Canad. Math. Bull. Vol. **27** (3), 1984

ISOMETRIC IMAGES OF C^* ALGEBRAS

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ABSTRACT. It is shown that if the isometric image of a linear subspace of Hilbert space operators is irreducible in a strong sense, then the isometry is either a multiplicative or anti-multiplicative map, possibly followed by multiplication by a unitary.

1. **Introduction.** We investigate the structure of isometric images of sets of operators. Elementary examples show that some condition on the irreducibility of the image is necessary to get a good structure theory.

Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{K}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})$ denote the compact and bounded linear operators in $\mathcal{H}$, respectively. Recently, Hopenwasser and Plastiras proved:

LEMMA [6]. *Let $\phi:\mathcal{K}(\mathcal{H})\to\mathcal{L}(\mathcal{H})$ be a linear isometry with property*

$$(I_4)\ \textit{For all } x, y \textit{ in } \mathcal{H},\ \sup_{\|K\|=1}|(\phi(K)x, y)| = \|x\|\cdot\|y\|.$$

Then ϕ has a unique extension to an isometry of $\mathcal{L}(\mathcal{H})$ into itself. □

If $\mathcal{I}$ is any ideal in a C^* algebra $\mathcal{A}$, and if ϕ is a * preserving multiplicative or anti-multiplicative map of $\mathcal{I}$ into $\mathcal{L}(\mathcal{H})$, then there is a standard method for uniquely extending ϕ to a multiplicative or anti-multiplicative map on all of $\mathcal{A}$. See, for example [1, Sect. 1.3]. We show that it is exactly this situation which occurs in the lemma above. Our theorem is considerably stronger, but it was the above considerations which led to our results, so we give them pride of place.

THEOREM 1.1. *Let $\mathcal{A}$ be any subspace of operators on $\mathcal{H}$ which contains $\mathcal{K}(\mathcal{H})$. Let ϕ be a linear isometry of $\mathcal{A}$ into $\mathcal{L}(\mathcal{H})$ with property*

$$(I_4)\ \sup\{|(\phi(A)x, y)| : A\in\mathcal{A}, \|A\| = 1\} = \|x\|\cdot\|y\|$$

for all $x, y\in\mathcal{H}$. Then ϕ is the restriction of either a *-automorphism or a *-anti-automorphism of $\mathcal{L}(\mathcal{H})$, followed by multiplication by a unitary. □

† Research partially supported by NSERC grant #A3488

Received by the editors September 28, 1982 and, in revised form, September 19, 1983.
Subject Classification Number 46L05 47D15

Before proceeding with the proof, we wish to indicate the place that this theorem occupies in a historical string of such theorems. The first result of this type was:

THEOREM (Banach-Stone, 1930's). *Let X and Y be compact Hausdorff spaces. Let ϕ be a linear isometry of $C(X)$ onto $C(Y)$. Then there is a multiplicative *-preserving map χ of $C(X)$ onto $C(Y)$, and a function h in $C(Y)$ with $|h| = 1$ such that $\phi(f) = h \cdot \chi(f)$ for all f in $C(X)$.* □

The next result seems to be

THEOREM [7] (Kadison, 1951). *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^* algebras. Let ϕ be a linear isometry of $\mathcal{A}$ onto $\mathcal{B}$. Then $\phi(I)$ is a unitary operator, and there is a *-preserving map χ which is multiplicative on powers such that $\phi(A) = \phi(I) \cdot \chi(A)$ for all A in $\mathcal{A}$.* □

The proof of this theorem uses a characterization of the extreme points of the unit ball of a C^* algebra. To use this, it is essential that $\phi(\mathcal{A})$ be a C^* algebra. It is the lack of this that makes the proof of our theorem difficult. It is to compensate for this lack that we must suppose that $\mathcal{A}$ contains the compact operators.

Using the notions of Class 0 and Class 1 positive linear maps, it was shown:

THEOREM [9] (Störmer, 1963). *Let $\mathcal{A}$ and $\mathcal{B}$ be C^* algebras. Let ϕ be a * preserving linear map of $\mathcal{A}$ onto $\mathcal{B}$ which is multiplicative on powers. Then for any irreducible representation π of $\mathcal{B}$, $\pi \circ \phi$ is a *-homomorphism or a *-anti-isomorphism.*

Putting the results of Kadison and Störmer together, one has

THEOREM. *Let $\mathcal{A}$ be a unital C^* algebra, and let ϕ be a linear isometry of $\mathcal{A}$ onto a unital irreducible C^* algebra $\mathcal{B}$. Then there is a *-isomorphism or *-anti-isomorphism χ of $\mathcal{A}$ onto $\mathcal{B}$ such that $\phi(A) = \phi(I) \cdot \chi(A)$ for all A in $\mathcal{A}$.*

To see how our theorem relates to this, we recall that a C^* algebra $\mathcal{B}$ contained in $\mathcal{L}(\mathcal{H})$ is irreducible if and only if any of the following conditions hold.

(I_1) $TB = BT$ for all B in $\mathcal{B}$ implies $T = \lambda I$.

(I_2) $\overline{\text{span}\, \mathcal{B}x} = \mathcal{H}$ for all $x \neq 0$ in $\mathcal{H}$.

(I_3) $\text{span}\, \mathcal{B}x = \mathcal{H}$ for all $x \neq 0$ in $\mathcal{H}$.

(I_4) $\sup\{|(Bx, y)| : \|B\| = 1,\ B \in \mathcal{B}\} = \|x\| \cdot \|y\|$ for x, y in $\mathcal{H}$.

The equivalence of I_1, I_2 and I_3 is well-known [c.f. 2, Prop. 2.3.1 and Cor. 2.8.4]. The equivalence of I_4 follows from Kaplansky's density theorem.

If $\mathcal{B}$ is not an algebra, as in our theorem, then these conditions are no longer equivalent. The strongest condition (I_4) appears to be necessary for our purposes. Some further discussion of this point is included at the conclusion of this paper.

Finally, for completeness, we should mention some similar results. In 1972, Paterson and Sinclair [8] showed that unital is not needed in the hypotheses of Kadison's theorem. And:

THEOREM [5] (Harris, 1969). *Let ϕ be a linear isometry of a power algebra $\mathcal{A}$ onto a power algebra $\mathcal{B}$ with $\phi(I)^*$ belonging to $\mathcal{B}$. Then $\phi(I)$ is unitary and $\phi(A)=\phi(I)\cdot\chi(A)$ where $\chi(A^2)=\chi(A)^2$ and $\chi(A^*)=\chi(A)^*$ when A and A^* belong to $\mathcal{A}$. (A power algebra is a unital subspace of $\mathcal{L}(\mathcal{H})$ closed under squaring.)* □

The Theorem 1.1 is originally due to the first author. Substantial simplifications in the proof are due to the second author.

2. **The proof.** Our strategy will be to prove that when P is a rank one partial isometry, then $\phi(P)$ is also a rank one partial isometry. The rest will follow fairly easily.

LEMMA 2.1. *Let $\{e, f\}$ be an orthonormal basis for the domain of a rank two operator A. Let $Ae=u$ and $Af=v$. Then*

$$\|A\|=\frac{1}{\sqrt{2}}[\|u\|^2+\|v\|^2+\{(\|u\|^2-\|v\|^2)^2+4\,|(u,v)|^2\}^{1/2}]^{1/2}$$
$$\geqq[\max\{\|u\|,\|v\|\}^4+|(u,v)|^2]^{1/4}.$$

Proof. Compute

$$A^*A=\begin{bmatrix}\|u\|^2 & \overline{(u,v)}\\ (u,v) & \|v\|^2\end{bmatrix}.$$

The rest is routine. □

COROLLARY 2.1. *Let $\{e, f\}$ be orthonormal and let A be any operator with $\|A\|\leqq 1$. Let $Ae=u$, $Af=v$, and suppose $\|u\|^2\geqq 1-\varepsilon^2$. Then $|(u,v)|<\sqrt{2}\varepsilon$.* □

If P is a partial isometry, write $D=D_P=P^*P$ and $R=R_P=PP^*$. If Q is a projection, let $Q^\perp$ denote $I-Q$.

LEMMA 2.2. *Let P be a finite rank partial isometry. Let $0<\varepsilon<1$, and let u be a unit vector such that $\|\phi(P)u\|^2\geqq 1-\varepsilon^2$. Suppose A belongs to $\mathcal{A}$, $\|A\|=1$, and $A=R^\perp AD^\perp$. Then*

(i) $\|\phi(A)u\|<\varepsilon$.

(ii) $|(\phi(A)w, \phi(P)u)|<5\varepsilon$ *for all unit vectors w in $\mathcal{H}$.*

https://doi.org/10.4153/CMB-1984-043-5 Published online by Cambridge University Press

Proof.

$$1=\sup_{|\lambda|*1}\|P+\lambda A\|^2\geqq\sup_{|\lambda|=1}\|\phi(P)u+\lambda\phi(A)u\|^2$$
$$=\|\phi(P)u\|^2+\|\phi(A)u\|^2+2\,|(\phi(A)u,\phi(P)u)|$$

Hence $\|\phi(A)u\|<\varepsilon$ and $|(\phi(A)u,\phi(P)u)|<\varepsilon^2/2$.

First suppose w is orthogonal to u. Let B be the norm one operator $\phi(P+A)$. Then

$$\|Bu\|^2\geq|(Bu,\phi(P)u)|\geq(\|\phi(P)u\|^2-|(\phi(A)u,\phi(P)u)|)^2$$
$$\geq(1-3\varepsilon^2/2)^2>1-3\varepsilon^2.$$

By Corollary 2.1, $|(\phi(P)u,\phi(P)w)|<\sqrt{2}\varepsilon$ and

$$\sqrt{6}\varepsilon>|(Bu,Bw)|=|(\phi(P)u,\phi(P)w)+(\phi(P)u,\phi(A)w)+(\phi(A)u,Bw)|$$

Hence $|(\phi(P)u,\phi(A)w|<\sqrt{6}\varepsilon+\sqrt{2}\varepsilon+\varepsilon<2\sqrt{6}\varepsilon$

In general, write $w=cv+su$ where v is orthogonal to u and $|c|^2+|s|^2=1$. Then

$$|(\phi(A)w,\phi(P)u)|\leq|c|\,2\sqrt{6}\varepsilon+|s|\,\varepsilon^2/2<(24\varepsilon^2+\varepsilon^4/4)^{1/2}<5\varepsilon.\quad\square$$

COROLLARY 2.2. *Suppose P is a finite rank partial isometry in $\mathcal{A}$ and $\|\phi(P)u\|=\|u\|=1$. Then for all $A=R^\perp AD^\perp$ in $\mathcal{A}$, $\phi(A)u=0$ and $(\phi(A)w,\phi(P)u)=0$ for all w in $\mathcal{H}$.* $\square$

LEMMA 2.3. *Let P_1 and P_2 be finite rank partial isometries with domain and range projections D_1, D_2 and R_1, R_2 respectively such that $D_1D_2=0=R_1R_2$. Suppose u_i are unit vectors such that $\|\phi(P_i)u_i\|^2\geq1-\varepsilon^2$, $0<\varepsilon<1$. Then there is an operator $A=R_1AD_2+R_2AD_1$ of norm one such that*

$$|(\phi(A)u_1,\phi(P_2)u_2)|\geq1-13\varepsilon.$$

Proof. By Lemma 2.2, $\|\phi(P_i)u_j\|<\varepsilon$ for $i\neq j$. By (I_4), we can choose B in $\mathcal{A}$ of norm one so that $|(\phi(B)u_1,\phi(P_2)u_2)|\geq1-\varepsilon$. Again by 2.2, we have $\|\phi(R_1^\perp BD_1^\perp)u_1\|<\varepsilon$ and $|(\phi(R_1BD_1)u_1,\phi(P_2)u_2)|<5\varepsilon$. So let $B_1=B-R_1^\perp BD_1^\perp-R_1BD_1=R_1BD_1^\perp+R_1^\perp BD_1$. We find that $\|B_1\|\leq1$ and $|(\phi(B_1)u_1,\phi(P_2)u_2)|\geq1-7\varepsilon$.

Again by Lemma 2.2, $|(\phi(R_2B_1D_2)u_1,\phi(P_2)u_2)|\leq\|\phi(R_2B_1D_2)u_1\|<\varepsilon$ and $|(\phi(R_2^\perp B_1D_2^\perp)u_1,\phi(P_2)u_2)|<5\varepsilon$. So let $A=B_1-R_2^\perp B_1D_2^\perp-R_2B_1D_2=R_2BD_1+R_1BD_2$. Then $\|A\|\leq1$ and $|(\phi(A)u_1,\phi(P_2)u_2|\geq1-13\varepsilon$. $\square$

COROLLARY 2.3. *Let P_1, P_2 be orthogonal finite rank partial isometries. Suppose there are unit vectors u_i such that $\|\phi(P_i)u_i\|=1$. Then $u_1\perp u_2$, $\phi(P_1)u_1\perp\phi(P_2)u_2$, and there exists an operator $A=R_1AD_2+R_2AD_1$ of norm one such that $\phi(A)u_1=\phi(P_2)u_2$.*

Proof. A simple compactness argument and Lemma 2.3 gives an operator A of norm one with $(\phi(A)u_1, \phi(P_2)u_2) = 1$. Since $\|\phi(A)\| = 1$, we must have $\phi(A)u_1 = \phi(P_2)u_2$. Corollary 2.2 shows that $\phi(P_i)u_i$ are orthogonal. It also implies that

$$\phi(P_i)^*\phi(P_j)u_j = \delta_{ij}u_j \quad \text{for} \quad i = 1, 2, j = 1, 2.$$

Finally, Corollary 2.1 applied to $\phi(P_1 + P_2)^*$ and the vectors $\phi(P_i)u_i$ shows that u_1 is orthogonal to u_2. □

LEMMA 2.4. *Let P be a finite rank partial isometry. Then $\phi(P)$ attains its norm, and the norm is bounded away from one on the complement of a finite dimensional space.*

Proof. We know that $\|\phi(P)\| = 1$. The lemma is false only if there is an orthonormal set $\{u_i, i \geq 1\}$ such that $\lim \|\phi(P)u_i\| = 1$. Choose a finite rank partial isometry Q orthogonal to P. Choose a unit vector v so that $\|\phi(Q)v\|^2 >$ 0.995. By Lemma 2.3, there are norm one operators $A_i = R_P A_i D_Q + R_Q A_i D_P$ such that $|(\phi(A_i)u_i, \phi(Q)v)| > 0.05$. But A_i lie in a compact set of operators and u_i tend weakly to zero. Thus $\lim |(\phi(A_i)u_i, \phi(Q)v)| = 0$. This contradiction establishes the lemma. □

LEMMA 2.5. *Let P be a finite rank partial isometry, and let $\mathcal{M}$ be the finite dimensional subspace on which $\phi(P)$ achieves its norm. Suppose A belongs to $\mathcal{A}$ such that $RAD = 0$. If u belongs to $\mathcal{M}$ and v belongs to $\phi(P)\mathcal{M}$, then $(\phi(A)u, v) = 0$.*

Proof. Corollary 2.2 establishing the lemma for $A = R^\perp AD^\perp$.

Consider $A = R^\perp AD$. To simplify notation, replace ϕ by $U \cdot \phi$ where U is any unitary extending $(\phi(P)|_{\mathcal{M}})^*$. This allows us to suppose $\phi(P)|_{\mathcal{M}} = I|_{\mathcal{M}}$. Let B be the compression of $\phi(A)$ to $\mathcal{M}$, and let $\beta = \|B\|$. The numerical radius theorem [3, pg. 114] gives $\sup_{\|m\|=1} |(Bm, m)| \geq \frac{1}{2}\|B\| = \frac{1}{2}\beta$. So for $t > 0$, $|\lambda| = 1$, and $\|A\| = 1$,

$$\begin{aligned} t^2 + 1 = \sup_{|\lambda|=1} \|tP + \lambda A\|^2 &\geq \sup_{|\lambda|=1} \|tI + \lambda B\|^2. \\ &\geq \sup_{\|x\|=1,\, |\lambda|=1} |(tI + \lambda Bx, x)|^2 \\ &\geq (t + \tfrac{1}{2}\beta)^2 = t^2 + \beta t + \beta^2/4 \end{aligned}$$

Letting t tend to infinity, we get $\beta = 0$. Hence $(\phi(R^\perp AD)u, v) = 0$.

Similarly, $(\phi(RAD^\perp)u, v) = 0$. □

COROLLARY 2.5. *If P is a rank one partial isometry, then $\phi(P)$ attains its norm only on a one dimensional space.*

Proof. Suppose u and v are orthonormal vectors such that $\|\phi(P)u\|=\|\phi(P)v\|=1$. By (I_4), there is an operator A in $\mathcal{A}$ such that $(\phi(A)u, \phi(P)v)\neq 0$. By Lemma 2.5, it follows that $(\phi(RAD)u, \phi(P)v)\neq 0$. But $RAD=\lambda P$ since P is rank one. By Corollary 2.1, $\|\phi(P)\|>1$ which is impossible. $\square$

LEMMA 2.6. *If P_1 is a rank one partial isometry, then so is $\phi(P_1)$.*

Proof. Let u_1 and v_1 be unit vectors such that $\phi(P_1)u_1=v_1$. Suppose u is orthogonal to u_1, yet $\phi(P_1)u=v\neq 0$. By Corollary 2.1, v is orthogonal to v_1. As in the proof of Lemma 2.3, there is an operator $A=R_1^\perp AD_1+R_1AD_1^\perp$ of norm at most one with $|(\phi(A)u_1, v)|>(1-\varepsilon)\|v\|$, where $\varepsilon>0$ will be fixed but arbitrary. To see this, use (I_4) to obtain a norm one operator B with $|(\phi(B)u, v)|>(1-\varepsilon)\|v\|$. By Lemma 2.2, $\phi(R_1^\perp BD_1^\perp)u_1=0$. Also, R_1BD_1 is a multiple of P_1, so $(\phi(R_1BD_1)u_1, v)=0$. Thus $A=R_1^\perp BD_1+R_1BD_1^\perp$ will suffice.

Let $v'=\phi(A)u_1$. By Lemma 2.5, v' is orthogonal to v_1. Also

$$|(\phi(P_1)u, v')|=|(v, \phi(A)u_1)|>(1-\varepsilon)\|v\|\geq\tfrac{1}{2}\|v\|$$

if $\varepsilon<\frac{1}{2}$.

Let P_2 be the rank one partial isometry orthogonal to P_1 with domain equal to the initial space of $R_1AD_1^\perp$ and range equal to the range of $R_1^\perp AD_1$. Should $R_1AD_1^\perp=0$, any domain in $D_1^\perp\mathcal{H}$ is allowed. Similarly, if $R_1^\perp AD_1=0$, any range in $R_1^\perp\mathcal{H}$ is acceptable. Let e_i, f_i, $i=1,2$ be the unit vectors in the initial and final spaces of P_i, respectively, such that $P_ie_i=f_i$. Let u_2, v_2 be unit vectors such that $\phi(P_2)u_2=v_2$. By Corollaries 2.2 and 2.3, we have $u_1\perp u_2$, $v_1\perp v_2$, and $\phi(P_i)u_j=0$ for $i\neq j$. By Lemma 2.5, $v=\phi(P_1)u$ is orthogonal to v_2. Thus $v'=\phi(A)u_1$ is "almost" orthogonal to v_2. Quantitatively,

$$|(v', v_2)|^2\leq\|v'\|^2-|(v', v/\|v\|)|^2\leq 1-(1-\varepsilon)^2<2\varepsilon.$$

Now we choose a unit vector u_3 extending $\{u_1, u_2\}$ to an orthonormal basis of span$\{u_1, u_2, u\}$; and a unit vector v_3 extending $\{v_1, v_2\}$ to an orthonormal basis of span$\{v_1, v_2, v'\}$. By Corollaries 2.1 and 2.2, $\phi(P_i)u_3$ are orthogonal to span$\{v_1, v_2\}$. By Lemma 2.5, $(\phi(A)u_i, v_i)=0$ for $i=1$ and 2.

We collect this information in matrix form. P_1, P_2 and A map span$\{e_1, e_2\}$ to span$\{f_1, f_2\}$ and with respect to these bases, they have matrices:

$$P_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\qquad P_2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\quad\text{and}\quad A=\begin{pmatrix}0 & a\\ b & 0\end{pmatrix}.$$

Since $1=\|A\|=\max\{|a|, |b|\}$, with no real loss we may suppose $b=1$ and $|a|\leq 1$. So $A=\begin{pmatrix}0 & a\\ 1 & 0\end{pmatrix}$. (The case $|a|=1$ is similar.)

Let Q and R be the projections onto span$\{v_1, v_2, v_3\}$ and span$\{u_1, u_2, u_3\}$ respectively. Let $\psi(B)=Q\phi(B)R$ be the compression of ϕ to this range and domain. With respect to the given orthonormal bases, one gets the matrices:

$$\psi(P_1)=\begin{pmatrix}1&0&0\\0&0&0\\0&0&\alpha\end{pmatrix},\qquad \psi(P_2)=\begin{pmatrix}0&0&0\\0&1&0\\0&0&\beta\end{pmatrix},\qquad \psi(A)=\begin{pmatrix}0&p&q\\x&0&r\\y&z&s\end{pmatrix}.$$

The construction guarantees that

$$|\alpha|=|(\phi(P_1)u_3,v_3)|\geq|(\phi(P_1)u,v_3)|\geq|(v,v')|\geq\tfrac{1}{2}\|v\|$$

and

$$|y|=|(\phi(A)u_1,v_3)|\geq|(v',v)|\geq 1-\varepsilon.$$

Now for $|\lambda|=1$, $\|\lambda P_1+A-a\bar{\lambda}P_2\|=\left\|\begin{pmatrix}\lambda&a\\1&-a\bar{\lambda}\end{pmatrix}\right\|=\sqrt{2}$, and ψ is norm decreasing, so for $T=\psi(\lambda P_1+A-a\bar{\lambda}P_2)$,

$$\sqrt{2}\geq\left\|\begin{bmatrix}\lambda&p&q\\x&-a\bar{\lambda}&r\\y&z&s+\alpha\lambda-a\beta\bar{\lambda}\end{bmatrix}\right\|=\|T\|$$

Apply Corollary 2.1 to $B=T/\sqrt{2}$, u_1 and u_3.

$$\|Bu_1\|^2=\tfrac{1}{2}(1+|x|^2+|y|^2)\geq\tfrac{1}{2}(1+(1-\varepsilon)^2)\geq 1-\varepsilon.$$

Thus $|(Bu_1,Bu_3)|<(2\varepsilon)^{1/2}$. However,

$$|(Bu_1,Bu_3)|=\tfrac{1}{2}\,|(x\bar{r}+y\bar{s})+\lambda(\bar{q}-\overline{ya\beta})+\bar{\lambda}(y\bar{\alpha})|$$

We can choose the argument of λ so that $|(Bu_1,Bu_3)|\geq\frac{1}{2}|y\alpha|$. Hence $\frac{1}{2}\|v\|<|\alpha|<2(2\varepsilon)^{1/2}(1-\varepsilon)^{-1}$. Now letting ε tend to zero, we get $v=0$ contradicting the original hypothesis. □

Proof of Theorem 1.1. Fix an orthonormal basis $\{e_n;n\geq1\}$ for $\mathcal{H}$. Let P_n be the rank one orthogonal projections onto $\mathbb{C}e_n$. By Lemma 2.6 and Corollary 2.3, $\phi(P_n)$ are pairwise orthogonal rank one partial isometries. Let $U=\sum_{n=1}^{\infty}\phi(P_n)$. This converges in the strong operator topology to a partial isometry U. Suppose P is a rank one orthogonal projection dominated by $Q_n=\sum_{k=1}^{n}P_k$. Since $\phi(P)$ is a rank one partial isometry, there are unit vectors u and v so that $\phi(P)u=v$. By Lemma 2.5, $\phi(Q_n)u=\phi(PQ_nP)u=v$. So the rank n partial isometry $\phi(Q_n)$ attains its norm at u and thus $U^*\phi(P)$ is a projection dominated by $U^*\phi(Q_n)$. In particular, the range of $\phi(P)$ is contained in the range of U.

The linear span of all projections dominated by some Q_n is dense in $\mathcal{K}(\mathcal{H})$. We choose unit vectors u_1, v_1 so that $\phi(P_1)u_1=v_1$. We can conclude that $\phi(K)u_1$ belongs to the range of U for all compact K. By Corollary 2.2, $\phi(A)u_1$ belongs to the range of U for all A in $\mathcal{A}$. By (I_4), U must be surjective. Similarly, if u is orthogonal to the domain of U, $\phi(A)u$ is orthogonal to the

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range of U and thus is zero. Again by (I_4), we conclude $u=0$ and thus U is unitary.

Let $\psi(A)=U^*\phi(A)$ for A in $\mathcal{A}$. Then $\psi(P_n)$ are a spanning, orthonormal family of rank one projections. Standard arguments show that the restriction of ψ to $\mathcal{K}(\mathcal{H})$ is either of the form $\psi(K)=V^*KV$ for some unitary V or the real transpose of such a map $\psi(K)=V^*K^tV$. However, we do not have a convenient reference so the details are included.

First, let W be a unitary taking the range of each P_n onto the range of $\psi(P_n)$. Replace ψ by $\psi_1(A)=W^*\psi(A)W$. Thus we have that $\psi_1(P_n)=P_n$. Let E_{mn} be the rank one partial isometry taking e_m to e_n.

Every rank one operator is a scalar multiple of a partial isometry, and thus by Lemma 2.6, ψ_1 takes it to a rank one operator. In particular, $F_{mn}=\psi_1(E_{mn})$ is a rank one partial isometry. By Lemma 2.5, F_{mn} acts on the span of $\{e_m, e_n\}$. Since $P_m \pm F_{mn}$ and $P_n \pm F_{mn}$ are all rank one, a simple calculation shows that F_{mn} equals $\lambda_{mn}E_{mn}$ or $\lambda_{mn}E_{nm}$ for some constant λ_{mn} of modulus one. If $\psi_1(E_{12})=\lambda_{12}E_{21}$, replace ψ_1 by its real transpose. So, without loss of generality, $\psi_1(E_{12})=\lambda_{12}E_{12}$. Then because $E_{12}+E_{1n}$ is rank one, so is $\lambda_{12}E_{12}+F_{1n}$ and thus $F_{1n}=\lambda_{1n}E_{1n}$. Finally, $E_{1n}+E_{mn}$ is rank one and $\psi_1(E_{1n}+E_{mn})=\lambda_{1n}E_{1n}+F_{mn}$, so $F_{mn}=\lambda_{mn}E_{mn}$.

Next, consider the span$\{e_1, e_m, e_n\}$. The operator

$$G=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

is rank one, and hence so is

$$\psi_1(G)=\begin{pmatrix}1&\lambda_{1m}&\lambda_{1n}\\ \lambda_{m1}&1&\lambda_{mn}\\ \lambda_{n1}&\lambda_{nm}&1\end{pmatrix}.$$

Hence $\lambda_{nm}=\bar{\lambda}_{mn}$ and $\lambda_{mn}=\bar{\lambda}_{1m}\lambda_{1n}$.

Let V be the unitary operator which takes e_n to $\lambda_{1n}e_n$. Set $\psi_2(A)=V^*\psi_1(A)V$. It is easily seen from the relations above that $\psi_2(E_{mn})=E_{mn}$. The span of $\{E_{mn}\}$ is dense in $\mathcal{K}(\mathcal{H})$ and ψ_2 is linear, so $\psi_2(K)=K$ for all compact operators.

Lastly, let A belong to $\mathcal{A}$ and u and v belong to $\mathcal{H}$. Let P and Q be the rank one projections onto the span of u and v respectively. By Lemma 2.5,

$$(\psi_2(A)u, v)=(\psi_2(QAP)u, v)=(QAPu, v)=(Au, v).$$

Hence $\psi_2(A)=A$. □

COROLLARY 2.7. *Let ϕ be a unital isometry of a subspace $\mathcal{A}$ containing $\mathcal{K}(\mathcal{H})$ into $\mathcal{L}(\mathcal{H})$ which has irreducibility property (I_4). Then ϕ is unitarily implemented or ϕ transpose (ϕ^t) is unitarily implemented.* □

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3. **Concluding remarks.** Theorem 1.1 and the comments in the introduction raise several questions. Does this theorem remain valid if (I_4) is replaced by the apparently weaker conditions (I_3) or (I_2)? A review of the proof shows that the strong (I_4) hypothesis was used in a crucial way, so a new argument would be needed. One might ask if (I_2) implies (I_4) for isometric images of $\mathcal{K}(\mathcal{H})$. This doesn't hold in general, as the following example shows.

Let μ be normalized Lebesgue measure on the unit circle S^1. Let H^2 be the Hardy space in $L^2(S^1, \mu)$ and let T_f be the Toeplitz operator for f in $L^\infty(S^1, \mu)$. The map taking $L^\infty(S^1, \mu)$ into $\mathcal{L}(H^2)$ by $\phi(f) = T_f$ is isometric [3]. Now $\mathcal{T} = \{T_f\}$ clearly has property (I_1). If g is a non-zero element of H^2, it is possible to find a sequence f_n in L^∞ so that $\|f_n g - 1\|_2$ tends to zero; so that $\mathcal{T}$ has property (I_2). It fails to have property (I_3) since $\{T_f 1 : f \in L^\infty\}$ is not all of H^2. It also fails to have (I_4). To see this, consider $u = z$ and $v = (z + z^2)/\sqrt{2}$. If $T_f u = v$, then $f = z/\sqrt{2} + 1/\sqrt{2} + \sum_{n=2}^{\infty} a_n \bar{z}^n$. If $\sigma_n(f)$ denotes the n^{th} Cesaro mean of f, [4], then $\|\sigma_n(f)\|_\infty \leq \|f\|_\infty$. Hence

$$\begin{aligned}\|f\|_\infty = \|\bar{z}f\|_\infty \geq \|\sigma_2(\bar{z}f)\|_\infty &= \|1/\sqrt{2} + \sqrt{2}\bar{z}/3\|_\infty \\ &\geq |\sigma_2(\bar{z}f)(1)| = 5/3\sqrt{2}.\end{aligned}$$

Hence $\sup\{|(T_f u, v)| : \|f\|_\infty \leq 1\} \leq \dfrac{3\sqrt{2}}{5} < 1$.

This example does not answer the following question: Is every unital isometry of a C^* algebra with property (I_4) either multiplicative or anti-multiplicative?

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