify statements below, we suppose throughout this note that the variables appearing in Proposition A always satisfy the constraints given there. The operator Δ_h^n will always act on functions of the variable t. To prove Proposition A we first show that $g(t+\delta)/g(t)$ is non-decreasing in t for a completely monotonic function g (a further condition gives 'strictly increasing'), which is the content of D. We then show that the function $(-1)^n \Delta_h^n t^p$ is completely monotonic, and Proposition A is a simple consequence.

The idea of stochastic ordering of probability measures underlies the proof of Proposition B (see [3]).

PROPOSITION B. *Let* $g\colon [0,\infty) \to \mathbb{R}$ *be Borel measurable and let* $\mu \not\equiv 0$ *be a positive Borel measure on* $[0,\infty)$ *for which the function*

$$h(t) = \frac{\int_{[0,\infty)} e^{-xt} g(x)\, d\mu(x)}{\int_{[0,\infty)} e^{-xt}\, d\mu(x)}$$

is well defined for all $t > 0$. *Then* h *is non-decreasing (non-increasing) whenever* g *is non-increasing (non-decreasing). Furthermore, strict monotonicity of* h *follows if we also assume that* g *is not* μ*-a.e. constant (i.e.* $\mu(\{x \mid g(x) \neq c\}) \neq 0$ *for all* $c \in R$).

NOTE. All integrals here are Lebesgue integrals.

PROOF. For each $t > 0$ define a probability measure ν_t by

$$\nu_t(E) = \frac{\int_E e^{-xt}\, d\mu(x)}{\int_{[0,\infty)} e^{-xt}\, d\mu(x)}, \qquad E \subset [0,\infty).$$

It is clear that for each $y \geqslant 0$, either $\nu_t([y,\infty)) = 1$ for all t or that $\nu_t([y,\infty)) < 1$ for all t. In the latter case we have

$$\frac{\nu_t([y,\infty))}{1 - \nu_t([y,\infty))} = \frac{\int_{[y,\infty)} e^{(y-x)t}\, d\mu(x)}{\int_{[0,y)} e^{(y-x)t}\, d\mu(x)}.$$

The numerator of this quotient is non-increasing in t since $y - x \leqslant 0$ on the range of integration, while the denominator is strictly increasing since $y - x > 0$ on the range of integration. We conclude that $\nu_t([y,\infty))$ is either identically 0, identically 1, or strictly decreasing in t, for each y. The same may be said of $\nu_t((y,\infty))$ by a similar argument.

Let us suppose now that g is non-negative and non-decreasing. Then

$$h(t) = \int_{[0,\infty)} g(x)\, d\nu_t(x) = \int_0^\infty \nu_t(\{x \mid g(x) > a\})\, da. \tag{1}$$

Since the integrand of the latter is either of the form $\nu_t((y,\infty))$ or of the form $\nu_t([y,\infty))$, it is non-increasing in t, and so $h(t)$ is non-increasing.

Now suppose further that g is not μ-a.e. constant. Then for some $a_0 \geqslant 0$, we have $0 < \nu_t(\{x \mid g(x) > a_0\}) < 1$. Since $\nu_t(\{x \mid g(x) > a_0 + \frac{1}{n}\}) \to \nu_t(\{x \mid g(x) > a_0\})$ as $n \to \infty$, we see that for some $\varepsilon > 0$, $0 < \nu_t(\{x \mid g(x) > a\}) < 1$ for $a_0 \leqslant a \leqslant a_0 + \varepsilon$. Thus on this interval the integrand of (1) is strictly decreasing.

For other values of a it is non-increasing, and we conclude that $h(t)$ is strictly decreasing.

To complete the proof, a similar argument gives the corresponding result for g non-negative and non-increasing. The general case follows by taking the positive and negative parts of g separately.

A theorem of Bernstein, discussed in [2], states that a function $f(t)$ is completely monotonic if and only if it has the representation

$$f(t) = \int_{[0,\infty)} e^{-xt}\, d\mu(x) \tag{2}$$

for some positive Borel measure μ on $[0, \infty)$. This fact will be used in no essential way, but it allows us to simplify many statements below. The following result is obvious, with $f(t)$ as in (2).

PROPOSITION C. $(-1)^n \Delta_h^n f(t) = \int_{[0,\infty)} e^{-xt}(1 - e^{-xh})^n\, d\mu(x)$.

It follows that $(-1)^n \Delta_h^n f(t)$ is completely monotonic if $f(t)$ is.

PROPOSITION D. *If $f(t)$ is completely monotonic, then either $f(t) = ce^{-dt}$ for some $c \geqslant 0$, $d \geqslant 0$, or $f(t + \delta)/f(t)$ is strictly increasing in t.*

PROOF. Let μ be the measure that represents f in the sense of (2). Then we have

$$\frac{f(t+\delta)}{f(t)} = \frac{\int_{[0,\infty)} e^{-xt} e^{-\delta x}\, d\mu(x)}{\int_{[0,\infty)} e^{-xt}\, d\mu(x)},$$

and the result follows from Proposition B with $g(x) = e^{-\delta x}$, upon noting that $g(x)$ is μ-a.e. constant if and only if μ is a point mass.

Proposition A now follows from Proposition D and

PROPOSITION E. *The function $(-1)^n \Delta_h^n t^p$ is completely monotonic.*

PROOF. It is elementary that

$$\int_0^\infty e^{-xt} x^{-z-1}\, dx = \Gamma(-z) t^z,$$

so that by Proposition C, we have

$$(-1)^n \Delta_h^n t^z = \frac{1}{\Gamma(-z)} \int_0^\infty e^{-xt}(1 - e^{-xh})^n x^{-z-1}\, dx \tag{3}$$

whenever $z < 0$. The left hand side of (3) defines an entire function of z, while the right hand side defines an analytic function of z in the domain $\mathrm{Re}(z) < n$, $z \neq 0, 1, \ldots, n-1$. Thus the two sides agree in the latter domain. Replacing z by p in (3) and using Bernstein's theorem gives the result.

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