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ON NILPOTENT GROUPS OF ALGEBRA AUTOMORPHISMS

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Introduction

The main purpose of this paper is to derive conclusions about the structure of a nilpotent group of algebra automorphisms and, in the case of a Lie algebra, about the influence of this nilpotence on the structure of the algebra. A motivation for this study is a well known theorem due to Kolchin: A unipotent linear group can be triangularized and is thus nilpotent. The converse is manifestly false, but we have (as an immediate consequence of Theorem 2.7):

THEOREM (*). *Let G be the automorphism group of an n-dimensional nilpotent Lie algebra over a perfect field, $n \neq 1$. If G is nilpotent then it is unipotent.*

This theorem does not generalize completely to non-associative algebras. We exhibit (Remark 2.3), in fact, a nilpotent four-dimensional algebra whose automorphism group is nilpotent and yet contains a semi-simple element of period two. However, the example pinpoints precisely what can happen for we have (from Theorem 2.2):

THEOREM (**). *Let G be the automorphism group of an n-dimensional nilpotent algebra over a perfect field, $n \neq 1$. If G is nilpotent then it is the direct product of a finite cyclic group and a unipotent group.*

Section 2 continues with some consequences of Theorem (*). It includes, also, the following theorem (see Theorem 2.11) which appeared in [7] where the ground field was assumed algebraically closed:

THEOREM (***). *Let L be a finite dimensional Lie algebra over any field of characteristic* 0. *Then L is characteristically nilpotent if and only if every semi-simple*

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automorphism is of finite order.

Section 3 contains examples which illustrate, among other things, that

(i) A nilpotent Lie algebra, even over an algebraically closed field, can have a nilpotent automorphism group and yet have a non-nilpotent derivation algebra.

(ii) The property of nilpotence of the automorphism group of a Lie algebra is not independent of extensions of the base field, even in characteristic 0.

(iii) The automorphism group of a non-nilpotent Lie algebra over an algebraically closed field need not contain a torus of positive dimension. (This answers a question of Winter [9, p. 141]).

Unlike derivation algebras, automorphism groups are difficult to compute and section 3 may also be considered as an illustration of techniques which may be used in this regard.

It is our pleasure to thank Professor G. Hochschild for reading an earlier draft of this paper and for his valuable suggestions.

1. Preliminaries

By an algebra, in this paper, we mean a finite dimensional non-associative algebra. If S is a subset of an algebra, B, S^r is the subspace spanned by all products of r elements of S, no matter how associated. B is called nilpotent of index r if $B^{r-1} \neq 0$, $B^r = 0$. $\Delta(B)$ denotes the Lie algebra of derivations of B. $A(B)$ is the automorphism group of B; $A_t(B)$, for $t \geq 2$, the subgroup of $A(B)$ of automorphisms which induce the identity on B/B^t; if B is nilpotent, $A_t(B)$ consists of unipotent transformations.

If G is a linear group over a field k, i.e., a subgroup of some $GL(n, k)$, G_u denotes the set of all unipotent elements in G, G_s the set of all semi-simple elements, and $\bar{G}$ the smallest k-closed subgroup of $GL(n, k)$ containing G. We denote by G^K the algebraic K-group deduced from an algebraic k-group G by an extension to K of the base field [4, p. 105]. If G is algebraic, G_0 denotes the connected component of the identity.

We call a k-linear transformation absolutely semi-simple if the natural extensions, σ^K, of σ are semi-simple for all extensions, K, of k. If G is a nilpotent linear group over k then G_u is a subgroup of G and is centralized by any absolutely semi-simple element $\sigma \in G$. (To see this, note we may assume, by the hypothesis on σ, that k is algebraically closed and since $\bar{G}$ is nilpotent, that G is algebraic. The assertions then follow from well-known theorems [8, section 6]).

2. The Main Theorems

The proof of the following lemma is immediate, yet it will be a most useful result.

2.1 LEMMA. *If D is a nilpotent derivation of an algebra, B, such that $(D(B))^2 = (0)$ then $I + D$ is a unipotent automorphism of B.*

2.2 THEOREM. *Let B be a nilpotent algebra of index $r+1$, $r \geq 2$, over a perfect field k. Suppose G is a nilpotent group of automorphisms of B containing $A_r(B)$. Then G_s is a finite cyclic group whose order divides $r-1$ and its elements induce scalar multiples of the identity on B/B^2.*

Proof. Since $G_u \subseteq (\bar{G})_u$, $G_s \subseteq (\bar{G})_s$ and $\bar{G}$ is nilpotent, we may assume G is k-closed. Let $\sigma \in G_s$. Choose a σ-stable subspace, U, of B such that $U \cap B^2 = (0)$. Then $B = U + U^2 + \cdots + U^r$ with U^r non-zero and contained in the annihilator of B. Let H be the subspace of $\mathrm{Hom}_k(B, B)$ consisting of transformations which are zero on B^2 and map U to U^r. Then $H \subseteq \Delta(B)$, and by Lemma 2.1, if $h \in H$, $I + h \in A_r(B) \subseteq G_u$ which is centralized by σ. Thus σ centralizes H and is necessarily a scalar, a, times the identity on $U + U^r$. It follows that σ is a^t times the identity on U^t, $t = 1, 2, \cdots, r$, in particular, $a^r = a$, i.e., $a^{r-1} = 1$. Since σ induces a scalar multiple of the identity on B/B^2, one has for $\tau \in G$, $\sigma\tau\sigma^{-1}\tau^{-1} \in G \cap A_2(B) \subseteq G_u$. Then, by [1, Lemma 9.7], σ centralizes G. But now G_s is an abelian subgroup of G, and therefore U may have been chosen to be a G_s-stable subspace. Since each element of G_s must be an $(r-1)st$ root of unity times the identity on U, the assertion follows.

We observe that Theorem (**) of the introduction follows immediately.

2.3 *Remark.* In view of the stronger statement to be proved below (Theorem 2.7) in the case of a Lie algebra, it is worth exhibiting an example in which G_s is not trivial. Let B be the four-dimensional algebra over any field of characteristic not 2 with basis x_1, x_2, x_3, x_4 and multiplication determined by $x_1x_1 = x_3$, $x_1x_2 = x_4$, $x_2x_2 = x_4$, $x_3x_3 = x_4$ and all unlisted products, x_ix_j, zero. Then $A(B)$ is nilpotent. In this case $A(B)_s = \{I, \sigma\}$ where, with respect to the given basis, $\sigma = \mathrm{diag}\,(-1, -1, 1, 1)$. It is interesting to note also that $A(B)_u$ is comprised entirely of the automorphisms, $I + h \in A_4(B)$, which were employed in the proof of Theorem 2.2. (See also 2.6 (ii)).

2.4 THEOREM. *Let L be a nilpotent Lie algebra over any field k, dim $L > 1$, G a nilpotent group of automorphisms of L containing $A(L)_u$. Then every absolutely semi-simple element of G is the identity.*

Proof. Let σ be an absolutely semi-simple element of G. Choose a σ-stable subspace, U, of L such that $L = U + L^2$, $U \cap L^2 = (0)$. $L^2 \neq (0)$ since the subgroup of $GL(n, k)$, for $n > 1$, generated by the unipotent elements is not nilpotent. As in the proof of Theorem 2.2, there is a non-zero $a \in k$ such that σ is a^t times the identity on each U^t, and, if $r+1$ is the index of nilpotency of L, $a^{r-1} = 1$. We must show $a = 1$. If $r = 2$, we already have $a^1 = 1$. Suppose $r \geq 3$. Choose x in U^{r-2} and not in the center of L. By Lemma 2.1, $I + (\text{ad}\, x)$ is in $A(L)_u$ and therefore commutes with σ. Thus $\text{ad}\, x = \sigma(\text{ad}\, x)\sigma^{-1} = \text{ad}\, \sigma(x) = \text{ad}\,(a^{r-2}x)$, which implies $a^{r-2} = 1$ and so $a = 1$.

It is customary to call nilpotent Lie algebra of index $r+1$ quasi-cyclic, if, for some subspace U, $L = U \dotplus U^2 \dotplus \cdots \dotplus U^r$ (where $\dotplus$ denotes vector space direct sum).

2.5 COROLLARY. *Let L be a quasi-cyclic Lie algebra over a field, k, with more than two elements, dim $L > 1$. Then $A(L)$ is not nilpotent.*

Proof. Let U be as above. Then, for any non-zero $a \in k$, the linear transformation σ_a, such that $\sigma_a(x) = a^i x$ for $x \in U^i$, is an automorphism of L. By Theorem 2.4, $A(L)$ could not be nilpotent.

2.6 *Remarks.* (i) We shall construct (see 3.5) an example indicating the necessity in 2.5 of hypothesizing k to have more than two elements.

(ii) The corollary almost extends to general non-associative algebras. For, recalling again the proof of 2.2, one finds, assuming a nilpotent automorphism group, the order of each σ_a must divide $r-1$. Thus, if k is infinite or if the order of the multiplicative groups, k^*, does not divide $r-1$, the automorphism group cannot be nilpotent. Note, however, that if B is the four dimensional algebra over Z_3 with basis x_1, x_2, x_3, x_4 such that the only non-zero products of these are $x_1x_1 = x_3$, $x_2x_3 = x_3x_1 = x_4$, then, letting $U = (x_1, x_2)$, $B = U \dotplus U^2 \dotplus U^3$; nevertheless, $A(B)$ is nilpotent.

Recalling Kolchin's Theorem, we restate Theorem 2.4 as follows:

2.7 THEOREM. *Let L be a nilpotent Lie algebra,* dim $L > 1$, *over a perfect field k, G a subgroup of $A(L)$ containing $A(L)_u$. Then G is nilpotent if and only if*

$G = A(L)_u$.

2.8 *Remarks.* (i) If the group, G, in the theorem is $A(L)$ (as in Theorem (*) of the introduction), and k has characteristic 0, then the hypothesis of nilpotence of L is superfluous. For, in characteristic 0, $\Delta(L)$ is the Lie algebra of $A(L)$ and, if $A(L)$ is nilpotent, $\Delta(L)$ is nilpotent. But letting $Z(L)$ denote the center of L, $L/Z(L)$ is isomorphic to the algebra of inner derivations, whose nilpotence then implies the nilpotence of L. The assertion fails in characteristic $p > 0$; for example, the automorphism group of the two-dimensional non-abelian Lie algebra, S, over a two element field is nilpotent although S is not. Furthermore, it will be seen in 3.2 that such examples can exist over an algebraically closed field. Note, we are still unable to justify the aforementioned hypothesis since the automorphism groups of these examples are, in fact, unipotent. Incidently, the statement "$A(L)$ nilpotent implies $\Delta(L)$ nilpotent" may fail in characteristic $p \neq 0$ even for a nilpotent L over an algebraically closed field as will also be illustrated in 3.2.

(ii) A Lie algebra, L, is called characteristically nilpotent if all its derivations are nilpotent. It is shown in [7, Theorem 1] that this is equivalent for $\dim L \neq 1$ to the nilpotence of $\Delta(L)$. The first example of such an algebra was given by Dixmier-Lister [5]. It is remarked in [6] that, although the Dixmier-Lister example has a nilpotent derivation algebra, its automorphism group is not nilpotent. We note that this is an immediate consequence of 2.7 for the example was shown in [5] to have a semi-simple automorphism of period 2.

(iii) Professor G. Hochschild mentioned, in a personal communication, his interest in the question of when the automorphism group of a nilpotent Lie algebra is connected, having noticed this is not the case for the Dixmier-Lister example. This too follows from Theorem 2.7, for, in characteristic 0, all the unipotent automorphisms are in the connected component of the identity which is nilpotent, since $\Delta(L)$ is nilpotent, and so it could not contain the automorphism of period 2. In a positive direction, we can conclude from 2.7 that, in characteristic 0, if $A(L)$ is nilpotent then it is connected (a unipotent group is connected).

We have the following elementary corollary of Theorem 2.7:

2.9 COROLLARY. *Let L be a nilpotent Lie algebra over a perfect field, k, with*

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more than two elements, $\dim L > 1$. *Suppose* $A(L)$ *is nilpotent. Then the center of* L *is contained in the derived algebra.*

Proof. If z were in the center of L but not in L^2, (z) would be an abelian direct summand and so the elements of $GL((z))$ would extend to automorphisms of L. However, $GL((z))$ is not unipotent.

We note that the assumption that k contains more than two elements is essential. (See 3.5)

As indicated in the introduction, we conclude this section with a strengthening of a theorem in [7]. For this we need the following lemma which is the generalization of [1, Corollary 16.7] to k-groups for certain k.

2.10 LEMMA. *Let* G *be a connected linear algebraic* k*-group such that every element of* G_s *has finite order. Suppose that* k^* *contains an element,* x*, of infinite order (thus, either* k *has characteristic* 0 *or* k *is a non-algebraic extension of its prime field). Then* G *is unipotent.*

Proof. It suffices to show that every element of $(G^\Omega)_s$ has finite order, where Ω is a universal domain containing k, for then by [1, 16.7] G^Ω, and therefore G, consists of unipotent transformations. Let k_0 be the minimal field of definition of G^Ω which contains x. By [2, Theorem A] G^Ω has a maximal torus, T, which is defined over k_0 and since k_0 is infinite, the set of k_0-rational points of T, which we denote by H, is Zariski-dense in T. Now H is an algebraic k_0-group and, since $H \subseteq G_s$, every element of H is of finite order. If, for $\sigma \in H$, $\sigma^r = 1$ then the characteristic roots of σ are rth roots of unity. Furthermore they are each of degree $\leq m$ over k_0, where $H \subseteq GL(m, k_0)$. However, k_0 is a finitely generated extension of a prime field and so there are only a finite number of roots of unity of degree $\leq m$ over k_0. Thus, there is some N such that $\sigma^N = I$ for all $\sigma \in H$. Since H is dense in T, the same relation is satisfied by the elements of T. Then, since each element of $(G^\Omega)_s$ lies in some maximal torus which is necessarily conjugate to T, the elements of $(G^\Omega)_s$ are all of finite order.

Theorem (***) of the introduction can, with no additional difficulties, be generalized to arbitrary algebras:

2.11 THEOREM. *Let* B *be an algebra over a field,* k*, of characteristic* 0. *Then* B *has only nilpotent derivations if and only if all the semi-simple automorphisms of* B *are of finite order.*

Proof. The elements of $A(B)_s$ are all of finite order if and only if the same holds for the semi-simple elements of $A(B)_0$. Then, considering Lemma 2.10, we need only recall that $\Delta(B)$ is the Lie algebra of $A(B)_0$ and therefore it consists of nilpotent transformations if and only if $A(B)_0$ consists of unipotent transformations.

3. Examples

We present, in this section, the examples promised in the introduction and sections 2.

Consider first the nine-dimensional Lie algebra, L_1, over any field k: L_1 has basis $x_1, x_2, \cdots, x_9$ and

$$[x_1, x_2] = x_4,\ [x_2, x_3] = x_5,\ [x_1, x_3] = x_6,\ [x_2, x_4] = x_6,\ [x_1, x_4] = x_7,$$

$$[x_3, x_5] = x_8,\ [x_2, x_5] = x_9,\ [x_1, x_7] = x_9,\ [x_3, x_8] = x_9,$$

with $[x_i, x_j] = 0$ for $i < j$ if it is not listed above.

3.1 PROPOSITION (i) *If k does not contain a primitive 3rd root of unity, $A(L_1)$ is unipotent.*

(ii) *If $\omega^3 = 1$, for $\omega \in k$, then, with respect to the given basis,* diag $(\omega, \omega, \omega^2, \omega^2, 1, 1, 1, \omega^2, \omega)$ *is an automorphism of L_1.*

Proof. (ii) is clear. (i) is an easy computation after one notes that the ideals $B_j = (x_i) + L_1^2$, $i = 1, 2, 3$ are invariant under all automorphisms (and so, for $\alpha \in A(L_1)$, $\alpha(x_i) \equiv c_i x_i \mod (L_1^2)$, with $c_i \in k$, for $i = 1, 2, 3$). For this observe that B_2 is the centralizer of L_1^3. Then, letting $C = \{z \in L_1 \mid (adw)^2(z) \in L_1^4$ for all $w \in B_2\}$, one finds $C = (x_2, x_3) + L_1^2$ and B_1 is the centralizer of C^2 while B_3 is the centralizer of $[B_1, B_2]$.

Applying Proposition 3.1, we find L_1 has some remarkable properties:

3.2 If k is any field of characteristic 3, $A(L_1)$ is nilpotent. However $\Delta(L_1)$ contains non-trivial semi-simple elements, e.g., $\delta=\mathrm{diag}\,(1, 1, -1, -1, 0, 0, 0, -1, 1)$, and is by [7, Theorem 1] not nilpotent. It is interesting to note also that, in this characteristic, $\Delta(L_1)$ cannot be the Lie algebra of $A(L_1)$ since the latter, having only unipotent elements, must have a nilpotent Lie algebra. In [4, p. 143] an algebra, V, over an infinite field is exhibited such that $\Delta(V)$ is not the Lie algebra of $A(V)$. However, the example requires an imperfect ground field and is not a Lie algebra. The present example is not thus restricted.

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Now let δ be as above and let H denote the semi-direct sum $(\delta)+L_1$. Suppose $\alpha \in A(H)$. Since $H^2 = L_1$, α leaves L_1 invariant so its restriction to L_1 is necessarily unipotent and, since $L_1^4 = (x_9)$, $\alpha(x_9) = x_9$. Then $x_9 = \alpha(\delta x_9) = \alpha(\delta)x_9$, whence $\alpha(\delta) \equiv \delta \bmod (L_1)$. Thus, although H is non-nilpotent, $A(H)$ is unipotent. (See (iii) in the introduction).

3.3 We see also from Proposition 3.1 that the property of nilpotence of $A(L)$ is not independent of extensions of the ground field, even in characteristic 0. Thus, in particular, the usual extension, $A(L)^K$, of $A(L)$ to an algebraic group over an extension, K, of k may not be the automorphism group of the extended algebra $L \otimes K$.

There is a well-known example of a characteristically nilpotent algebra given in [3, p. 123]. This algebra, L_2, has dimension eight over a field k of characteristic $\neq 2$ and the following mulitiplication.

$$[x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\quad [x_1, x_4] = x_5,\quad [x_1, x_5] = x_6,$$
$$[x_1, x_6] = x_8,\ [x_1, x_7] = x_8,\quad [x_2, x_3] = x_5,\quad [x_2, x_4] = x_6,$$
$$[x_2, x_5] = x_7,\ [x_2, x_6] = 2x_8,\ [x_3, x_4] = -x_7 + x_8,\ [x_3, x_5] = -x_8,$$

with $[x_i, x_j] = 0$ for $i < j$ and $i + j > 8$. In view of [6], it seems relevant to prove

3.4 PROPOSITION. *$A(L_2)$ is nilpotent.*

Proof. It suffices to prove $A(L_2) = A_2(L_2)$. For this, note the following:

(i) $(x_2)+L_2^2$ is the transporter of L_2^2 into L_2^4, i.e. $\{z \in L_2 \mid [z, L_2^2] \subseteq L_2^4\}$, and hence is invariant under every automorphism.

(ii) If $y_1, y_2 \in L_2$ with $y_1 \notin (x_2) + L_2^2$ then $(\operatorname{ad} y_1)^2 - \operatorname{ad} y_2$ maps L_2^2 into $(L_2^2)^2 = (x_7, x_8)$ if and only if $y_i \equiv a^i x_i \bmod L_2^2$, $i = 1, 2$, for some $0 \neq a \in k$.

(iii) By (ii), $(x_1) + L_2^2$ is invariant under all automorphisms and therefore so is $[(x_1) + L_2^2, L_2^4] = (x_6, x_8)$. But, for $z \notin (x_2) + L_2^2$, $\operatorname{ad} z|_{(x_6, x_8)}$ is zero if and only if $z \equiv b\left(x_1 - \frac{1}{2}x_2\right) \bmod (L_2^2)$, for some $0 \neq b \in k$.

Now let $\alpha \in A(L_2)$. By (ii) and (iii) we have

$$\alpha(x_1) \equiv a x_1 \bmod (L_2^2)$$
$$\alpha(x_2) \equiv a^2 x_2 \bmod (L_2^2)$$
$$\alpha\left(x_1 - \frac{1}{2}x_2\right) \equiv b\left(x_1 - \frac{1}{2}x_2\right) \bmod (L_2^2)$$

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with $a, b \in k$. Then we must have $a = b = 1$.

We conclude with an example of a nilpotent Lie algebra, L_3, over Z_2, which is quasi-cyclic and whose center is not contained in the derived algebra, yet $A(L_3)$ is nilpotent (see corollaries 2.5 and 2.9). L_3 has basis $x_1, x_2, \cdots, x_8$ with $[x_1, x_2] = x_6$, $[x_1, x_3] = x_7$, $[x_2, x_4] = x_8$, and $[x_i, x_j] = 0$ for $i < j$ otherwise.

3.5 PROPOSITION. *$A(L_3)$ is nilpotent.*

Proof. $L_3^2 = (x_6, x_7, x_8)$; $Z(L_3)$, the center, $= (x_5) + L_3^2$. For $y \in L_3$, ad y has rank 1 if and only if $y \equiv$ either x_3 or x_4 mod $(Z(L_3))$. Thus, for $\alpha \in A(L_3)$,

$$\alpha^2(x_i) \equiv x_i \bmod (Z(L_3)), \text{ for } i = 3, 4.$$

Further, the centralizer of $(x_3) + Z(L_3)$ is $(x_2, x_3, x_4, \cdots, x_8)$ while that of $(x_4) + Z(L_3)$ is $(x_1, x_3, x_4,, \cdots, x_8)$.

Thus,

$\alpha^2(x_i) \equiv x_i \bmod ((x_3, x_4, \cdots, x_8))$, for $i = 1, 2$. It follows easily that α^2, and therefore α, is unipotent.

REFERENCES

[1] A. Borel, Groupes linéaires algébriques, Annals of Mathematics, Vol. **64** (1956), pp. 20–82.
[2] A. Borel and T.A. Springer, Rationality properties of linear algebraic groups, Proc. Symp. Pure Math., vol. **9,** Amer. Math. Soc., Providence, R.I., 1966, pp. 26–32.
[3] N. Bourbaki, Groupes et Algèbres de Lie, Hermann, Paris, 1960.
[4] C. Chevalley, Théorie des Groupes de Lie, Hermann, Paris, 1968.
[5] J. Dixmier and W.G. Lister, Derivations of nilpotent Lie algebras, Proc. Amer. Math. Soc., vol. **8** (1957) pp. 155–158.
[6] J. Dyer, A nilpotent Lie algebra with nilpotent automorphism group, Bull. Amer. Math. Soc., vol. **76** (1970) pp. 52–56.
[7] G. Leger and S. Tôgô, Characteristically nilpotent Lie algebras, Duke Math. J., vol. **26** (1959) pp. 623–628.
[8] M. Rosenlicht, Nilpotent linear algebraic groups, Seminari 1962/63 Anal. Alg. Geom. Topol., vol. 1, Ist. Naz. Alta Mat., Ediz. Cremonese, Rome, 1965, pp. 133–152.
[9] D. Winter, On groups of automorphisms of Lie algebras, J. Algebra, vol. **8** (1968), pp. 131–142.

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