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BAER AND QUASI-BAER PROPERTIES OF GROUP RINGS

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Abstract

A ring R is said to be a Baer (respectively, quasi-Baer) ring if the left annihilator of any nonempty subset (respectively, any ideal) of R is generated by an idempotent. It is first proved that for a ring R and a group G, if a group ring RG is (quasi-) Baer then so is R; if in addition G is finite then $|G|^{-1} \in R$. Counter examples are then given to answer Hirano's question which asks whether the group ring RG is (quasi-) Baer if R is (quasi-) Baer and G is a finite group with $|G|^{-1} \in R$. Further, efforts have been made towards answering the question of when the group ring RG of a finite group G is (quasi-) Baer, and various (quasi-) Baer group rings are identified. For the case where G is a group acting on R as automorphisms, some sufficient conditions are given for the fixed ring R^G to be Baer.

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1. Introduction

Throughout this paper R is assumed to be an associative ring with unity. For a subset X of R, let $\mathbf{l}_R(X)$ denote the left annihilator of X in R. A ring R is said to be a Baer (respectively, quasi-Baer) ring if for any nonempty subset (respectively, any ideal) X of R we have $\mathbf{l}_R(X) = Re$ where $e^2 = e \in R$. The concept of a Baer ring was introduced by Kaplansky in [9] to abstract properties of rings of operators on a Hilbert space, while the notion of a quasi-Baer ring was first used by Clark [5] in 1967 to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. The definitions of Baer and quasi-Baer rings are indeed left-right symmetric by [9] and [5]. For the development and an up-to-date account of the study of quasi-Baer and Baer rings, we refer to the article of Birkenmeier, Kim and Park [1].

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The objective of this paper is to consider the question of when a group ring is (quasi-) Baer. Several related results can be recalled. If R is a quasi-Baer ring and C_∞ is the infinite cyclic group and H is the discrete Heisenberg group, then the group rings RC_∞ and RH are quasi-Baer. This result was obtained in [2], following the authors' result that a ring R is quasi-Baer if and only if $R[x]$ is quasi-Baer, and if and only if $R[x, x^{-1}]$ is quasi-Baer. For an ordered monoid G, it was proved in Hirano [7] that if R is a quasi-Baer ring then the monoid ring RG is quasi-Baer and that RG is a reduced Baer ring if and only if the same is true of R. It was proved in [6] that if R is a reduced ring and G is a so called 'u.p.' semigroup then the semigroup ring RG is Baer if and only if the same is true of R. In [3], the authors proved that for a so-called 'u.p.' monoid G, the monoid ring RG is quasi-Baer if and only if the same is true of R. The main idea in proving all these results is similar to that used in the cases of (Laurent) polynomial rings and it does not help for the question of when a group ring is (quasi-) Baer (which was raised in [1, Question 2.12]). In the Open Problem Section of the Third International Symposium on Ring Theory (Kyongju, South Korea, 1999), Hirano asked whether the group ring RG is quasi-Baer if R is quasi-Baer and G is a finite group with $|G|^{-1} \in R$.

The group ring of a group G over a ring R is denoted by RG. Write C_n for the cyclic group of order n. The following results are obtained: If RG is (quasi-) Baer then so is R; if RG is quasi-Baer and G is a finite group then $|G|^{-1} \in R$. As a response to Hirano's question, two integral domains R_1, R_2 with $2^{-1} \in R_1$ and $3^{-1} \in R_2$ are constructed such that $R_1C_{2^k}$ and $R_2C_{3^l}$ are not quasi-Baer for any $k \geq 2$ or any $l \geq 1$. We also construct a Baer ring R with $6^{-1} \in R$ such that RS_3 is not Baer. In addition, we prove that Hirano's question has a positive answer when $G = C_2$ or $G = S_3$ and that if D_∞ is the infinite dihedral group then RD_∞ is quasi-Baer if and only if R is quasi-Baer. Two sufficient conditions are obtained for a fixed ring to be Baer.

For any finite subgroup H of a group G, we let $\hat{H} = \sum_{h\in H} h$. If $g \in G$ has finite order, we define $\hat{g} = \hat{H}$ where $H = \langle g \rangle$. We write $\mathbb{Z}$ for the ring of integers and $\mathbb{Z}_n$ for the ring of integers modulo n. As usual, $\mathbb{Q}$ is the field of rationals and $\mathbb{C}$ denotes the field of complex numbers. The imaginary unit is denoted by $\mathbf{i}$. The $n \times n$ matrix ring over R is denoted $\mathbb{M}_n(R)$.

2. Necessary conditions

We start by proving the following.

THEOREM 2.1. *Let R be a subring of a ring S such that both share the same identity. Suppose that S is a free left R-module with a basis G such that $1 \in G$ and $ag = ga$ for all $a \in R$ and all $g \in G$. If S is (quasi-) Baer then so is R.*

PROOF. We give the proof for the case of quasi-Baer rings and the proof for the case of Baer rings is similar. Let I be an ideal of R. Since S is quasi-Baer, $\mathbf{l}_S(SI) = Se$ where $e^2 = e \in S$. Write $e = a_0 g_{\alpha(0)} + \cdots + a_n g_{\alpha(n)}$ where $g_{\alpha(0)} = 1$ and the $g_{\alpha(i)} \in G$ are distinct and $a_i \in R$. Then for all $a \in I$ we have

$$0 = ea = (a_0 g_{\alpha(0)} + \cdots + a_n g_{\alpha(n)})a = a_0 a g_{\alpha(0)} + \cdots + a_n a g_{\alpha(n)},$$

which shows that $a_i a = 0$. Therefore $a_i I = 0$ for $i = 0, \ldots, n$. Thus

$$a_i SI = a_i(\oplus_{g \in G} Rg)I = a_i \sum (RI)g = \sum a_i Ig = 0.$$

So $a_i \in \mathbf{l}_S(SI) = Se$, which implies that $a_i = a_i e$. It follows that $a_0^2 = a_0 \in R$. Because $a_0 I = 0$, we have $Ra_0 \in \mathbf{l}_R(I)$. If $r \in \mathbf{l}_R(I)$ then

$$rSI = r(\oplus_{g \in G} Rg)I = r \sum (RI)g = \sum rIg = 0.$$

So $r \in \mathbf{l}_S(SI) = Se$. This shows that

$$r = re = r(a_0 g_{\alpha(0)} + \cdots + a_n g_{\alpha(n)}) = ra_0 g_{\alpha(0)} + \cdots + ra_n g_{\alpha(n)}.$$

So $r = ra_0 \in Ra_0$. Hence $\mathbf{l}_R(I) = Ra_0$. □

COROLLARY 2.2. *Let R be a ring and G be a group. If the group ring RG is (quasi-) Baer then so is R.*

PROOF. Note that $S = RG = \oplus_{g \in G} Rg$ is a free left R-module with a basis G satisfying the assumptions of Theorem 2.1. □

COROLLARY 2.3. [2] *If $R[x]$ or $R[x, x^{-1}]$ is (quasi-) Baer then so is R.*

PROOF. $R[x]$ and $R[x, x^{-1}]$ are free R-modules with bases $\{x^i : i = 0, 1, \ldots\}$ and $\{x^i : i = 0, \pm 1, \ldots\}$ satisfying the assumptions of Theorem 2.1. □

THEOREM 2.4. *If G is a finite group and the group ring RG is quasi-Baer then $|G|^{-1} \in R$.*

PROOF. It is well known that the augmentation ideal is $\omega(RG) = \sum_{g \in G} R(1 - g)$ and $\mathbf{l}_{RG}(\omega(RG)) = RG\hat{G}$ (see [11, Lemma 1.2, p.68]). Since RG is quasi-Baer, we have

$$RG\hat{G} = RGe \tag{2.1}$$

where $e^2 = e \in RG$. There exists $\sum r_g g \in RG$ such that $e = \left(\sum r_g g\right)\hat{G} = \left(\sum r_g\right)\hat{G}$. Thus $\left(\sum r_g\right)\hat{G} = e = e^2 = |G|\left(\sum r_g\right)^2 \hat{G}$, which shows that

$$\sum r_g = |G| \left(\sum r_g\right)^2. \tag{2.2}$$

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Since $RG\hat{G} \neq 0$, we have $e \neq 0$, so $|G| \neq 0$. Hence the following claim has been proved.

CLAIM 2.5. *If a group ring of a finite group is quasi-Baer then the order of the group is not zero in the coefficient ring.*

Now let $n = |G|$ and $r = \sum r_g$. By (2.1), $\hat{G} = \left(\sum s_g g\right)e = \left(\sum s_g g\right)r\hat{G}$. Applying augmentation mapping to both sides yields

$$n = \left(\sum s_g\right) rn. \tag{2.3}$$

By (2.2) and (2.3), it suffices to show that $\mathbf{l}_R(n) = 0$. Suppose that $\mathbf{l}_R(n) \neq 0$. Then $na = 0$ for some nonzero $a \in R$. Thus $n(Ra) = R(na) = 0$, so $n \in \mathbf{l}_R(Ra)$. Since R is quasi-Baer by Corollary 2.2, $\mathbf{l}_R(Ra) = Rf$ where $f^2 = f \in R$. Clearly $f \neq 1$, so $1 - f \neq 0$. Moreover, $n(1 - f) = 0$. But

$$(1 - f)(RG)(1 - f) = (1 - f)R(1 - f)G.$$

Since RG is quasi-Baer, it follows by Clark [5] that $(1 - f)(RG)(1 - f)$ is quasi-Baer. Therefore, SG is quasi-Baer where $S = (1 - f)R(1 - f)$. So $n \neq 0$ in S by the Claim. This contradicts the fact that $n(1 - f) = 0$. Hence $\mathbf{l}_R(n) = 0$. The proof is complete. □

The next fact is an immediate consequence of Theorem 2.4.

EXAMPLE 1. $\mathbb{Z}G$ is not quasi-Baer for any nontrivial finite group G.

EXAMPLE 2. Let G be a finite group and n be an integer with $n > 1$. Then the following are equivalent:

(1) $\mathbb{Z}_n G$ is Baer.
(2) $\mathbb{Z}_n G$ is quasi-Baer.
(3) $\gcd(n, |G|) = 1$ and n is square-free.

PROOF. (1) clearly implies (2).

Suppose that (2) holds. Write $n = p_1^{s_1} \cdots p_k^{s_k}$ where all p_i are prime numbers and $s_i > 0$. Then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{s_1}} \times \cdots \times \mathbb{Z}_{p_k^{s_k}}$, and $\mathbb{Z}_n G \cong \mathbb{Z}_{p_1^{s_1}} G \times \cdots \times \mathbb{Z}_{p_k^{s_k}} G$. It follows from (2) that each $\mathbb{Z}_{p_i^{s_i}} G$ is quasi-Baer. So $\mathbb{Z}_{p_i^{s_i}}$ is quasi-Baer by Corollary 2.2 and $p_i^{s_i}$ does not divide $|G|$ by Theorem 2.4. It follows that $s_i = 1$ and p_i does not divide $|G|$. Hence (3) holds.

If (3) is satisfied then $\mathbb{Z}_n G$ is a semisimple ring by Maschke's Theorem, so (1) holds. □

3. Group rings of finite groups and Hirano's question

Let R be a ring and G be a finite group. If RG is (quasi-) Baer then R is (quasi-) Baer and $|G|^{-1} \in R$. Thus it is natural to ask whether the converse holds true. This question on quasi-Baer rings has been raised by Hirano [8]. In this section, counter-examples to these questions are given and various (quasi-) Baer group rings are identified.

LEMMA 3.1. *If $2^{-1} \in R$, then $RC_2 \cong R \times R$.*

PROOF. Write $C_2 = \{1, g\}$. If $2^{-1} \in R$, then the mapping $RC_2 \to R \times R$ given by $a + bg \mapsto (a + b, a - b)$ is a ring isomorphism. □

COROLLARY 3.2. *If $2^{-1} \in R$, then RC_2 is (quasi-) Baer if and only if the same is true of R.*

LEMMA 3.3. *If $2^{-1} \in R$ then $RC_4 \cong R \times R \times R[x]/(x^2 + 1)$.*

PROOF. Write $C_4 = \{1, g, g^2, g^3\}$ and let $e = (1 + g^2)/2$. Since e is a central idempotent of RC_4, we have $RC_4 = RC_4e \times RC_4(1 - e)$. Direct calculation shows that $RC_4e = \{re + sge : r, s \in R\}$ and $RC_4(1 - e) = \{r(1 - e) + sg(1 - e) : r, s \in R\}$. The mapping $RC_4e \to R[x]/(x^2 - 1)$ given by $re + sge \mapsto r + s\bar{x}$ is a ring isomorphism. Similarly, the mapping $RC_4(1 - e) \to R[x]/(x^2 + 1)$ given by $r(1 - e) + sg(1 - e) \mapsto r + s\bar{x}$ is a ring isomorphism. Moreover, by Lemma 3.1 $R[x]/(x^2 - 1) \cong RC_2 \cong R \times R$. □

COROLLARY 3.4. *If $2^{-1} \in R$ then RC_4 is (quasi-) Baer if and only if the same is true of $R[x]/(x^2 + 1)$.*

LEMMA 3.5. *Let R be a ring with $3^{-1} \in R$ and $C_3 = \{1, g, g^2\}$. Then the following statements hold:*

(1) *$e = \frac{1}{3}\hat{g}$ is a central idempotent of RC_3 and $RC_3 = (RC_3)e \times (RC_3)(1 - e)$, where $(RC_3)e = \{re : r \in R\} \cong R$ and*

$$(RC_3)(1 - e) = \{r + sg + (-r - s)g^2 : r, s \in R\}.$$

(2) *If $R \subseteq \mathbb{C}$ then*

$$RC_3 \cong R[x]/(x^3 - 1) \cong R[x]/(x - 1) \times R[x]/(x^2 + x + 1)$$
$$\cong R \times R[x]/(x^2 + x + 1).$$

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PROOF. The verification of (1) is straightforward. If $1/3 \in R \subseteq \mathbb{C}$, then the ideals $(x-1)$ and (x^2+x+1) are coprime in $R[x]$, so (2) follows by Chinese Remainder Theorem. □

It follows that if R is a subring of $\mathbb{C}$ with $1/3 \in R$ then RC_3 is Baer if and only if the same is true of $R[x]/(x^2+x+1)$.

THEOREM 3.6. *Let R be a subring of $\mathbb{C}$ and let $Q(R)$ denote the quotient field of R. Consider the polynomial $x^2+ax+b \in R[x]$ with $a^2-4b \neq 0$. Let w be a solution of $x^2+ax+b=0$ in $\mathbb{C}$. Then $R[x]/(x^2+ax+b)$ is (quasi-) Baer if and only if either $w \in R$ or $Rw \cap R = 0$ (that is, $w \notin Q(R)$).*

PROOF. Let S denote the ring $R[x]/(x^2+ax+b)$. Let $x^2+ax+b=(x-w)(x-v)$ where $w, v \in \mathbb{C}$. By hypothesis, $w \neq v$. First suppose that $w \notin Q(R)$. Then S is a subring of $\mathbb{C}$ and hence is a domain. In particular, S is Baer. Next suppose that $w \in Q(R)$. Then $v \in Q(R)$. Define the map $\varphi : R[x] \to Q(R) \times Q(R)$ by $\varphi(f(x)) = (f(w), f(v))$. Then the kernel of φ is (x^2+ax+b). Hence S can be regarded as a subring of $Q(R) \times Q(R)$. Clearly S is not a domain. We can easily see that S is Baer if and only if S contains the idempotent $(1,0) \in Q(R) \times Q(R)$ and that $(1,0) \in S$ if and only if there exists $rx+s \in R[x]$ such that $rw+s=1$ and $rv+s=0$. Since $x^2+ax+b=(x-w)(x-v)$, we deduce that $(ar-1)s = [-(w+v)r-1]s = [-(1-2s)-1]s = 2s(s-1) = 2(-rv)(-rw) = 2r^2b$. This implies that s is divisible by r in R. Hence $v = -s/r \in R$ and so $w = -a-v \in R$. □

Next we give counter-examples to Hirano's question for $G = C_3$ and C_4.

EXAMPLE 3. Let $R_0 = \{n/2^k : n \in \mathbb{Z},\ k \text{ a non-negative integer}\}$. Then R_0 is a subring of $\mathbb{Q}$. Set

$$R = \{a + 3b\mathbf{i} : a, b \in R_0\}.$$

Then R is a subring of $\mathbb{C}$ with $1/2 \in R$. Because R is a domain, it is certainly Baer. Clearly $\mathbf{i} \notin R$. Moreover, for $r=3$ and $s=3\mathbf{i}$, we have $s = r\mathbf{i} \in R\mathbf{i} \cap R$. So, by Theorem 3.6, $R[x]/(x^2+1)$ is not quasi-Baer. Hence RC_4 is not quasi-Baer by Corollary 3.4.

EXAMPLE 4. Let $R_0 = \{n/3^k : n \in \mathbb{Z},\ k \text{ a non-negative integer}\}$. Then R_0 is a subring of $\mathbb{Q}$. Set

$$R = \{a + b\sqrt{3}\mathbf{i} : a, b \in R_0\}.$$

Then R is a subring of $\mathbb{C}$ with $1/3 \in R$. Because R is a domain, it is certainly Baer. Let $a = 2\sqrt{3}\mathbf{i}$, $b = -(3+\sqrt{3}\mathbf{i})$ and $w = b/a$. Then $a, b \in R$ and $w = (-1+\sqrt{3}\mathbf{i})/2$

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is a root of x^2+x+1. So $Rw \cap R \neq 0$. Moreover, it is easy to verify that the equation $x^2+x+1=0$ is not solvable in R. Hence it follows by Theorem 3.6 and Lemma 3.5 that RC_3 is not quasi-Baer.

THEOREM 3.7. *If RG is Baer then so is RH for every subgroup H of G.*

PROOF. Let A be a nonempty subset of RH. Because RG is Baer and $RH \subseteq RG$, we have $\mathbf{l}_{RG}(A) = RGe$, where $e^2 = e \in RG$. Write $e = \sum_{h\in H} a_h h + \sum_{g\notin H} b_g g$. Then for all $\beta \in A$,

$$0 = e\beta = \left(\sum_{h\in H} a_h h\right)\beta + \left(\sum_{g\notin H} b_g g\right)\beta. \tag{3.1}$$

Note that if $h \in H$ and $g \notin H$ then $hg \notin H$. This shows that the support of $\left(\sum_{g\notin H} b_g g\right)\beta$ is contained in $G\backslash H$. So it follows by (3.1) that if $\alpha = \sum_{h\in H} a_h h$ then $\alpha \in \mathbf{l}_{RH}(A) \subseteq \mathbf{l}_{RG}(A) = RGe$, and hence

$$\sum_{h\in H} a_h h = \left(\sum_{h\in H} a_h h\right) e = \left(\sum_{h\in H} a_h h\right)^2 + \left(\sum_{h\in H} a_h h\right)\left(\sum_{g\notin H} b_g g\right).$$

Therefore $\alpha^2 = \alpha = \alpha e$ and $RH\alpha \subseteq \mathbf{l}_{RH}(A)$. If $\gamma \in \mathbf{l}_{RH}(A)$ then $\gamma A = 0$. So $\gamma = \gamma e = \gamma\left(\sum_{h\in H} a_h h\right) + \gamma\left(\sum_{g\notin H} b_g g\right)$, hence $\gamma = \gamma\left(\sum_{h\in H} a_h h\right) = \gamma\alpha$. So $RH\alpha = \mathbf{l}_{RH}(A)$. Hence R is Baer. □

EXAMPLE 5. If R is the ring in Example 3 and G is a group containing a subgroup isomorphic to C_4, then RG is not Baer by Theorem 3.7. In particular, for all $k \geq 2$ the group ring RC_{2^k} is not Baer and hence not quasi-Baer. Similarly, if R is the ring in Example 4 and G is a group containing a subgroup isomorphic to C_3, then RG is not Baer. In particular, for all $k \geq 1$ the group ring RC_{3^k} is not quasi-Baer.

LEMMA 3.8. [4, Lemma 4.7] *If $6^{-1} \in R$, then $RS_3 \cong R \times R \times \mathbb{M}_2(R)$.*

A new family of quasi-Baer rings can be obtained as group rings of S_3.

COROLLARY 3.9. *Let $6^{-1} \in R$. Then RS_3 is quasi-Baer if and only if the same is true of R, and RS_3 is Baer if and only if the same is true of $\mathbb{M}_2(R)$.*

By Pollingher and Zaks [12, p.134], there exists a Baer ring R such that $6^{-1} \in R$ and $\mathbb{M}_2(R)$ is not Baer, so RS_3 is not Baer.

The next theorem gives another family of quasi-Baer group rings.

THEOREM 3.10. *Let* $D_\infty = \langle x, y : o(x) = 2, o(y) = \infty, xyx = y^{-1}\rangle$ *be the infinite dihedral group. Then* RD_∞ *is quasi-Baer if and only if* R *is quasi-Baer.*

PROOF. The implication in one direction follows from Corollary 2.2. To prove the converse, suppose that R is quasi-Baer. First notice that $RD_\infty \cong S[x;\sigma]/(x^2-1)$ where $S = R[y, y^{-1}]$ and $\sigma \in \mathrm{Aut}(R)$ with $\sigma(y) = y^{-1}$ and $\sigma(r) = r$ for all $r \in R$ (see [10, p.22]). Let $T = S[x;\sigma]/(x^2-1)$. We next show that T is a quasi-Baer ring. Let A be a nonzero ideal of T and set

$$I = \{a \in S : a + b\bar{x} \in A \quad \text{for some } b \in S\},$$
$$J = \{b \in S : a + b\bar{x} \in A \quad \text{for some } a \in S\}.$$

Then $I = J$ is an ideal of S. Because R is quasi-Baer, S is quasi-Baer by [2, Theorem 1.2]. Thus $\mathbf{l}_S(I) = Se$ where $e^2 = e \in S$. We verify next that $\mathbf{l}_T(A) = Te$. Because $eI = 0$, we have $eA = 0$, so $Te \subseteq \mathbf{l}_T(A)$. Let $c + d\bar{x} \in \mathbf{l}_T(A)$ where $c, d \in S$ and let $a_0 \in I$. Then there exists $b_0 \in I$ such that $a_0 + b_0\bar{x} \in A$. Therefore, for all $a \in S$, we have

$$0 = (c + d\bar{x})a(a_0 + b_0\bar{x}) = (c + d\bar{x})(aa_0 + ab_0\bar{x})$$
$$= [caa_0 + d\sigma(a)\sigma(b_0)] + [cab_0 + d\sigma(a)\sigma(a_0)]\bar{x}.$$

It follows that, for all $a \in S$,

$$caa_0 + d\sigma(a)\sigma(b_0) = 0 \quad \text{and} \quad cab_0 + d\sigma(a)\sigma(a_0) = 0.$$

Thus, letting $a = y^n$ ($n \in \mathbb{Z}$) yields

$$cy^n a_0 + dy^{-n}\sigma(b_0) = 0 \quad \text{and} \quad cy^n b_0 + dy^{-n}\sigma(a_0) = 0.$$

Since y^n is in the center of S, it follows that

$$y^{2n}ca_0 = -d\sigma(b_0) \quad \text{and} \quad y^{2n}cb_0 = -d\sigma(a_0). \tag{3.2}$$

Because (3.2) holds for all $n \in \mathbb{Z}$ and because c, d, a_0, b_0 are fixed elements of S, we obtain

$$ca_0 = d\sigma(a_0) = 0.$$

Thus $\sigma^{-1}(d)a_0 = 0 = ca_0$. Since a_0 is an arbitrary element of I, we deduce that c and $\sigma^{-1}(d)$ are in $\mathbf{l}_S(I) = Se$. Write $c = s_1e$, $\sigma^{-1}(d) = s_2e$ where $s_1, s_2 \in S$. Then $d = \sigma(s_2)\sigma(e)$ and so $c + d\bar{x} = s_1e + \sigma(s_2)\sigma(e)\bar{x} = [s_1 + \sigma(s_2)\bar{x}]e \in Te$. Hence $\mathbf{l}_T(A) = Te$ and T is a quasi-Baer ring. □

REMARK 1. (1) RD_∞ may not be Baer even for an integral domain R: because $\mathbb{Z}C_2$ is not Baer (Example 1), it follows, by Theorem 3.7, that $\mathbb{Z}D_\infty$ is not Baer.
(2) Since $\mathbb{Z}D_\infty$ is quasi-Baer but $\mathbb{Z}C_2$ is not, the quasi-Baer analog of Theorem 3.7 does not hold. In Example 8 below, an integral domain R is given such that RC_3 is not quasi-Baer but RS_3 is quasi-Baer (so $6^{-1} \in R$).

REMARK 2. In view of Corollary 3.9 and Theorem 3.10, it would be interesting to know when the group ring RD_n of the dihedral group D_n of order $2n$ is quasi-Baer. The method used in proving Theorem 3.10 can be used to show that if RC_n is quasi-Baer and $2^{-1} \in R$ then RD_n is quasi-Baer, but the converse does not hold because of Remark 1(2).

4. Fixed rings

Let G be a group acting on a ring R as automorphisms and let R^G be the fixed ring of G acting on R. Here we study the conditions under which R^G becomes (quasi-) Baer.

THEOREM 4.1. *Let R be a ring and G be a group acting on R as automorphisms such that either* (i) $ee^g = e^g e$ *for all $g \in G$ and all $e^2 = e \in R$ or* (ii) *G is finite with $|G|^{-1} \in R$. If R is Baer then so is R^G.*

PROOF. Let A be a nonempty subset of R^G. Since R is Baer, we have $\mathbf{l}_R(A) = Re$ where $e^2 = e \in R$. For $g \in G$ we have $Re^g = R^g e^g = (Re)^g = (\mathbf{l}_R(A))^g = \mathbf{l}_{R^g}(A^g) = \mathbf{l}_R(A) = Re$. It follows that

$$e^g = e^g e \quad \text{and} \quad e = ee^g \quad \text{for all } g \in G. \tag{4.1}$$

Suppose that (i) holds. It follows from (4.1) that $e = e^g$ for all $g \in G$, so $e \in R^G$. Since $eA = 0$, we have that $R^G e \subseteq \mathbf{l}_{R^G}(A)$. For $r \in \mathbf{l}_{R^G}(A)$, we have $rA = 0$, so $r \in \mathbf{l}_R(A) = Re$. Thus, $r = re \in R^G e$. Hence $\mathbf{l}_{R^G}(A) = R^G e$.

Suppose that (ii) holds. Let $f = (1/|G|)\sum_{g \in G} e^g$. Note that, for all $g, h \in G$, (4.1) implies $e^h e^g = (e^h e)e^g = e^h(ee^g) = e^h e = e^h$. This shows that

$$\begin{aligned} f^2 &= \left(\frac{1}{|G|}\sum_{h\in G} e^h\right)\left(\frac{1}{|G|}\sum_{g\in G} e^g\right) = \frac{1}{|G|^2}\sum_{h\in G}\sum_{g\in G} e^h e^g = \frac{1}{|G|^2}\sum_{h\in G}\sum_{g\in G} e^h \\ &= \frac{1}{|G|^2}\sum_{h\in G} |G| e^h = f. \end{aligned}$$

Moreover, $f^g = f$ for all $g \in G$. So $f \in R^G$. Because $eA = 0$ and $f \in Re$ (by (4.1)), we have $R^G f \subseteq \mathbf{l}_{R^G}(A)$. Note that $\mathbf{l}_{R^G}(A) \subseteq \mathbf{l}_R(A) = Re^g$ for all $g \in G$. Thus, for $r \in \mathbf{l}_{R^G}(A)$, we have $r = re^g$ for all $g \in G$. Hence

$$r = \frac{1}{|G|}|G|r = \frac{1}{|G|}\sum_{g\in G} re^g = r\frac{1}{|G|}\sum_{g\in G} e^g = rf \in R^G f.$$

So $\mathbf{l}_{R^G}(A) = R^G f$. So R^G is Baer. □

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The next example shows that the assumptions (i) and (ii) in the previous theorem are necessary.

EXAMPLE 6. [13, Example 6.4] Let K be a field of characteristic $p > 0$. Let $R = \mathbb{M}_2(K)$ and $G = \langle g \rangle$ where $g : R \to R, r \mapsto u^{-1}ru$, with $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then R is Baer (indeed simple Artinian). Direct calculation shows that $R^G = \left\{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) : a, b \in K\right\}$. So $J(R^G) = \left\{\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) : b \in K\right\}$. If $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ then $\mathbf{l}_{R^G}(x) = J(R^G)$. Because $J(R^G)$ cannot be generated by an idempotent, R^G is not Baer. If $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in R$ then $e^2 = e$ and $e^g = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It is clear that $ee^g = e \neq e^g = e^g e$. Moreover, $|G| = p$ is zero in R.

The next example shows that R being Baer is not necessary for R^G to be Baer.

EXAMPLE 7. Let K be a field with $2^{-1} \in K$ and $R = \left\{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) : a, b \in K\right\}$. Let $g : R \to R$ be given by $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & -b \\ 0 & a \end{smallmatrix}\right)$, and $G = \langle g \rangle$. It is seen that $R^G = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) : a \in K\right\} \cong K$. So R^G is Baer but R is not quasi-Baer.

In contrast to Theorem 4.1, we give in our concluding example a quasi-Baer ring S and a finite group G acting on S as automorphisms such that $|G|^{-1} \in S$ and S^G is not quasi-Baer.

LEMMA 4.2. *Suppose that R is a ring with $3^{-1} \in R$. Let $g = (123) \in S_3$ and $G = \langle \gamma \rangle$ where $\gamma : RS_3 \to RS_3$ is given by $\xi \mapsto g^{-1}\xi g$. Then $|G| = 3$ and $(RS_3)^G \cong R \times RC_3$.*

PROOF. It is clear that $|G| = 3$. Let $f = (12) + (13) + (23) \in RS_3$. Then

$$(RS_3)^G = \{\xi \in RS_3 : \xi g = g\xi\} = \left\{a + bf + cg + dg^2 : a, b, c, d \in R\right\}.$$

Let $e = (1/3)\hat{g}$. Then $e^2 = e \in (RS_3)^G$ and $ef = fe = f$. So e is a central idempotent of $(RS_3)^G$. This shows that

$$(RS_3)^G = (RS_3)^G e \times (RS_3)^G(1 - e),$$

where

$$\begin{aligned}
(RS_3)^G(1 - e) &= \left\{(a + bf + cg + dg^2)(1 - e) : a, b, c, d \in R\right\} \\
&= \left\{\frac{2a - c - d}{3} + \frac{-a + 2c - d}{3}g + \frac{-a - c + 2d}{3}g^2 : a, c, d \in R\right\} \\
&= \left\{r + sg + (-r - s)g^2 : r, s \in R\right\}, \\
(RS_3)^G e &= \left\{(a + bf + cg + dg^2)e : a, b, c, d \in R\right\} \\
&= \{(a + c + d)e + bf : a, b, c, d \in R\} = \left\{re + \frac{s}{3}f : r, s \in R\right\} \\
&\cong RC_2 \cong R \times R.
\end{aligned}$$

The last isomorphism is by Lemma 3.1. To see the second last isomorphism, note that $f^2 = 9e$ and $fe = ef = f$, so $re + (s/3)f \mapsto r + sh$ (where $C_2 = \{1, h\}$) is the required isomorphism. Therefore, it follows by Lemma 3.5 that

$$(RS_3)^G \cong R \times RC_3. \qquad \square$$

EXAMPLE 8. Let $R_0 = \{n/6^k : n \in \mathbb{Z},\ k \text{ a nonnegative integer}\}$ and set

$$R = \{a + 5b\sqrt{3}\mathbf{i} : a, b \in R_0\}.$$

Then R is a subring of $\mathbb{C}$ and $6^{-1} \in R$. It is easy to see that $x^2 + x + 1 = 0$ is not solvable in R. Moreover, if $w = (-1 \pm \sqrt{3}\mathbf{i})/2$ (a root of $x^2 + x + 1$), then $10w = -5 \pm 5\sqrt{3}\mathbf{i} \in R$. So $Rw \cap R \neq 0$. Hence, by Lemma 3.5 and Theorem 3.6, RC_3 is not quasi-Baer. Let G be the group in Lemma 4.2. Then $|G| = 3$ and $(RS_3)^G \cong R \times RC_3$ by Lemma 4.2. So it follows that $|G|^{-1} \in RS_3$ and $(RS_3)^G$ is not quasi-Baer. However, RS_3 is quasi-Baer by Corollary 3.9. In summary, (1) RS_3 is quasi-Baer (so $6^{-1} \in R$), (2) RC_3 is not quasi-Baer, (3) $(RS_3)^G$ is not quasi-Baer where $|G| = 3$ is a unit of RS_3.

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References

[1] G. F. Birkenmeier, J. Y. Kim and J. K. Park, 'On quasi-Baer rings', in: *Algebra and its applications (Athens, OH, 1999), Contemp. Math.* **259** (Amer. Math. Soc., Providence, RI, 2000) pp. 67–92.

[2] ———, 'Polynomial extensions of Baer and quasi-Baer rings', *J. Pure Appl. Algebra* **159** (2001), 25–42.

[3] G.F. Birkenmeier and J.K. Park, 'Triangular matrix representations of ring extensions', *J. Algebra* **265** (2003), 457–477.

[4] J. Chen, Y. Li and Y. Zhou, 'Morphic group rings', *J. Pure Appl. Algebra* **205** (2006), 621–639.

[5] W. E. Clark, 'Twisted matrix units semigroup algebras', *Duke Math. J.* **34** (1967), 417–424.

[6] N. J. Groenewald, 'A note on extensions of Baer and p.p.-rings', *Publ. de L'institut Math.* **34** (1983), 71–72.

[7] Y. Hirano, 'On ordered monoid rings over a quasi-Baer ring', *Comm. Algebra* **29** (2001), 2089–2095.

[8] ———, 'Open problems', in: *Proceedings of the 3rd Korea-China-Japan Symposium and the 2nd Korea-Japan Joint Seminar held in Kyongju, June 28–July 3, 1999*, Trends Math. (2001) pp. 442.
[9] I. Kaplansky, *Rings of Operators*, Math. Lecture Notes Series (Benjamin, New York, 1965).
[10] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings* (John Wiley & Sons, Chichester, 1987).
[11] D. Passman, *The Algebraic Structure of Group Rings* (John Wiley & Sons, New York-London-Sydney, 1977).
[12] A. Pollingher and A. Zaks, 'On Baer and quasi-Baer rings', *Duke Math. J.* **37** (1970), 127–138.
[13] Z. Yi, 'Homological dimension of skew group rings and crossed products', *J. Algebra* **164** (1994), 101–123.

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