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Global Geometrical Coordinates on Falbel's Cross-Ratio Variety

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Abstract. Falbel has shown that four pairwise distinct points on the boundary of a complex hyperbolic 2-space are completely determined, up to conjugation in $\mathrm{PU}(2,1)$, by three complex cross-ratios satisfying two real equations. We give global geometrical coordinates on the resulting variety.

1 Introduction

It is well known that a set of four pairwise distinct points on the Riemann sphere is determined up to Möbius equivalence by their cross-ratio. Moreover, permuting these points determines a new cross-ratio that may be expressed as a simple function of the first one; see [1, Section 4.4]. The cross-ratio was generalised to sets of four points in the boundary of complex hyperbolic space by Korányi and Reimann [8]. By simply counting dimensions, it is easy to see that this complex number cannot completely determine the four points up to $\mathrm{PU}(2,1)$ equivalence. By permuting the points we obtain 24 cross-ratios. There are certain relations between them; see [6] or [12]. After factoring out these relations, one is left with three complex cross-ratios satisfying two real relations; see [3]. Falbel's *cross-ratio variety* $\mathfrak{X}$ is the subset of $\mathbb{C}^3$ where these relations are satisfied. Falbel has shown in [3, Proposition 2.4] that these three complex numbers uniquely determine our initial set of four points up to $\mathrm{PU}(2,1)$ equivalence and, moreover, it is not possible to merely use two of the cross-ratios to do this. He goes on to discuss cross-ratios in a much more general setting. We will not be concerned with this level of generality here.

In [10] we used points of $\mathfrak{X}$ in our generalisation of Fenchel–Nielsen coordinates to the complex hyperbolic setting. There it was more convenient to use a slightly different normalisation from that of Falbel. In this paper we maintain the notation of [10]. Thus, we take Falbel's cross-ratio variety $\mathfrak{X}$ to be parametrised by three non-zero complex numbers $\mathbb{X}_1$, $\mathbb{X}_2$, and $\mathbb{X}_3$ satisfying the following identities:

$$|\mathbb{X}_3| = |\mathbb{X}_2|/|\mathbb{X}_1|, \tag{1.1}$$

$$2|\mathbb{X}_1|^2\Re(\mathbb{X}_3) = |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 - 2\Re(\mathbb{X}_1+\mathbb{X}_2) + 1. \tag{1.2}$$

(These equations are equivalent to the equations in [3, Proposition 2.3] under the substitution $\omega_0 = 1/\mathbb{X}_3$, $\omega_1 = 1/\mathbb{X}_1$ and $\omega_2 = \overline{\mathbb{X}}_2$.) From this description the topology and the geometry of the cross-ratio variety are mysterious. The purpose of this

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paper is to use global geometrical coordinates to give an alternative description of $\mathfrak{X}$ that makes its topology somewhat more transparent. In [4] the differential geometric structure of $\mathfrak{X}$ is discussed in more detail. A further application of cross-ratios is their use in the generalisation of Jørgensen's inequality to complex hyperbolic space [7]. We can interpret those results geometrically using the coordinates developed here.

We shall assume some background knowledge of complex hyperbolic geometry. Further details concerning the wider context may be found in Goldman [6]. Specific details, including many of the conventions we use, are given in [10].

2 The Main Theorems

Let $\langle\cdot,\cdot\rangle$ be a Hermitian form on $\mathbb{C}^3$ of signature $(2,1)$ and let $\mathbf{H}^2_{\mathbb{C}}$ be the image of its negative vectors under complex projection on $\mathbb{C}^3 - \{0\}$; see [6]. Let p_1, q_1, p_2, q_2 be four pairwise distinct points in $\partial\mathbf{H}^2_{\mathbb{C}}$ (and so each of them may be lifted to a null vector in $\mathbb{C}^3 - \{0\}$). Let L_i denote the complex line spanned by p_i and q_i for $i = 1, 2$. Each of these complex lines intersects $\mathbf{H}^2_{\mathbb{C}}$ in a disc which is naturally identified with the Poincaré model of the hyperbolic plane; see [6, Theorem 3.1.9]. There are four possibilities for the relative position of these two complex lines: they may be the same; they may intersect in a point of $\mathbf{H}^2_{\mathbb{C}}$; their ideal boundaries may intersect in a point of $\partial\mathbf{H}^2_{\mathbb{C}}$, in which case we say they are *asymptotic*, or finally they may be a positive distance apart, in which case we say they are *ultra-parallel*. These four possibilities are completely analogous to the more familiar case of a pair of geodesics in the hyperbolic plane. When L_1 and L_2 are ultra-parallel, there is a unique complex line $L^\perp$ orthogonal to both of them. In this case, let z_i denote the point of intersection of $L_\perp$ and L_i for $i = 1, 2$ and let $2d$ denote the distance between z_1 and z_2. When L_1 and L_2 intersect in a single point, let $z_1 = z_2$ denote this point of intersection and let ϕ denote the angle between L_1 and L_2. If $\mathbf{n}_i$ is a polar vector to L_i normalised so that $\langle\mathbf{n}_i,\mathbf{n}_i\rangle = 1$, then $\bigl|\langle\mathbf{n}_2,\mathbf{n}_1\rangle\bigr|$ equals $\cosh d$ or $\cos\phi$ in our two cases respectively; see [6, page 100]. In order not to have to write out formulae in the separate cases, we use the quantity $\bigl|\langle\mathbf{n}_2,\mathbf{n}_1\rangle\bigr|$, which we denote by r, as one of our coordinates. While doing so, we keep the geometrical significance of r in mind.

The points p_i, q_i and z_i form a hyperbolic triangle in the natural Poincaré metric on L_i. Let $2\theta_i$ be the angle at z_i measured from the side joining z_i, q_i to the side joining z_i, p_i; see Figure 2.1. It is the internal angle of the triangle at z_i if the triple (z_i, q_i, p_i) is positively oriented with respect to the natural complex structure on L_i. Otherwise, it is the external angle. The angle θ_i lies in the interval $(0,\pi)$. Drop a perpendicular from z_i to the geodesic joining p_i and q_i and call it γ_i. Orient the geodesic containing γ_i so that θ_i is the angle between its positive direction and the sides of the triangle with endpoint z_i. The length of γ_i is related to θ_i by the angle of parallelism formulae [1, Theorem 7.9].

We now define one more parameter, an angle ψ varying in $[0,2\pi)$. In the ultra-parallel case, using parallel transport, move γ_2 along $L_\perp$ to L_1. Then ψ is the angle in L_1 from the parallel transport of the positive direction of γ_2 to the positive direction of γ_1. In the spirit of the classical complex distance [9, 11], we use the following convention for the sign of ψ. When looking along the geodesic perpendicular to L_1 and L_2 the sign of ψ should be positive; see Figure 2.1 and compare it with Figure 1

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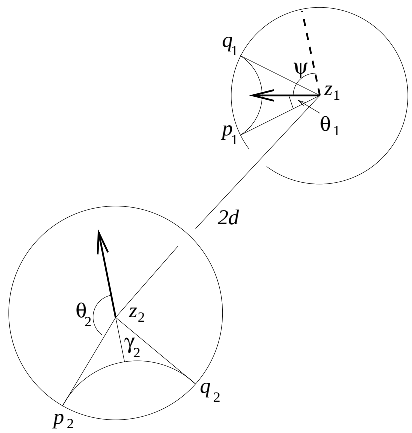

Figure 2.1: *The parameters for a pair of ultra-parallel geodesics. Here $d > 0$, $0 < \theta_1 < \pi/2$, $\pi/2 < \theta_2 < \pi$, $0 < \psi < \pi/2$.*

of [11]. In the case where L_1 and L_2 intersect in a point, we let Π_1 be an orthogonal projection onto L_1. Then ψ is defined to be the angle in L_1 measured from γ_1 to $\Pi_1(\gamma_2)$. Goldman calls this angle the *phase*; see [6, Section 2.2.2]. Note that when L_1 and L_2 are orthogonal, that is $r = \left|\langle \mathbf{n}_2, \mathbf{n}_1\rangle\right| = 0$, we have $\Pi_1(\gamma_2) = z_1$ and so ψ is not defined. In fact, r and ψ form polar coordinates on $\mathbb{C}$.

In the degenerate case where $L_1 = L_2$ we have $r = \left|\langle \mathbf{n}_2, \mathbf{n}_1\rangle\right| = 1$. In this case we take $z_1 = z_2$ to be any point of $L_1 = L_2$ and we use it to define the other parameters θ_1, θ_2 and ψ as above (ψ is simply the angle between γ_1 and γ_2). By making a different choice of point $\tilde{z}_1 = \tilde{z}_2$ we obtain different parameters $\tilde{\theta}_1$, $\tilde{\theta}_2$ and $\tilde{\psi}$. There is an equivalence relation relating two sets of parameters which correspond to the same four points p_i and q_i but to different choices of z_i and $\tilde{z}_i$. We shall show that this leads to a collapse of the parameter space where $r = 1$ to a 1-dimensional set. We give a geometrical interpretation of this collapse in the next section.

Writing $\mathbf{p}_i$ and $\mathbf{q}_i$ for lifts of p_i and q_i to $\mathbb{C}^3 - \{0\}$, we define the three cross-ratios of our points p_i and q_i as follows:

$$\mathbb{X}_1 = [p_2, p_1, q_1, q_2] = \frac{\langle \mathbf{q}_1, \mathbf{p}_2\rangle \langle \mathbf{q}_2, \mathbf{p}_1\rangle}{\langle \mathbf{q}_2, \mathbf{p}_2\rangle \langle \mathbf{q}_1, \mathbf{p}_1\rangle}, \tag{2.1}$$

$$\mathbb{X}_2 = [p_2, q_1, p_1, q_2] = \frac{\langle \mathbf{p}_1, \mathbf{p}_2\rangle \langle \mathbf{q}_2, \mathbf{q}_1\rangle}{\langle \mathbf{q}_2, \mathbf{p}_2\rangle \langle \mathbf{p}_1, \mathbf{q}_1\rangle}, \tag{2.2}$$

$$\mathbb{X}_3 = [p_1, q_1, p_2, q_2] = \frac{\langle \mathbf{p}_2, \mathbf{p}_1\rangle \langle \mathbf{q}_2, \mathbf{q}_1\rangle}{\langle \mathbf{q}_2, \mathbf{p}_1\rangle \langle \mathbf{p}_2, \mathbf{q}_1\rangle}. \tag{2.3}$$

These cross-ratios are 0 or ∞ only if a pair of the points coincide. By hypothesis we exclude this possibility. Our coordinates depend on grouping our points into two pairs. If we had chosen different pairs we would obtain different cross-ratios; see [12, Section 4.3] for further details. For example, the involution interchanging p_1 and p_2 and fixing q_1 and q_2 sends $\mathbb{X}_1$ to $1/\mathbb{X}_1$ and interchanges $\mathbb{X}_2$ and $\mathbb{X}_3$.

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Our main theorem is that we can express these three cross-ratios in terms of our geometrical coordinates $r, \theta_1, \theta_2, \psi$.

Theorem 2.1 *Let p_1, q_1, p_2, and q_2 be four pairwise distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$. Let L_i be the complex line spanned by p_i and q_i. Suppose that L_1 and L_2 are not asymptotic. Let r, θ_1, θ_2, and ψ be the geometrical coordinates defined above. Then*

$$\mathbb{X}_1 = \frac{r^2\, e^{i\theta_1+i\theta_2} - 2r\cos\psi + e^{-i\theta_1-i\theta_2}}{-4\sin\theta_1\sin\theta_2}, \tag{2.4}$$

$$\mathbb{X}_2 = \frac{r^2\, e^{-i\theta_1+i\theta_2} - 2r\cos\psi + e^{i\theta_1-i\theta_2}}{4\sin\theta_1\sin\theta_2}, \tag{2.5}$$

$$\mathbb{X}_3 = \frac{r^2\, e^{i\psi} - 2r\cos(\theta_1-\theta_2) + e^{-i\psi}}{r^2\, e^{i\psi} - 2r\cos(\theta_1+\theta_2) + e^{-i\psi}}. \tag{2.6}$$

Moreover, these expressions satisfy the cross-ratio identities (1.1) *and* (1.2), *and, when $r \neq 1$, the quantities $re^{i\psi}$, θ_1, and θ_2 may be written uniquely in terms of $\mathbb{X}_1$, $\mathbb{X}_2$, and $\mathbb{X}_3$.*

Finally, we consider the case where L_1 and L_2 are asymptotic. Again we have $r = \left|\langle \mathbf{n}_2, \mathbf{n}_1\rangle\right| = 1$. Let $z_1 = z_2$ be the intersection point of $\overline{L}_1$ and $\overline{L}_2$. The point z_i now lies on ∂L_i and so, provided it does not coincide with one of our four original points, z_i, p_i, and q_i are the vertices of an ideal triangle in L_i. This implies that θ_1 and θ_2 are each either 0 or π, and $\psi = |\theta_1 - \theta_2|$. Hence both the numerator and the denominator of each of (2.1), (2.2), and (2.3) vanish. We shall see that this configuration corresponds to a smooth point of $\mathfrak{X}$, that is a removable singularity of our coordinates. We use a blow-up at each of these points to evaluate $\mathbb{X}_1$, $\mathbb{X}_2$, and $\mathbb{X}_3$. In other words, we separate out the different tangent directions. Let $(r', \theta_1', \theta_2', \psi')$ be a tangent vector at one of the points where $r = 1$, each θ_i is either 0 or π, and $\psi = |\theta_1 - \theta_2|$. For each such tangent vector with $\theta_i' \neq 0$ we can express $\mathbb{X}_i$ in terms of r', θ_i', and ψ'.

Theorem 2.2 *Let p_1, q_1, p_2, and q_2 be four pairwise distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$. Let L_i be the complex line spanned by p_i and q_i. Suppose that L_1 and L_2 are asymptotic at a point distinct from p_i and q_i. Let $(r', \theta_1', \theta_2', \psi')$ be a tangent vector to the corresponding point of the space of coordinates defined above with $\theta_i' \neq 0$. Then $r' \neq 0$ and*

$$\mathbb{X}_1 = \frac{r'^2 + 2ir'(\theta_1' + \theta_2') - (\theta_1' + \theta_2')^2 + \psi'^2}{-4\theta_1'\theta_2'}, \tag{2.7}$$

$$\mathbb{X}_2 = \frac{r'^2 - 2ir'(\theta_1' - \theta_2') - (\theta_1' - \theta_2')^2 + \psi'^2}{4\theta_1'\theta_2'}, \tag{2.8}$$

$$\mathbb{X}_3 = \frac{r'^2 - 2ir'\psi' - \psi'^2 + (\theta_1' - \theta_2')^2}{r'^2 - 2ir'\psi' - \psi'^2 + (\theta_1' + \theta_2')^2}. \tag{2.9}$$

Moreover, these expressions satisfy the cross-ratio identities (1.1) *and* (1.2), *and θ_1'/r', θ_2'/r' and ψ'/r' may be expressed uniquely in terms of $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$.*

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The last case to consider is when L_1 and L_2 are asymptotic at one of our four points. In what follows, we suppose that this point is p_2. The other cases are similar and we discuss them briefly in the final section. The points p_1, q_1, and $z_1 = p_2$ again form an ideal triangle and so $\theta_1 = 0$ or π. Once again the denominator of (2.1) and (2.2) vanishes so we need to use a blow up. The difference is that this time we have $\theta_1' = 0$ as well. In order for $\mathbb{X}_1$ and $\mathbb{X}_2$ to be finite, their numerator must vanish as well. This means that we must use second derivatives of $r(t)$, $\theta_i(t)$, and $\psi(t)$.

Theorem 2.3 *Let p_1, q_1, p_2, and q_2 be four pairwise distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$. Let L_i be the complex line spanned by p_i and q_i. Suppose that L_1 and L_2 are asymptotic at p_2. Let $(r', \theta_1', \theta_2', \psi')$ be a tangent vector to the corresponding point of the space of coordinates. Then $\theta_1' = 0$, $r' = 0$, and $\psi' = \theta_2' \neq 0$. Let $(r'', \theta_1'', \theta_2'', \psi'')$ be the corresponding vector of second derivatives. Then*

$$\mathbb{X}_1 = \frac{ir'' - \theta_1'' - \theta_2'' + \psi''}{-2\theta_1''}, \quad \mathbb{X}_2 = \frac{ir'' + \theta_1'' - \theta_2'' + \psi''}{2\theta_1''}, \tag{2.10}$$
$$\mathbb{X}_3 = \frac{ir'' + \psi'' + \theta_1'' - \theta_2''}{ir'' + \psi'' - \theta_1'' - \theta_2''}.$$

In particular, $\mathbb{X}_2 = 1 - \mathbb{X}_1$ *and* $\mathbb{X}_3 = -\mathbb{X}_2/\mathbb{X}_1 = 1 - 1/\mathbb{X}_1$.

3 The Generic Cases

In this section we prove Theorem 2.1. We first treat the case where L_1 and L_2 are ultra-parallel and then the case where they intersect.

In the ultra-parallel case we normalise the four points p_1, q_1, p_2, and q_2 in terms of the parameters d, θ_1, θ_2 and ψ as follows. We do this both in terms of Heisenberg coordinates on $\partial\mathbf{H}^2_{\mathbb{C}}$ and as lifts of these points to $\mathbb{C}^3$. We begin by supposing that $\partial L_\perp$ is the vertical axis in the Heisenberg group. A complex line L is orthogonal to $L_\perp$ if and only if ∂L is a Euclidean circle centred on the vertical axis. We choose ∂L_1 to be the circle of radius 1 and height 0. Then the points p_1 and q_1 may be taken to be any points on this circle subtending an angle $2\theta_1$ at the centre. We take them to be

$$p_1 = (e^{i\theta_1}, 0), \quad q_2 = (e^{-i\theta_1}, 0).$$

We choose ∂L_2 to be another circle of height 0. The fact that the distance between L_1 and L_2 is $2d$ implies that the radius of ∂L_2 is e^d. Again, p_2 and q_2 are points on this circle subtending an angle of $2\theta_2$. The angle ψ is the angle between the centre of this interval and that of the interval between p_1 and q_1. We must be careful about the direction of ψ; see Figure 2.1. Since $e^d > 1$, when looking along the geodesic from z_1 to z_2, a positive angle ψ is negative with respect to the natural orientation on L_2. Thus we have

$$p_2 = (e^{d+i\theta_2-i\psi}, 0) \quad q_2 = (e^{d-i\theta_2-i\psi}, 0).$$

To calculate cross-ratios it is convenient to take the standard lifts of these four

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points. They are

$$\mathbf{p}_1 = \begin{bmatrix} -1 \\ \sqrt{2}e^{i\theta_1} \\ 1 \end{bmatrix}, \quad \mathbf{q}_1 = \begin{bmatrix} -1 \\ \sqrt{2}e^{-i\theta_1} \\ 1 \end{bmatrix}, \tag{3.1}$$

$$\mathbf{p}_2 = \begin{bmatrix} -e^{2d} \\ \sqrt{2}e^{d+i\theta_2-i\psi} \\ 1 \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} -e^{2d} \\ \sqrt{2}e^{d-i\theta_2-i\psi} \\ 1 \end{bmatrix}.$$

Substituting the vectors (3.1) into (2.1) gives:

$$\begin{aligned}\mathbb{X}_1 &= \frac{\langle \mathbf{q}_1, \mathbf{p}_2\rangle\langle \mathbf{q}_2, \mathbf{p}_1\rangle}{\langle \mathbf{q}_2, \mathbf{p}_2\rangle\langle \mathbf{q}_1, \mathbf{p}_1\rangle} \\ &= \frac{\left\langle \begin{bmatrix} -1 \\ \sqrt{2}e^{-i\theta_1} \\ 1 \end{bmatrix}, \begin{bmatrix} -e^{2d} \\ \sqrt{2}e^{d+i\theta_2-i\psi} \\ 1 \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} -e^{2d} \\ \sqrt{2}e^{d-i\theta_2-i\psi} \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ \sqrt{2}e^{i\theta_1} \\ 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} -e^{2d} \\ \sqrt{2}e^{d-i\theta_2-i\psi} \\ 1 \end{bmatrix}, \begin{bmatrix} -e^{2d} \\ \sqrt{2}e^{d+i\theta_2-i\psi} \\ 1 \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} -1 \\ \sqrt{2}e^{-i\theta_1} \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ \sqrt{2}e^{i\theta_1} \\ 1 \end{bmatrix} \right\rangle} \\ &= \frac{\cosh^2 d\, e^{i\theta_1+i\theta_2} - 2\cosh d \cos\psi + e^{-i\theta_1-i\theta_2}}{-4\sin\theta_1 \sin\theta_2}.\end{aligned}$$

This has proved (2.4). Substituting (3.1) into (2.2) and (2.3) enables us to prove (2.5) and (2.6) similarly:

$$\begin{aligned}\mathbb{X}_2 &= \frac{\langle \mathbf{p}_1, \mathbf{p}_2\rangle\langle \mathbf{q}_2, \mathbf{q}_1\rangle}{\langle \mathbf{q}_2, \mathbf{p}_2\rangle\langle \mathbf{p}_1, \mathbf{q}_1\rangle} = \frac{\cosh^2 d\, e^{-i\theta_1+i\theta_2} - 2\cosh d\cos\psi + e^{i\theta_1-i\theta_2}}{4\sin\theta_1\sin\theta_2}, \\ \mathbb{X}_3 &= \frac{\langle \mathbf{p}_2, \mathbf{p}_1\rangle\langle \mathbf{q}_2, \mathbf{q}_1\rangle}{\langle \mathbf{q}_2, \mathbf{p}_1\rangle\langle \mathbf{p}_2, \mathbf{q}_1\rangle} = \frac{\cosh^2 d\, e^{i\psi} - 2\cosh d\cos(\theta_1-\theta_2) + e^{-i\psi}}{\cosh^2 d\, e^{i\psi} - 2\cosh d\cos(\theta_1+\theta_2) + e^{-i\psi}}.\end{aligned}$$

The details of the case where L_1 and L_2 intersect are very similar and we leave them to the reader. In this case we have $r = \cos\phi < 1$ and, using the ball model, we normalise so that

$$\begin{aligned}p_1 &= (e^{i\theta_1}, 0), & q_1 &= (e^{-i\theta_1}, 0), \\ p_2 &= \left(re^{i\psi+i\theta_2}, \sqrt{1-r^2}e^{i\theta_2}\right), & q_2 &= \left(re^{i\psi-i\theta_2}, \sqrt{1-r^2}e^{-i\theta_2}\right).\end{aligned}$$

Lifting these points to $\mathbb{C}^3$ and substituting them into (2.1), (2.2) and (2.3) gives the expressions (2.4), (2.5) and (2.6).

When $L_1 = L_2$, that is $r = 1$, we choose $z_1 = z_2$ to be any point of $L_1 = L_2$. Then we may use the same formulae as in the intersecting case to find expressions for $\mathbb{X}_1$, $\mathbb{X}_2$, and $\mathbb{X}_3$. Namely:

$$\mathbb{X}_1 = \frac{\cos(\theta_1+\theta_2) - \cos\psi}{-2\sin\theta_1\sin\theta_2}, \quad \mathbb{X}_2 = \frac{\cos(\theta_1-\theta_2) - \cos\psi}{2\sin\theta_1\sin\theta_2},$$

$$\mathbb{X}_3 = \frac{\cos\psi - \cos(\theta_1-\theta_2)}{\cos\psi - \cos(\theta_1+\theta_2)}.$$

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Hence, it is clear that the $\mathbb{X}_i$ are real and satisfy $\mathbb{X}_2 = 1 - \mathbb{X}_1$ and $\mathbb{X}_3 = -\mathbb{X}_2/\mathbb{X}_1 = 1 - 1/\mathbb{X}_1$ and the cross-ratio variety has collapsed to a 1-dimensional set. These are the identities satisfied by the classical cross-ratio; see [1, page 76]. As we discussed above, making a different choice of $z_1 = z_2$ yields different parameters, also satisfying the above identities. We now interpret this geometrically. Let α_i be the geodesic joining p_i and q_i. If α_1 and α_2 intersect, let β be the angle between them. If α_1 and α_2 are ultra-parallel, let b be the distance between them. The parameters β and b are independent of our choice of $z_1 = z_2$. The geodesics γ_i and α_i (together with their common perpendicular if appropriate) form a hyperbolic quadrilateral or pentagon. Using hyperbolic trigonometry we can show that our coordinates give us β or b respectively in terms of $\mathbb{X}_1$. Namely, if c_i is the length of γ_i, then the angle of parallelism formulae imply:

$$1 - 2\mathbb{X}_1 = \frac{\cos\theta_1\cos\theta_2 - \cos\psi}{\sin\theta_1\sin\theta_2} = \sinh c_1 \sinh c_2 - \cos\psi\cosh c_1\cosh c_2.$$

Thus, using the identities in [5, Sections VI.3.4 and VI.3.2], we see that $1 - 2\mathbb{X}_1 = \cos\beta$ or $1 - 2\mathbb{X}_1 = \cosh b$ respectively. These identities are satisfied by the classical cross-ratio (see [1, Sections 7.24 and 7.23]) and this shows that, in this case, the Korányi–Reimann cross-ratio and the classical one coincide. The parameters θ_1, θ_2, ψ and $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\psi}$ are equivalent if and only if they determine quadrilaterals or pentagons with the same angle β or side length b.

We now show that, in the case where $r \neq 1$, we may solve for $re^{i\psi}$, θ_1, and θ_2 in terms of $\mathbb{X}_1, \mathbb{X}_2$, and $\mathbb{X}_3$. From (2.4), (2.5) and (2.6) we obtain

$$\begin{aligned}
2\Re(\mathbb{X}_1 + \mathbb{X}_2) &= r^2 + 1,\\
2\Im(\mathbb{X}_1 + \mathbb{X}_2) &= -(r^2 - 1)\cot\theta_2 \;=\; -\big(2\Re(\mathbb{X}_1 + \mathbb{X}_2) - 2\big)\cot\theta_2,\\
2\Im(\mathbb{X}_1 - \mathbb{X}_2) &= -(r^2 - 1)\cot\theta_1 \;=\; -\big(2\Re(\mathbb{X}_1 + \mathbb{X}_2) - 2\big)\cot\theta_1,\\
2\Re(\mathbb{X}_1 - \mathbb{X}_2) &= -(r^2 + 1)\cot\theta_1\cot\theta_2 + 2r\csc\theta_1\csc\theta_2\cos\psi.
\end{aligned}$$

The first three of these expressions enable us to express r^2, $\cot\theta_1$, and $\cot\theta_2$ in terms of $\mathbb{X}_1$ and $\mathbb{X}_2$ (for the last two we use $r^2 \neq 1$). Since $r \geq 0$ and $\theta_i \in (0, \pi)$, this determines r, θ_1, and θ_2. When $r = 0$ we have already explained why ψ is undefined. When $r \neq 0$ the last equation enables us to express $\cos\psi$ in terms of the other variables. Since ψ varies in $[0, 2\pi)$, this does not determine ψ uniquely. In order to do so, we must use $\mathbb{X}_3$ to find $\sin\psi$. When $r \neq 0$, we have

$$\Im\Big(\frac{1}{1 - \mathbb{X}_3}\Big) = \Im\Big(\frac{r^2 e^{i\psi} - 2r\cos(\theta_1 + \theta_2) + e^{-i\psi}}{4r\sin\theta_1\sin\theta_2}\Big) = \frac{(r^2 - 1)\sin\psi}{4r\sin\theta_1\sin\theta_2}.$$

Thus we can write $re^{i\psi}$ as a function of $\mathbb{X}_1, \mathbb{X}_2$, and $\mathbb{X}_3$.

To complete the proof of Theorem 2.1, we now verify that our expressions for $\mathbb{X}_1$,

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$\mathbb{X}_2$ and $\mathbb{X}_3$ satisfy (1.1) and (1.2). From (2.4) we have:

$$|\mathbb{X}_1|^2 = \frac{(r^2-1)^2 - 4r(r^2+1)\cos(\theta_1+\theta_2)\cos\psi + 4r^2(\cos^2(\theta_1+\theta_2)+\cos^2\psi)}{16\sin^2\theta_1\sin^2\theta_2},$$

$$|\mathbb{X}_2|^2 = \frac{(r^2-1)^2 - 4r(r^2+1)\cos(\theta_1-\theta_2)\cos\psi + 4r^2(\cos^2(\theta_1-\theta_2)+\cos^2\psi)}{16\sin^2\theta_1\sin^2\theta_2},$$

$$|\mathbb{X}_3|^2 = \frac{(r^2-1)^2 - 4r(r^2+1)\cos(\theta_1-\theta_2)\cos\psi + 4r^2(\cos^2(\theta_1-\theta_2)+\cos^2\psi)}{(r^2-1)^2 - 4r(r^2+1)\cos(\theta_1+\theta_2)\cos\psi + 4r^2(\cos^2(\theta_1+\theta_2)+\cos^2\psi)}.$$

This immediately yields the identity $|\mathbb{X}_3| = |\mathbb{X}_2|/|\mathbb{X}_1|$, which is (1.1). Using the expressions derived above and simplifying, we have:

$$\begin{aligned}
&2|\mathbb{X}_1|^2\Re(\mathbb{X}_3)\\
&= \frac{2\Re\big((r^2e^{i\psi} - 2r\cos(\theta_1-\theta_2) + e^{-i\psi})(r^2e^{-i\psi} - 2r\cos(\theta_1+\theta_2) + e^{i\psi})\big)}{16\sin^2\theta_1\sin^2\theta_2}\\
&= \frac{(r^2-1)^2 - 4r(r^2+1)\cos(\theta_1+\theta_2)\cos\psi + 4r^2\cos^2\psi + 4r^2\cos(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}{16\sin^2\theta_1\sin^2\theta_2}\\
&\quad + \frac{(r^2-1)^2 - 4r(r^2+1)\cos(\theta_1-\theta_2)\cos\psi + 4r^2\cos^2\psi + 4r^2\cos(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}{16\sin^2\theta_1\sin^2\theta_2}\\
&= |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 - 2\Re(\mathbb{X}_1+\mathbb{X}_2) + 1.
\end{aligned}$$

This verifies (1.2), as required.

4 The Asymptotic Case

In this section we consider the case where L_1 and L_2 are asymptotic and our goal is to prove Theorems 2.2 and 2.3.

Since L_1 and L_2 are asymptotic, we have $r = 1$. Assume first that the point $z_1 = z_2$ which is $\partial L_1 \cap \partial L_2$ is distinct from p_i and q_i. In this case z_i, p_i, and q_i form an ideal triangle. The internal angle of this triangle at z_i is 0. Thus $\theta_i = 0$ or $\theta_i = \pi$ depending on the orientation of γ_i. The tangent vectors to γ_i at z_i are either equal or opposite and so $\psi = 0$ or π respectively. In other words, $\psi = |\theta_1 - \theta_2|$. Hence the numerator and denominator in (2.1), (2.2), and (2.3) vanish.

Consider a path parametrised by t that is given by twice differentiable real valued functions $r = r(t)$, $\theta_i = \theta_i(t)$, and $\psi = \psi(t)$. Suppose that $\theta_i(t) \in (0,\pi)$ for $t \neq 0$ and also $r(0) = 1$, each $\theta_i(0)$ is either 0 or π and $\psi(0) = \big|\theta_1(0) - \theta_2(0)\big|$. We evaluate the $\mathbb{X}_i$ by substituting these values into our expressions from Theorem 2.1 and letting t tend to zero. Geometrically this is the same as a blow-up. We write $r'(t)$, $\theta_i'(t)$, and $\psi'(t)$ for the derivatives. For ease of notation, we write r', θ_i', and ψ' for $r'(0)$, $\theta_i'(0)$, and $\psi'(0)$ respectively. By reparametrising if necessary, we may suppose that $(r', \theta_1', \theta_2', \psi')$ is non-trivial and so is the tangent vector to our path at $t = 0$. Using l'Hôpital's rule twice on each of (2.1), (2.2), and (2.3) yields (2.7), (2.8), and (2.9). If $r' = 0$, then $r = 1$ for all points of our path. In other words, since $\theta_i(t) \in (0,\pi)$ for $t \neq 0$, we have $L_1 = L_2$ when $t \neq 0$ and hence, by continuity, $L_1 = L_2$ when

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$t = 0$ as well. This can be seen directly by observing that the $\mathbb{X}_i$ are all real and satisfy $\mathbb{X}_2 = 1 - \mathbb{X}_1$ and $\mathbb{X}_3 = -\mathbb{X}_2/\mathbb{X}_1 = 1 - 1/\mathbb{X}_1$. Then Proposition 4.14 of [10] implies that $L_1 = L_2$.

The effect of changing θ_1 from 0 to π or vice versa while fixing θ_2 (and so changing $\psi = |\theta_1 - \theta_2|$ as appropriate) is to multiply the numerator and denominator of each of the $\mathbb{X}_i$ by -1. Hence, the formulae (2.7), (2.8), and (2.9) are the same for each choice of $\theta_i \in \{0, \pi\}$ with $\psi = |\theta_1 - \theta_2|$. Suppose that we are given $\theta_1' > 0$. Since $\theta_1(t) \in (0, \pi)$ for $t \neq 0$, this is is the limit as t tends to zero from above of points where $\theta_1(t)$ tends to 0 but is the limit as t tends to 0 from below of points where $\theta_2(t)$ tends to π; that is, points approximated by $\theta_1(t) = t\theta_1'$ for $t > 0$ and $\theta_1'(t) = \pi + t\theta_1'$ for $t < 0$. Similar arguments apply to the case of $\theta_1' < 0$, $\theta_2' > 0$ and $\theta_2' < 0$. This leads to an identification of each tangent direction to the points $(r, \theta_1, \theta_2, \psi) = (1, 0, 0, 0)$ and $(1, \pi, \pi, 0)$ and the points $(1, \pi, 0, \pi)$ and $(1, 0, \pi, \pi)$.

We can verify (2.7), (2.8), and (2.9) directly as follows. Normalise so that L_1 and L_2 are asymptotic at $z_1 = z_2 = \infty$ on the boundary of the Siegel domain, and then take p_i and q_i to be points in the Heisenberg group with the following coordinates

$$p_1 = (0, \theta_1'), \quad q_1 = (0, -\theta_1'), \quad p_2 = \left(\sqrt{r'}, \theta_2' + \psi'\right), \quad q_2 = \left(\sqrt{r'}, -\theta_2' + \psi'\right).$$

Lifting these points to vectors in $\mathbb{C}^3$ and evaluating directly gives (2.7), (2.8), and (2.9).

As in the generic case, by writing:

$$\Re(\mathbb{X}_1 + \mathbb{X}_2) = 1, \quad \Im(\mathbb{X}_1 + \mathbb{X}_2) = \frac{r'}{-\theta_2'},$$
$$\Im(\mathbb{X}_1 - \mathbb{X}_2) = \frac{r'}{-\theta_1'}, \quad \Im\left(\frac{1}{1 - \mathbb{X}_3}\right) = \frac{-r'\psi'}{2\theta_1'\theta_2'},$$

it is obvious that we can completely determine θ_1'/r', θ_2'/r' and ψ'/r' from $\mathbb{X}_1$, $\mathbb{X}_2$, and $\mathbb{X}_3$. Furthermore, it is straightforward to verify that (2.7), (2.8), and (2.9) satisfy (1.1) and (1.2). This completes the proof of Theorem 2.2.

We now show that the configuration of Theorem 2.2 corresponds to a removable singularity of the coordinates. Specifically, we claim that the complex lines spanned by p_2 and q_1 and by p_1 and q_2 are not asymptotic. In order to see this, recall that swapping p_1 and p_2 while fixing q_1 and q_2 involves sending $\mathbb{X}_1$ to $1/\mathbb{X}_1$ and swapping $\mathbb{X}_2$ and $\mathbb{X}_3$. A short calculation shows that $\Re(1/\mathbb{X}_1 + \mathbb{X}_3) = 1$ if and only if $r'\theta_1'\theta_2' = 0$ proving the claim.

We now suppose that one of the θ_i' is zero. We will show that, in this case, the point $z_1 = z_2$ at which L_1 and L_2 are asymptotic must be one of the p_i or the q_i. Suppose that $\theta_1' = 0$ but yet $\mathbb{X}_i$ are each finite. Since the denominators of $\mathbb{X}_1$ and $\mathbb{X}_2$ vanish, so must the numerators. Hence, we must have $r'^2 + 2ir'\theta_2' - \theta_2'^2 + \psi'^2 = 0$. If $\theta_2' = 0$, then $r' = 0$ and $\psi' = 0$, which we have supposed not to be the case. So we assume that $\theta_2' \neq 0$, and so $r' = 0$ and $\psi' = \pm\theta_2'$. We take the plus sign. Then the expressions (2.10) follow by applying l'Hôpitals's rule to (2.7), (2.8), and (2.9). From these expressions it is easy to see that $\mathbb{X}_2 = 1 - \mathbb{X}_1$ and $\mathbb{X}_3 = -\mathbb{X}_2/\mathbb{X}_1 = 1 - 1/\mathbb{X}_1$.

Using [10, Proposition 4.7] we see that the angular invariant of p_1, q_1, and p_2 is $\pm\pi/2$ and so these three points all lie on the boundary of the same complex geodesic, namely L_1. In other words, L_1 and L_2 are asymptotic at p_2.

Conversely, for any real numbers r''/θ_1'' and $(\psi''-\theta_2'')/\theta_1''$ the expressions (2.10) are realised by taking

$$p_1 = (0, \theta_1''), \quad q_1 = (0, -\theta_1''), \quad p_2 = \infty, \quad q_2 = \left(\sqrt{r''}, -\theta_2'' + \psi''\right).$$

When $r'' \neq 0$ the point q_2 does not lie on ∂L_1, and so L_1 and L_2 are asymptotic at p_2.

If we had chosen $\theta_1' = 0$ and $\psi' = -\theta_2' \neq 0$, then we would have had $\mathbb{X}_2 = 1 - \mathbb{X}_1$ and $\mathbb{X}_3 = -\overline{\mathbb{X}}_2/\overline{\mathbb{X}}_1 = 1 - 1/\overline{\mathbb{X}}_1$. This would have lead to the case where L_1 and L_2 were asymptotic at q_2. Likewise, the cases where $\theta_2' = 0$ and $\psi' = \pm\theta_1' \neq 0$ correspond to the cases where L_1 and L_2 are asymptotic at p_1 or q_1.

Note added in proof. Recently Cunha and Gusevskii have pointed out that in the case where $L_1 = L_2$ quadruples of points that differ by an antiholomorphic isometry have the same cross-ratios. Therefore, in this special case, a point in the cross-ratio variety does not uniquely determine the quadruple of points [2].

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