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ON CAYLEY-DICKSON RINGS

BY
DANIEL J. BRITTEN

M. Slater has shown that a prime alternative (not associative) ring R such that $3R\neq 0$ is a Cayley-Dickson ring, [7]. That is, if F is the field of quotients of the center, Z, of R then $F \otimes_Z R$ is a Cayley-Dickson algebra.

If $J=H(R_n, J_a)$ is a prime Jordan matrix algebra of characteristic $\neq 2$ with $n\geq 3$ and J_a is a canonical involution, then R is an involution prime alternative ring whose symmetric elements are in its nucleus (see [3], Theorem 1, page 127 and Theorem 2, page 129). We shall prove that any involution prime alternative (not associative) ring R whose symmetric elements are in its nucleus is a Cayley-Dickson ring. This result is of interest since it allows us to obtain a Jordan ring of quotients for a prime Jordan ring $J=H(R_3, J_n)$ where R is alternative (not associative). Our result is independent of characteristic and its proof is "elementary" in the sense that it is basically an application of a theorem due to E. Kleinfeld ([4], page 728, Lemma 5), and one due to W. S. Martindale, [6], but we also use the fact that a simple alternative (not associative) ring is a Cayley-Dickson algebra, [1], [5].

THEOREM (KLEINFELD). *If R is an arbitrary prime alternative (not associative) ring then its nucleus is equal to its center.*

THEOREM (MARTINDALE). *Let R be a nonassociative ring with involution *. R is *-prime if and only if R contains a prime ideal P such that $P \cap P^*=0$.*

Martindale's proof was for associative rings. Although the proof for the non-associative case shall not be included, one may obtain it from Martindale's proof by changing certain products of ideals to their intersection, [2].

Finally, we shall prove an analogue of the Faith Utumi Theorem for Cayley-Dickson rings.

We shall assume throughout that R is an alternative (not associative) rings with involution *. An ideal, A, of R is a *-ideal if $A^*=A$. An ideal, Q of R is prime (*-prime) if $AB\subseteq Q$ implies $A\subseteq Q$ or $B\subseteq Q$ for ideals (*-ideals) A, B of R. R is said to be involution prime or *-prime if 0 is a *-prime ideal. The nucleus of R is the set $N=\{x \in R: (x, y, z)=(xy)z-x(yz)=0 \text{ for all } y, z \in R\}$: the center of R is the set $Z=\{x \in N: xy=yx\}$; the set of symmetric elements in R is $H=\{x \in R: x^*=x\}$.

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LEMMA. *If R is *-prime and A is a nonzero *-ideal of R, then $A \cap H \neq 0$.*

Proof. Suppose $A \cap H=0$, so that for all x in A we have $xx^*=0$ and $x+x^*=0$ which implies $x^2=0$ for all x in A. Thus, A is anti-commutative.

It is easy to see that R has characteristic. If the characteristic of R is 2, then $x^*=x$ for all x in A so that $A \subseteq H$. Therefore, we assume that the characteristic of R is not two. For x, y, w, z in A, we have $(xy)(zx)=z(yz)x=-x(x(yz))=0$, by a Moufang identity, so that by anti-commutativity $(ab)(cd)=0$ for a, b, c, d in A whenever two arguments in different factors are the same. Hence for x, y, w, z in A we have $0=((x+w)y)((x+w)z)=(xy)(wz)+(wy)(xz)$ and $0=(x(y+z))((y+z)w)=(xy)(zw)+(xz)(yw)=-(xy)(wz)+(wy)(xz)$. By adding the two equations, we get $2(wy)(xz)=0$. Thus $A^2A^2=0$ so that $A=0$ which is contrary to the assumption.

THEOREM *If R is any involution prime alternative (not associative) ring R whose symmetric elements are in its nucleus then R is a Cayley-Dickson ring.*

Proof. Assume R is *-prime and $H \subseteq N$. Then R is a sub-direct sum of prime alternative (not associative) rings and $N \subseteq Z$ so that H is an associative integral domain. Let K be the field of quotients of H. It is easy to see that $K \otimes_H R$ is *-prime with involution defined, $(k^{-1} \otimes r)^* = k^{-1} \otimes r^*$. By our Lemma, it follows that $K \otimes_H R$ is *-simple. Therefore, $K \otimes_H R$ is simple or it contains an ideal I which is simple such that $K \otimes_H R$ is the direct sum $I+I^*$. The latter case implies that R is associative. Hence $K \otimes_H R$ is simple and therefore, a Cayley-Dickson algebra. It is easy to see that $K \otimes_H R$ is isomorphic to $F \otimes_Z R$.

THEOREM *If R is a Cayley-Dickson ring and F is the field of quotients of the center, Z, of R so that $R'=F \otimes_Z R$ is a Cayley-Dickson algebra, then if given any basis $v_1, \ldots, v_8$ of R' over F, there exists an integral domain $I \subseteq Z$ such that $\sum Iv_i \subseteq R$ and if I' is the field of quotients of I then $I'=F$. (Here we are identifying R with $1 \otimes R$.)*

Proof. Every element in R is of the form $\sum a_i v_i$ where a_i is in F. Since v_i is an element of R', we have $v_i = z_i^{-1}(\sum a_{ij} v_j)$, summing over j for some choice of z_i in Z and $\sum a_{ij} v_j$ in R. Hence $z_i v_i$ is in R, so that, letting $I=(z_1 \cdots z_8)Z$, $Iv_i \subseteq R$ for $i=1, \ldots, 8$. z_i^{-1} is in I', because $z_i^{-1}=((z_1 \cdots z_8)z_i^2)^{-1}((z_1 \cdots z_8)z_i)$. Thus $I'=F$, since Z is contained in I'.

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UNIVERSITY OF WINDSOR,
WINDSOR, ONTARIO